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Geometry of multilinear forms on l_1

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Abstract. We characterize extreme, exposed and smooth points in the Banach space $\mathcal{L}(^nE)$ of continuous n-linear forms on E, and in its subspace $\mathcal{L}_s(^nE)$ of symmetric n-linear forms on E when $E = l_1$ and $E = l_1^m$ for $n, m \in \mathbb{N}$ with $n, m \geq 2$.

1. Introduction

Throughout the paper, we let $n, m \in \mathbb{N}, n, m \geq 2$. We write B_E for the closed unit ball of a real Banach space E, and the dual space of E is denoted by E^*. An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y + z)$ implies $x = y = z$. An element $x \in B_E$ is called an *exposed point* of B_E if there is $f \in E^*$ such that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. An element $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. We denote by $\operatorname{ext} B_E, \exp B_E$ and $\operatorname{sm} B_E$ the set of extreme points, the set of exposed points and the set of smooth points of B_E, respectively. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form T on the product $E \times \cdots \times E$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \ldots, x_n)|$ and $\mathcal{L}_s(^nE)$ denotes the closed subspace of all continuous symmetric n-linear forms on E. Notice that $\mathcal{L}(^nE)$ is identified with the dual of n-fold projective tensor product $\hat{\bigotimes}_{\pi,n}E$. With this identification, the action of a continuous n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{\pi,n}E$ is given by

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$$\Big\langle \sum_{i=1}^{k} x^{(1),i}\otimes\cdots\otimes x^{(n),i},\ T\Big\rangle = \sum_{i=1}^{k} T\Big(x^{(1),i},\ldots,x^{(n),i}\Big).$$

Notice also that $\mathcal{L}_s({}^nE)$ is identified with the dual of n-fold symmetric projective tensor product $\hat{\bigotimes}_{s,\pi,n}E$. With this identification, the action of a continuous symmetric n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{s,\pi,n}E$ is given by

$$\Big\langle \sum_{i=1}^{k} \frac{1}{n!}\Big(\sum_{\sigma} x^{\sigma(1),i}\otimes\cdots\otimes x^{\sigma(n),i}\Big),\ T\Big\rangle = \sum_{i=1}^{k} T\Big(x^{(1),i},\ldots,x^{(n),i}\Big),$$

where σ goes over all permutations on $\{1,\ldots,n\}$. For more details about the theory of polynomials and multilinear mappings on Banach spaces, we refer to [6].

Let us say a little bit about the history of classification problems of the extreme points, the exposed points, and the smooth points of the unit ball of continuous n-homogeneous polynomials on a Banach space.

We let $l_p^n = \mathbb{R}^n$ for every $1 \le p \le \infty$ equipped with the l_p-norm. Choi et al. ([2], [3], [4]) classified $\operatorname{ext} B_{\mathcal{P}({}^2 l_p^2)}$ and $\operatorname{sm} B_{\mathcal{P}({}^2 l_p^2)}$ for $p = 1, 2$. Choi and Kim [5] classified $\operatorname{sm} B_{\mathcal{P}({}^2 l_1)}$. Grecu [7] classified the sets $\operatorname{ext} B_{\mathcal{P}({}^2 l_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. Kim et al. [26] showed that if E is a separable real Hilbert space with $\dim(E) \ge 2$, then, $\operatorname{ext} B_{\mathcal{P}({}^2E)} = \exp B_{\mathcal{P}({}^2E)}$. Kim [8] classified $\exp B_{\mathcal{P}({}^2 l_p^2)}$ for $1 \le p \le \infty$. Kim ([10], [12]) characterized $\operatorname{ext} B_{\mathcal{P}({}^2 d_*(1,w)^2)}$ and $\operatorname{sm} B_{\mathcal{P}({}^2 d_*(1,w)^2)}$, where $d_*(1,w)^2 = \mathbb{R}^2$ with an octagonal norm $\|(x,y)\|_w = \max\Big\{|x|, |y|, \frac{|x|+|y|}{1+w}\Big\}$ for $0 < w < 1$. Kim [16] classified $\exp B_{\mathcal{P}({}^2 d_*(1,w)^2)}$ and showed that $\exp B_{\mathcal{P}({}^2 d_*(1,w)^2)} \ne \operatorname{ext} B_{\mathcal{P}({}^2 d_*(1,w)^2)}$. Recently, Kim ([17], [21]) classified $\operatorname{ext} B_{\mathcal{P}({}^2\mathbb{R}^2_{h(\frac{1}{2})})}$ and $\exp B_{\mathcal{P}({}^2\mathbb{R}^2_{h(\frac{1}{2})})}$, where $\mathbb{R}^2_{h(\frac{1}{2})} = \mathbb{R}^2$ with a hexagonal norm $\|(x,y)\|_{h(\frac{1}{2})} = \max\Big\{|y|, |x| + \frac{1}{2}|y|\Big\}$.

Parallel to the classification problems of $\operatorname{ext} B_{\mathcal{P}({}^nE)}$, $\exp B_{\mathcal{P}({}^nE)}$, and $\operatorname{sm} B_{\mathcal{P}({}^nE)}$ it seems to be very natural to study the, classification problems of extreme and exposed points of the unit ball of continuous (symmetric) multilinear forms on a Banach space.

Kim [9] classified $\operatorname{ext} B_{\mathcal{L}_s({}^2 l_\infty^2)}, \exp B_{\mathcal{L}_s({}^2 l_\infty^2)}$ and $\operatorname{sm} B_{\mathcal{L}_s({}^2 l_\infty^2)}$. It was shown that $\operatorname{ext} B_{\mathcal{L}_s({}^2 l_\infty^2)} = \exp B_{\mathcal{L}_s({}^2 l_\infty^2)}$. Kim ([11], [13], [14], [15]) classified $\operatorname{ext} B_X$ and $\exp B_X$, where $X = \mathcal{L}({}^2 d_*(1,w)^2)$ or $\mathcal{L}_s({}^2 d_*(1,w)^2)$. Kim ([18], [19]) also classified $\operatorname{ext} B_{\mathcal{L}_s({}^2 l_\infty^3)}$, $\operatorname{ext} B_{\mathcal{L}_s({}^3 l_\infty^2)}$ and $\operatorname{sm} B_{\mathcal{L}_s({}^3 l_\infty^2)}$. It was shown that $\operatorname{ext} B_{\mathcal{L}_s({}^2 l_\infty^3)} = \exp B_{\mathcal{L}_s({}^2 l_\infty^3)}$ and $\operatorname{ext} B_{\mathcal{L}_s({}^3 l_\infty^2)} = \exp B_{\mathcal{L}_s({}^3 l_\infty^2)}$. Kim [20] characterized $\operatorname{ext} B_{\mathcal{L}({}^2 l_\infty^n)}$ and $\operatorname{ext} B_{\mathcal{L}_s({}^2 l_\infty^n)}$, and showed that $\exp B_{\mathcal{L}({}^2 l_\infty^n)} = \operatorname{ext} B_{\mathcal{L}({}^2 l_\infty^n)}$ and $\exp B_{\mathcal{L}_s({}^2 l_\infty^n)} = \operatorname{ext} B_{\mathcal{L}_s({}^2 l_\infty^n)}$. Kim [22] characterized $\operatorname{ext} B_{\mathcal{L}({}^2 l_\infty^3)}$ and $\exp B_{\mathcal{L}({}^2 l_\infty^3)}$. Kim [25] characterized $\operatorname{sm} B_{\mathcal{L}_s({}^n l_\infty^2)}$.

Kim [24] studied $\operatorname{ext} B_{\mathcal{L}(^2 l_\infty)}$. Cavalcante et al. [1] characterized $\operatorname{ext} B_{\mathcal{L}(^n l_\infty^m)}$. Recently, Kim [23] classified $\operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)}$ and $\operatorname{ext} B_{\mathcal{L}_s(^n l_\infty^2)}$. It was shown that $|\operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)}| = 2^{(2^n)}$ and $|\operatorname{ext} B_{\mathcal{L}_s(^n l_\infty^2)}| = 2^{n+1}$, and that $\exp B_{\mathcal{L}(^n l_\infty^2)} = \operatorname{ext} B_{\mathcal{L}(^n l_\infty^2)}$ and $\exp B_{\mathcal{L}_s(^n l_\infty^2)} = \operatorname{ext} B_{\mathcal{L}_s(^n l_\infty^2)}$.

In this paper, we characterize the extreme and exposed points of the unit balls of $\mathcal{L}(^n l_1)$ and $\mathcal{L}_s(^n l_1)$. We also characterize the smooth points of the unit balls of $\mathcal{L}(^n l_1^m)$ and $\mathcal{L}_s(^n l_1^m)$ for $m \geq 2$.

2. The unit ball of $\mathcal{L}(^n l_1)$

In this section we characterize $\operatorname{ext} B_{\mathcal{L}(^n l_1)}, \exp B_{\mathcal{L}(^n l_1)}$ and $\operatorname{sm} B_{\mathcal{L}(^n l_1^m)}$ for $n, m \geq 2$. First, we present an explicit formulae for the norm of $T \in \mathcal{L}(^n l_1)$.

Theorem 1. *Let $T \in \mathcal{L}(^n l_1)$ with*

$$T\left(\left(x_j^{(1)}\right), \ldots, \left(x_j^{(n)}\right)\right) = \sum_{j_1,\ldots,j_n \in \mathbb{N}} a_{j_1\ldots j_n} x_{j_1}^{(1)} \ldots x_{j_n}^{(n)} \tag{1}$$

for some $a_{j_1\ldots j_n} \in \mathbb{R}$. Then $\|T\| = \sup\{|a_{j_1\cdots j_n}| : j_1, \ldots, j_n \in \mathbb{N}\}$.

Proof. Note that, for $j_1, \ldots, j_n \in \mathbb{N}$, we have $|a_{j_1\ldots j_n}| = |T(e_{j_1}, \ldots, e_{j_n})| \leq \|T\|$. Hence,

$$\|T\| \geq \sup\{|a_{j_1\ldots j_n}| : j_1, \ldots, j_n \in \mathbb{N}\}.$$

On the other hand we get

$$\begin{aligned}
\|T\| &= \sup\left\{\left|T\left(\left(x_j^{(1)}\right), \ldots, \left(x_j^{(n)}\right)\right)\right| : \left\|\left(x_j^{(k)}\right)\right\|_1 = 1,\ k = 1, \ldots, n\right\} \\
&\leq \sup\left\{\sum_{j_1,\ldots,j_n \in \mathbb{N}} |a_{j_1\cdots j_n}| \left|x_{j_1}^{(1)}\right| \cdots \left|x_{j_n}^{(n)}\right| : \left\|\left(x_j^{(k)}\right)\right\|_1 = 1,\ k = 1, \ldots, n\right\} \\
&\leq \sup\{|a_{j_1\cdots j_n}| : j_1, \ldots, j_n \in \mathbb{N}\} \\
&\quad \times \sup\left\{\sum_{j \in \mathbb{N}} \left|x_j^{(1)}\right| \cdots \sum_{j \in \mathbb{N}} \left|x_j^{(n)}\right| : \left\|(x_j^{(k)})\right\|_1 = 1,\ k = 1, \ldots, n\right\} \\
&= \sup\{|a_{j_1\cdots j_n}| : j_1, \ldots, j_n \in \mathbb{N}\}.
\end{aligned}$$

□

Now we characterize all extreme points of the unit ball of $\mathcal{L}(^n l_1)$.

Theorem 2. *Let $T \in \mathcal{L}(^n l_1)$ be defined by* (1) *and let $\|T\| = 1$. Then $T \in \operatorname{ext} B_{\mathcal{L}(^n l_1)}$ if and only if $|a_{j_1\cdots j_n}| = 1$ for all $j_1, \ldots, j_n \in \mathbb{N}$.*

Proof. By Theorem 1, $|a_{j_1\cdots j_n}| \leq 1$ for all $j_1, \ldots, j_n \in \mathbb{N}$.

Necessity. Assume the contrary. Then there are $j_1^{'}, \ldots, j_n^{'} \in \mathbb{N}$ such that $\left|a_{j_1^{'}\cdots j_n^{'}}\right| < 1$. Let $S_{j_1^{'}\cdots j_n^{'}} \in \mathcal{L}(^n l_1)$ be defined by

$$S_{j_1^{'}\cdots j_n^{'}}\left(\left(x_j^{(1)}\right), \ldots, \left(x_j^{(n)}\right)\right) = x_{j_1^{'}}^{(1)} \cdots x_{j_n^{'}}^{(n)}. \tag{2}$$

Choose $\epsilon_0 > 0$ such that $\left|a_{j_1^{'}\cdots j_n^{'}}\right| + \epsilon_0 < 1$ and set $R^{\pm} = T \pm \epsilon_0 S_{j_1^{'}\cdots j_n^{'}}$. By Theorem 1, we have $\|R^{\pm}\| = 1$. Since $T = \frac{1}{2}(R^+ + R^-)$, T is not extreme. This is a contradiction. Hence, $|a_{j_1\cdots j_n}| = 1$ for all $j_1, \ldots, j_n \in \mathbb{N}$.

Sufficiency. Suppose that $T = \frac{1}{2}(T_1 + T_2)$ for some $T_j \in \mathcal{L}(^n l_1)$ with $\|T_j\| = 1$. Write

$$T_1\left(\left(x_j^{(1)}\right), \ldots, \left(x_j^{(n)}\right)\right) = \sum_{j_1,\ldots,j_n \in \mathbb{N}} b_{j_1\cdots j_n} x_{j_1}^{(1)} \cdots x_{j_n}^{(n)},$$

$$T_2\left(\left(x_j^{(1)}\right), \ldots, \left(x_j^{(n)}\right)\right) = \sum_{j_1,\ldots,j_n \in \mathbb{N}} c_{j_1\cdots j_n} x_{j_1}^{(1)} \cdots x_{j_n}^{(n)}$$

for some $b_{j_1\cdots j_n}, c_{j_1\cdots j_n} \in \mathbb{R}$ with $|b_{j_1\cdots j_n}| \leq 1,\ |c_{j_1\cdots j_n}| \leq 1$ for all $j_1, \ldots, j_n \in \mathbb{N}$. Then, $a_{j_1\cdots j_n} = \frac{1}{2}(b_{j_1\cdots j_n} + c_{j_1\cdots j_n})$ for all $j_1, \ldots, j_n \in \mathbb{N}$. Since $|a_{j_1\cdots j_n}| = 1$ for all $j_1, \ldots, j_n \in \mathbb{N}$, $a_{j_1\cdots j_n} = b_{j_1\cdots j_n} = c_{j_1\cdots j_n}$ for all $j_1, \ldots, j_n \in \mathbb{N}$. Therefore, $T = T_j$ for $j = 1, 2$. Hence $T \in \operatorname{ext} B_{\mathcal{L}(^n l_1)}$. □

The following theorem shows that every extreme point of the unit ball of $\mathcal{L}(^n l_1)$ is exposed.

Theorem 3. *The equality* $\exp B_{\mathcal{L}(^n l_1)} = \operatorname{ext} B_{\mathcal{L}(^n l_1)}$ *holds.*

Proof. Let $T \in \operatorname{ext} B_{\mathcal{L}(^n l_1)}$. By Theorem 2, the equality (1) holds for some $a_{j_1\cdots j_n} \in \mathbb{R}$ with $|a_{j_1\cdots j_n}| = 1$ for all $j_1, \ldots, j_n \in \mathbb{N}$. Let $\phi : \mathbb{N}^n \to \mathbb{N}$ be a bijection. Define $f \in \mathcal{L}(^n l_1)^*$ by

$$f(S) := \sum_{(j_1,\ldots,j_n)\in\mathbb{N}^n} \frac{1}{2^{\phi(j_1,\ldots,j_n)}} \operatorname{sign}(a_{j_1\cdots j_n}) S(e_{j_1}, \ldots, e_{j_n}).$$

Then, by Theorem 2,

$$f(T) = \sum_{(j_1,\ldots,j_n)\in\mathbb{N}^n} \frac{1}{2^{\phi(j_1,\ldots,j_n)}} |a_{j_1\cdots j_n}| = \sum_{(j_1,\ldots,j_n)\in\mathbb{N}^n} \frac{1}{2^{\phi(j_1,\ldots,j_n)}} = 1.$$

It follows that

$$\begin{aligned} 1 &= f(T) \leq \|f\| \\ &= \sup\left\{ \sum_{(j_1,\ldots,j_n)\in\mathbb{N}^n} \frac{1}{2^{\phi(j_1,\ldots,j_n)}} \left|S\left(e_{j_1}, \ldots, e_{j_n}\right)\right| : S \in \mathcal{L}(^n l_1), \|S\| = 1 \right\} \\ &\leq \sum_{(j_1,\ldots,j_n)\in\mathbb{N}^n} \frac{1}{2^{\phi(j_1,\ldots,j_n)}} = 1. \end{aligned}$$

Hence $\|f\| = 1$.

Claim: $f(S) < 1$ for every $S \in B_{\mathcal{L}(^n l_1)}$ with $S \neq T$.

It is enough to show that if $f(S) = 1$ for some $S \in B_{\mathcal{L}(^n l_1)}$, then $S = T$. We have

$$\sum_{(j_1,\ldots,j_n)\in\mathbb{N}^n} \frac{1}{2^{\phi(j_1,\ldots,j_n)}} |a_{j_1\cdots j_n}| = 1 = f(S)$$
$$= \sum_{(j_1,\ldots,j_n)\in\mathbb{N}^n} \frac{1}{2^{\phi(j_1,\ldots,j_n)}} \mathrm{sign}(a_{j_1\cdots j_n}) S(e_{j_1},\ldots,e_{j_n}),$$

which implies that

$$S(e_{j_1},\ldots,e_{j_n}) = a_{j_1\cdots j_n} = T(e_{j_1},\ldots,e_{j_n})$$

for all $j_1,\ldots,j_n \in \mathbb{N}$. By the n-linearity, $S = T$. Therefore, f exposes T. Hence $T \in \exp B_{\mathcal{L}(^n l_1)}$. □

We characterize all smooth points of the unit ball of $\mathcal{L}(^n l_1^m)$ for $n, m \geq 2$.

Theorem 4. *Let $n, m \geq 2$, and let $T \in \mathcal{L}(^n l_1^m)$ with $\|T\| = 1$ and*

$$T\left(\left(x_j^{(1)}\right),\ldots,\left(x_j^{(n)}\right)\right) = \sum_{1\leq j_1,\ldots,j_n\leq m} a_{j_1\cdots j_n} x_{j_1}^{(1)}\cdots x_{j_n}^{(n)}.$$

Then $T \in \mathrm{sm}\, B_{\mathcal{L}(^n l_1^m)}$ if and only if there are $j_1^{'},\ldots,j_n^{'} \in \{1,\ldots,m\}$ such that

$$1 = \left|a_{j_1^{'}\cdots j_n^{'}}\right| > |a_{j_1\cdots j_n}|$$

for all $j_1,\ldots,j_n \in \{1,\ldots,m\}$ with $(j_1,\ldots,j_n) \neq \left(j_1^{'},\ldots,j_n^{'}\right)$.

Proof. Sufficiency. Let $f \in \mathcal{L}(^n l_1^m)^*$ be such that $f(T) = 1 = \|f\|$. As $\mathcal{L}(^n l_1^m)^* = \otimes_{\pi,n} l_1^m$, there are $k \in \mathbb{N}$ and $f^{(1),i},\ldots,f^{(n),i} \in l_1^m$ such that

$$f = \sum_{i=1}^{k} f^{(1),i} \otimes \ldots \otimes f^{(n),i}.$$

We claim that $f = \mathrm{sign}\left(a_{j_1^{'}\cdots j_n^{'}}\right)\left(e_{j_1^{'}} \otimes \ldots \otimes e_{j_n^{'}}\right)$. Indeed, let $j_1,\ldots,j_n \in \{1,\ldots,m\}$ be such that $(j_1,\ldots,j_n) \neq \left(j_1^{'},\ldots,j_n^{'}\right)$, and let $S_{j_1\cdots j_n} \in \mathcal{L}(^n l_1^m)$ be defined by

$$S_{j_1\cdots j_n}\left(\left(x_j^{(1)}\right),\ldots,\left(x_j^{(n)}\right)\right) = x_{j_1}^{(1)}\cdots x_{j_n}^{(n)}. \tag{3}$$

Choose $\epsilon_0 > 0$ such that $|a_{j_1\cdots j_n}| + \epsilon_0 < 1$ and set $R^{\pm} = T \pm \epsilon_0 S_{j_1\cdots j_n}$. By Theorem 1 we have $\|R^{\pm}\| = 1$. Thus,

$$\begin{aligned} 1 &\geq \max\left\{\left|f(R^{\pm})\right|\right\} = \max\left\{|f(T) \pm \epsilon_0 f(S_{j_1\cdots j_n})|\right\} \\ &= |f(T)| + \epsilon_0\,|f(S_{j_1\cdots j_n})| = 1 + \epsilon_0\,|f(S_{j_1\cdots j_n})|, \end{aligned}$$

which implies that $f\left(S_{j_1\cdots j_n}\right)=0$ for all $j_1,\ldots,j_n\in\{1,\ldots,m\}$ such that $(j_1,\ldots,j_n)\neq\left(j_1^{'},\ldots,j_n^{'}\right)$. Hence

$$\left|a_{j_1^{'}\cdots j_n^{'}}\right|=1=f(T)=a_{j_1^{'}\cdots j_n^{'}}f\left(S_{j_1^{'}\cdots j_n^{'}}\right),$$

which shows that $f\left(S_{j_1^{'}\cdots j_n^{'}}\right)=\operatorname{sign}\left(a_{j_1^{'}\cdots j_n^{'}}\right)$. Thus, for $S\in\mathcal{L}(^n l_1^m)$,

$$\begin{aligned} f(S) &= \sum_{1\le i_1,\ldots,i_n\le m} S\left(e_{i_1},\ldots,e_{i_n}\right)f(S_{j_1\cdots j_n}) \\ &= S\left(e_{j_1^{'}},\ldots,e_{j_n^{'}}\right)f\left(S_{j_1^{'}\cdots j_n^{'}}\right)=S\left(e_{j_1^{'}},\ldots,e_{j_n^{'}}\right)\operatorname{sign}\left(a_{j_1^{'}\cdots j_n^{'}}\right) \\ &= \operatorname{sign}\left(a_{j_1^{'}\cdots j_n^{'}}\right)\left(e_{j_1^{'}}\otimes\ldots\otimes e_{j_n^{'}}\right)(S). \end{aligned}$$

Therefore, f is uniquely determined and $T\in\operatorname{sm}B_{\mathcal{L}(^n l_1^m)}$.

Necessity. By Theorem 1,

$$1=\|T\|=\max\left\{|a_{j_1\cdots j_n}|:1\le j_1,\ldots,j_n\le m\right\}.$$

Hence there are $j_1^{'},\ldots,j_n^{'}\in\{1,\ldots,m\}$ such that $1=\left|a_{j_1^{'}\cdots j_n^{'}}\right|$.

Claim: $|a_{j_1\cdots j_n}|<1$ for all $j_1,\ldots,j_n\in\{1,\ldots,m\}$ with $(j_1,\ldots,j_n)\neq\left(j_1^{'},\ldots,j_n^{'}\right)$.

If not, then there are $i_1^{'},\ldots,i_n^{'}\in\{1,\ldots,m\}$ such that $\left(i_1^{'},\ldots,i_n^{'}\right)\neq\left(j_1^{'},\ldots,j_n^{'}\right)$ and $1=\left|a_{i_1^{'}\cdots i_n^{'}}\right|$. Let $f_1,\ f_2\in\mathcal{L}(^n l_1^m)^*$ be defined by

$$f_1=\operatorname{sign}\left(a_{j_1^{'}\cdots j_n^{'}}\right)\left(e_{j_1^{'}}\otimes\ldots\otimes e_{j_n^{'}}\right),\quad f_2=\operatorname{sign}\left(a_{i_1^{'}\cdots i_n^{'}}\right)\left(e_{i_1^{'}}\otimes\ldots\otimes e_{i_n^{'}}\right). \tag{4}$$

Then

$$f_1\neq f_2,\ f_j(T)=1=\|f_j\|\ (j=1,2).$$

This is a contradiction. Therefore, we have proved the claim. □

Theorem 5. *Let $n\ge 2$ and let $T\in\mathcal{L}(^n l_1)$ be defined by* (1) *such that $\|T\|=1$. Suppose that $\left|a_{j_1^{'}\cdots j_n^{'}}\right|=1=\left|a_{i_1^{'}\cdots i_n^{'}}\right|$ for some $\left(j_1^{'},\ldots,j_n^{'}\right)\neq\left(i_1^{'},\ldots,i_n^{'}\right)\in\mathbb{N}^n$. Then $T\notin\operatorname{sm}B_{\mathcal{L}(^n l_1)}$.*

Proof. Let $f_1,\ f_2\in\mathcal{L}(^n l_1)^*$ be defined by (4). Then,

$$f_1\neq f_2,\ f_j(T)=1=\|f_j\|\ (j=1,2).$$

Therefore, $T\notin\operatorname{sm}B_{\mathcal{L}(^n l_1)}$. □

3. The unit ball of $\mathcal{L}_s(^n l_1)$

In this section we characterize $\operatorname{ext} B_{\mathcal{L}_s(^n l_1)}, \exp B_{\mathcal{L}_s(^n l_1)}$ and $\operatorname{sm} B_{\mathcal{L}_s(^n l_1^m)}$ for $n, m \geq 2$. The first theorem characterizes all extreme points of the unit ball of $\mathcal{L}_s(^n l_1)$.

Theorem 6. *Let $T \in \mathcal{L}_s(^n l_1)$ be defined by* (1) *and let $\|T\| = 1$. Then $T \in \operatorname{ext} B_{\mathcal{L}_s(^n l_1)}$ if and only if $|a_{j_1 \cdots j_n}| = 1$ for all $j_1, \ldots, j_n \in \mathbb{N}$.*

Proof. By Theorem 1, $|a_{j_1 \cdots j_n}| \leq 1$ for all $j_1, \ldots, j_n \in \mathbb{N}$.

Sufficiency. Assume the contrary. There are $j_1^{'}, \ldots, j_n^{'} \in \mathbb{N}$ such that $\left| a_{j_1^{'} \cdots j_n^{'}} \right| < 1$. Let $S_{j_1^{'} \cdots j_n^{'}} \in \mathcal{L}(^n l_1)$ be defined by (2). Choose $\epsilon_0 > 0$ such that $\left| a_{j_1^{'} \cdots j_n^{'}}, \right| + \epsilon_0 < 1$. Let $L^{\pm} \in \mathcal{L}_s(^n l_1)$ be defined by

$$L^{\pm} = T \pm \frac{\epsilon_0}{n!} \sum_{\sigma} S_{j_{\sigma(1)}^{'} \cdots j_{\sigma(n)}^{'}},$$

where σ is a permutation on $\{1, \ldots, n\}$. By Theorem 1 we have $\|L^{\pm}\| = 1$. Since $T = \frac{1}{2}(L^{+} + L^{-})$, T is not extreme. This is a contradiction. Hence, $|a_{j_1 \cdots j_n}| = 1$ for all $j_1, \ldots, j_n \in \mathbb{N}$.

The proof of necessity is identical to the proof of necessity of Theorem 2. □

We show that every extreme point of the unit ball of $\mathcal{L}_s(^n l_1)$ is exposed.

Theorem 7. *The equality* $\exp B_{\mathcal{L}_s(^n l_1)} = \operatorname{ext} B_{\mathcal{L}_s(^n l_1)}$ *is true.*

Proof. It is identical to that for Theorem 3. □

The following theorem shows a relation between the spaces $\mathcal{L}(^n l_1)$ and $\mathcal{L}_s(^n l_1)$.

Theorem 8. *Let $n, m \geq 2$. Then:*

(a) $\operatorname{ext} B_{\mathcal{L}_s(^n l_1)} = \operatorname{ext} B_{\mathcal{L}(^n l_1)} \cap \mathcal{L}_s(^n l_1)$;
(b) $\exp B_{\mathcal{L}_s(^n l_1)} = \exp B_{\mathcal{L}(^n l_1)} \cap \mathcal{L}_s(^n l_1)$.

Proof. The statement (a) follows from Theorems 2 and 6, and (b) follows from Theorems 3 and 7 in view of (a). □

We characterize all smooth points of the unit ball of $\mathcal{L}_s(^n l_1^m)$ for $n, m \geq 2$.

Theorem 9. *For $n, m \geq 2$, let $T \in \mathcal{L}_s(^n l_1^m)$ be the same as in Theorem 4. Then $T \in \operatorname{sm} B_{\mathcal{L}_s(^n l_1^m)}$ if and only if there are $j_1^{'}, \ldots, j_n^{'} \in \{1, \ldots, m\}$ such that*

$$1 = \left| a_{j_{\sigma(1)}^{'} \cdots j_{\sigma(n)}^{'}} \right| > |a_{j_1 \cdots j_n}|$$

for all $j_1, \ldots, j_n \in \{1, \ldots, m\}$ with $(j_1, \ldots, j_n) \neq \left(j_{\sigma(1)}^{'}, \ldots, j_{\sigma(n)}^{'} \right)$, where σ is a permutation on $\{1, \ldots, n\}$.

Proof. Sufficiency. Let $f \in \mathcal{L}_s({}^n l_1^m)^*$ be such that $f(T) = 1 = \|f\|$. We claim that

$$f = \text{sign}\left(a_{j_1'\cdots j_n'}\right) \frac{1}{n!} \sum_\sigma \left(e_{j'_{\sigma(1)}} \otimes \ldots \otimes e_{j'_{\sigma(n)}}\right).$$

Indeed, let $j_1, \ldots, j_n \in \{1, \ldots, m\}$ be such that $(j_1, \ldots, j_n) \neq \left(j'_{\sigma(1)}, \ldots, j'_{\sigma(n)}\right)$ for every permutation σ. Let $S_{j_1\cdots j_n} \in \mathcal{L}({}^n l_1^m)$ be defined by (3). Choose $\epsilon_0 > 0$ such that $|a_{j_1\cdots j_n}| + \epsilon_0 < 1$. Let $L^\pm \in \mathcal{L}_s({}^n l_1)$ be defined by

$$L^\pm = T \pm \frac{\epsilon_0}{n!} \sum_\sigma S_{j_{\sigma(1)}\cdots j_{\sigma(n)}}.$$

By Theorem 1, we have $\|L^\pm\| = 1$. Thus

$$\begin{aligned} 1 \quad &\geq \quad \max\{|f(L^\pm)|\} = \max\left\{\left|f(T) \pm \frac{\epsilon_0}{n!} f\left(\sum_\sigma S_{j_{\sigma(1)}\cdots j_{\sigma(n)}}\right)\right|\right\} \\ &= \quad |f(T)| + \frac{\epsilon_0}{n!}\left|f\left(\sum_\sigma S_{j_{\sigma(1)}\cdots j_{\sigma(n)}}\right)\right| = 1 + \frac{\epsilon_0}{n!}\left|f\left(\sum_\sigma S_{j_{\sigma(1)}\cdots j_{\sigma(n)}}\right)\right|, \end{aligned}$$

which implies that $f\left(\sum_\sigma S_{j_{\sigma(1)}\cdots j_{\sigma(n)}}\right) = 0$ for all $j_1, \ldots, j_n \in \{1, \ldots, m\}$ such that $(j_1, \ldots, j_n) \neq \left(j'_{\sigma(1)}, \ldots, j'_{\sigma(n)}\right)$ for every permutation σ. Hence,

$$\left|a_{j_1'\cdots j_n'}\right| = 1 = f(T) = a_{j_1'\cdots j_n'} \frac{1}{n!} f\left(\sum_\sigma S_{j'_{\sigma(1)}\cdots j'_{\sigma(n)}}\right),$$

which shows that $\frac{1}{n!} f\left(\sum_\sigma S_{j'_{\sigma(1)}\cdots j'_{\sigma(n)}}\right) = \text{sign}\left(a_{j_1'\cdots j_n'}\right)$. Thus, for $S \in \mathcal{L}({}^n l_1^m)$,

$$\begin{aligned} f(S) &= \sum_{1\leq i_1,\ldots,i_n\leq m} S(e_{i_1}, \ldots, e_{i_n}) \frac{1}{n!} f\left(\sum_\sigma S_{j_{\sigma(1)}\cdots j_{\sigma(n)}}\right) \\ &= S\left(e_{j_1'}, \ldots, e_{j_n'}\right) \frac{1}{n!} f\left(\sum_\sigma S_{j'_{\sigma(1)}\cdots j'_{\sigma(n)}}\right) = S\left(e_{j_1'}, \ldots, e_{j_n'}\right) \text{sign}\left(a_{j_1'\cdots j_n'}\right) \\ &= \text{sign}\left(a_{j_1'\cdots j_n'}\right) \frac{1}{n!} \sum_\sigma \left(e_{j'_{\sigma(1)}} \otimes \ldots \otimes e_{j'_{\sigma(n)}}\right)(S). \end{aligned}$$

Therefore, f is uniquely determined and $T \in \text{sm}\, B_{\mathcal{L}({}^n l_1^m)}$.

Necessary. By Theorem 1,

$$1 = \|T\| = \max\{|a_{j_1\cdots j_n}| : 1 \leq j_1, \ldots, j_n \leq m\}.$$

Hence, there are $j_1^{'}, \ldots, j_n^{'} \in \{1, \ldots, m\}$ such that $1 = \left|a_{j_1^{'} \cdots j_n^{'}}\right|$. Note that since T is symmetric, $1 = \left|a_{j_{\sigma(1)}^{'} \cdots j_{\sigma(n)}^{'}}\right|$ for every permutation σ on $\{1, \ldots, n\}$.

Claim: $|a_{j_1 \cdots j_n}| < 1$ for all $j_1, \ldots, j_n \in \{1, \ldots, m\}$ such that $(j_1, \ldots, j_n) \neq \left(j_{\sigma(1)}^{'}, \ldots, j_{\sigma(n)}^{'}\right)$ for every permutation σ on $\{1, \ldots, n\}$.

If not, then there are $i_1^{'}, \ldots, i_n^{'} \in \{1, \ldots, m\}$ such that $1 = \left|a_{i_1^{'} \cdots i_n^{'}}\right|$ and $(i_1^{'}, \ldots, i_n^{'}) \neq \left(j_{\sigma(1)}^{'}, \ldots, j_{\sigma(n)}^{'}\right)$ for every permutation σ on $\{1, \ldots, n\}$. Let $f_1,\ f_2 \in \mathcal{L}_s(^n l_1^m)^*$ be defined by

$$\begin{aligned} f_1 &= \operatorname{sign}\left(a_{j' \cdots j'}\right) \frac{1}{n!} \sum_{\sigma} \left(e_{j_{\sigma(1)}^{'}} \otimes \ldots \otimes e_{j_{\sigma(n)}^{'}}\right), \\ f_2 &= \operatorname{sign}\left(a_{i_1^{'} \cdots i_n^{'}}\right) \frac{1}{n!} \sum_{\sigma} \left(e_{i_{\sigma(1)}^{'}} \otimes \ldots \otimes e_{i_{\sigma(n)}^{'}}\right). \end{aligned} \tag{5}$$

Then,

$$f_1 \neq f_2,\ f_j(T) = 1 = \|f_j\| \ (j = 1, 2).$$

This is a contradiction. Therefore, we have proved the claim. □

Theorem 10. *For $n \geq 2$, let $T \in \mathcal{L}_s(^n l_1)$ be the same as in Theorem 5. Suppose that $\left|a_{j_1^{'} \cdots j_n^{'}}\right| = 1 = \left|a_{i_1^{'} \cdots i_n^{'}}\right|$ for some $\left(j_1^{'}, \ldots, j_n^{'}\right)$, $\left(i_1^{'}, \ldots, i_n^{'}\right)$ satisfying $\left(i_1^{'}, \ldots, i_n^{'}\right) \notin \left\{\left(j_{\sigma(1)}^{'}, \ldots, j_{\sigma(n)}^{'}\right) : \sigma \text{ is a permutation on} \{1, \ldots, n\}\right\}$. Then $T \notin \operatorname{sm} B_{\mathcal{L}_s(^n l_1)}$.*

Proof. Let $f_1,\ f_2 \in \mathcal{L}_s(^n l_1)^*$ be defined by (5). Then

$$f_1 \neq f_2,\ f_j(T) = 1 = \|f_j\| \ (j = 1, 2).$$

Therefore, $T \notin \operatorname{sm} B_{\mathcal{L}_s(^n l_1)}$. □

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