ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 25, Number 1, June 2021
Available online at http://acutm.math.ut.ee

Thue's equation as a tool to solve two different problems

Sadek Bouroubi and Ali Debbache

To all health care workers, front line soldiers facing the COVID-19 virus

Abstract. A Thue equation is a Diophantine equation of the form $f(x, y) = r$, where f is an irreducible binary form of degree at least 3, and r is a given nonzero rational number. A set S of at least three positive integers is called a D_1^3-set if the product of any of its three distinct elements is a perfect cube minus one. We prove that any D_1^3-set is finite and, for any positive integer a, the two-tuple $\{a, 2a\}$ is extendible to a D_1^3-set 3-tuple, but not to a 4-tuple. Using the well-known Thue equation $2x^3 - y^3 = 1$, we show that the only cubic-triangular number is 1.

1. Introduction

Let $S = \{x_1, \ldots, x_m\}$ be a set of m positive integers, $m \geq 2$. The set S is called a Diophantine m-tuple if the product of any two distinct elements increased by one is a perfect square, i.e., $x_i x_j + 1 = u_{ij}^2$, where $u_{ij} \in \mathbb{N} = \{1, 2, 3, \ldots\}$, $1 \leq i < j \leq m$. Diophantus of Alexandria was the first to look for such sets. He found a set of four positive rational numbers $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$ with the above property. However, Fermat was the first to give $\{1, 3, 8, 120\}$ as an example of a Diophantine quadruple. For a detailed history of Diophantine m-tuples and corresponding results, we refer the reader to Dujella's webpage [3]. Throughout the following the notion of a D_1^3-set is essential.

Received January 31, 2021.
2020 *Mathematics Subject Classification.* Primary 11D45; Secondary 11D09.
Key words and phrases. D_1^3-set, Diophantine equation, cubic-triangular number, Thue's equation.
https://doi.org/10.12697/ACUTM.2021.25.10
Corresponding author: Sadek Bouroubi

Definition 1. A set S of at least three positive integers is called a D_1^3-set if the product of any of its three distinct elements is a perfect cube minus one.

Definition 2. A D_1^3-set S is said to be extendible if there exists an integer $y \notin S$ such that $S \cup \{y\}$ is still a D_1^3-set.

Example 1. The set $\{1, 2, 13\}$ is a D_1^3-set, which is not extendible to four terms (see Theorem 2).

2. Main results

Theorem 1. *Any D_1^3-set is finite.*

Proof. Let $S = \{x_1, x_2, x_3, \ldots\}$ be a D_1^3-set. Suppose that there exists an integer $y \notin S$ such that $S \cup \{y\}$ is still a D_1^3-set. Then, by setting

$$\begin{cases} a = x_3 x_2, \\ b = x_3 x_1, \\ c = x_2 x_1, \end{cases}$$

we get

$$\begin{cases} ay + 1 = x_3 x_2 y + 1 = u^3, \\ by + 1 = x_3 x_1 y + 1 = v^3, \\ cy + 1 = x_2 x_1 y + 1 = w^3, \end{cases}$$

for some positive integers u, v and w.

Hence

$$(ay + 1)(by + 1)(cy + 1) = (uvw)^3.$$

We recognize here an elliptic curve, which has only finitely many solutions (see [1]). □

Proposition 1. *Any set $\{x, y\}$ of two elements can be a subset of a D_1^3-set of three elements.*

Proof. Thanks to the identity $(xy + 1)^3 = xy(x^2y^2 + 3xy + 3) + 1$, it is clear that the triple $\{x, y, x^2y^2 + 3xy + 3\}$ is a D_1^3-set. □

Corollary 1. *For any positive integer a, the set $\{a, 2a, 4a^4 + 6a^2 + 3\}$ is a D_1^3-set.*

Proof. To get the result, it is enough to substitute x by a and y by $2a$ in Proposition 1. □

Theorem 2. *For any positive integer a, the set $\{a, 2a\}$ is not extendible to a D_1^3-set of four terms.*

Proof. Suppose there exist two positive integers b and c such that the quadruple $\{a, 2a, b, c\}$ is a D_1^3-set. Then the following system of equations has a solution $(u, v, w, t) \in \mathbb{N}^4$:

$$(S)\begin{cases} 2a^2b + 1 = u^3, \\ 2a^2c + 1 = v^3, \\ abc + 1 = w^3, \\ 2abc + 1 = t^3. \end{cases}$$

The system (S) yields

$$2w^3 - t^3 = 1. \tag{1}$$

We recognize here a Thue's equation, which has the unique positive integer solution, $(w, t) = (1, 1)$ (see [2]), which is impossible in (S). This completes the proof. □

3. The cubic-triangular numbers

A triangular number is a famous figurate number that can be represented in the form of an equilateral triangle of points, where the first row contains a single element and each subsequent row contains one more element than the previous one (see Figure 1). Let T_n denote the n^{th} triangular number, then T_n is equal to the sum of the n natural numbers from 1 to n, whose initial values are listed as the sequence A000217 in [4]. We have

$$T_n = \frac{n(n+1)}{2} = \binom{n+1}{2},$$

where $\binom{n}{k}$ is a binomial coefficient.

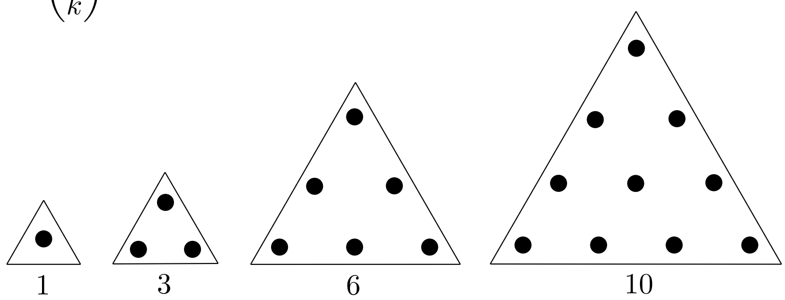

Figure 1. The first four triangular numbers.

Definition 3. A cubic-triangular number T_u is a positive integer that is simultaneously cubic and triangular, i.e., for some positive integer v,

$$T_u = \frac{u(u+1)}{2} = v^3. \tag{2}$$

Theorem 3. *The only cubic-triangular number is 1.*

Proof. Let n be a cubic-triangular number. According to equation (2), there exist two positive integers u and v such that $2n = u(u+1) = 2v^3$. Since u and $u+1$ are coprime, there exist two positive integers x and y such that $u = x^3$ and $u+1 = 2y^3$, so in that case, we get the Thue equation $2y^3 - x^3 = 1$ that has $(x, y) = (1, 1)$ as the unique positive integer solution, or $u = 2x^3$ and $u+1 = y^3$ which implies the equation $y^3 - 2x^3 = 1$ which is equivalent to $2(-x)^3 - (-y)^3 = 1$, that has $(x, y) = (-1, -1)$ as the unique positive integer solution. Thus, $u = 1$ or $u = -2$ and then $n = 1$, which is the unique cubic-triangular number. □

Remark 1. As we can see, Thue's equation (1) is useful in two problems mentioned above that seem to be a priori different.

Acknowledgements

The authors would like to thank the referees for their pertinent comments and valuable suggestions, which significantly improved the manuscript. The authors would also like to thank DGRSDT for its valuable support.

References

[1] A. Baker, *Bounds for the solutions of the hyperelliptic equation*, Proc. Cambridge Philos. Soc. **65**(2) (1969), 439–444.
[2] M. Bennett, *Rational approximation to algebraic numbers of small height: the Diophantine equation* $|ax^n - by^n| = 1$, J. Reine Angew. Math. **535** (2001), 1–49.
[3] A. Dujella. *Diophantine m-tuples*, in: http://web.math.hr/ duje/.
[4] *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/search?q=A000217.

USTHB University, Faculty of Mathematics, L'IFORCE Laboratory, P.O. Box 32, El Alia 16111, Bab-Ezzouar, Algiers, Algeria
E-mail address: sbouroubi@usthb.dz, bouroubis@gmail.com

USTHB University, Faculty of Mathematics, LATN Laboratory, P.O. Box 32, El Alia 16111, Bab-Ezzouar, Algiers, Algeria
E-mail address: a_debbache2003@yahoo.fr
URL: https://liforce.usthb.dz/node/11