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Tauberian theorems for weighted mean statistical summability of double sequences of fuzzy numbers

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Abstract. We discuss Tauberian conditions under which the statistical convergence of double sequences of fuzzy numbers follows from the statistical convergence of their weighted means. We also prove some other results which are necessary to establish the main results.

1. Introduction

Since the introduction by Zadeh [27], the concept of fuzzy set theory and its applications have attracted the attention of many researchers from various branches of mathematics. Dubois and Prade [8] introduced the notion of fuzzy numbers and defined the basic operations of addition, subtraction, multiplication and division. Goetschel and Voxman gave a less restrictive definition of fuzzy numbers in [13]. Matloka [15] introduced the concepts of bounded and convergent sequences of fuzzy numbers and studied their properties.

In 1935, Zygmund [28] defined a new type of convergence known as almost convergence that formed the foundation of the concept of statistical convergence which was formally introduced by Fast [11] and reintroduced by Schoenberg [23] and also, independently by Buck [4]. Later this idea was associated with summability theory. Nuray and Savaş [18] extended the concept of statistical convergence to sequences of fuzzy numbers and showed that a sequence of fuzzy numbers is statistically convergent if and only if it is statistically Cauchy.

Recently there has been an increasing interest in summability methods of sequences of fuzzy numbers. In [24], Subrahmanyam defined the Cesàro

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summability method for sequences of fuzzy numbers and proved fuzzy analogues of some classical Tauberian theorems. For a detailed study and some results related to convergence of sequences of fuzzy numbers and Tauberian conditions, we refer to the papers [1, 3, 5, 9, 10, 14, 19, 26].

We now give some preliminary definitions and notations which are required in the later part of the paper.

Definition 1.1 (see [8, 13])**.** A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \to [0, 1]$ which satisfies the following four conditions:

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,
(ii) u is fuzzy convex, i.e., $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), v(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$,
(iii) u is upper semi-continuous,
(iv) the set $[u]_0 = \overline{\{x \in \mathbb{R}(x) > 0\}}$ is compact, where $\overline{\{x \in \mathbb{R}(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R}(x) > 0\}$ in the usual topology of $\mathbb{R}$.

The set of all fuzzy numbers on $\mathbb{R}$ is denoted by E^1 and called the space of fuzzy numbers. The α-level set $[u]_\alpha$ of $u \in E^1$ is defined by

$$[u]_\alpha = \begin{cases} \{t \in \mathbb{R} : u(t) \geq \alpha\} & \text{if } 0 < \alpha \leq 1, \\ \overline{\{t \in \mathbb{R} : u(t) > \alpha\}} & \text{if } \alpha = 0. \end{cases}$$

The set $[u]_\alpha$ is a closed, bounded, and non-empty interval for each $\alpha \in [0, 1]$, it is defined by $[u]_\alpha = [u^-(\alpha), u^+(\alpha)]$. The set $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ may be regarded as the fuzzy number

$$\overline{r}(t) = \begin{cases} 1 & \text{if } t = r, \\ 0 & \text{if } t \neq r. \end{cases}$$

Definition 1.2 (see [2, 8])**.** Let $u, v, w \in E^1$ and $k \in \mathbb{R}$. The addition, scalar multiplication, and product in E^1 are defined as follows:

$$\begin{aligned} u + v = w &\Leftrightarrow [w]_\alpha = [u]_\alpha + [v]_\alpha \quad (\alpha \in [0, 1]), \\ ku = w &\Leftrightarrow [w]_\alpha = k[u]_\alpha \quad (\alpha \in 0, 1]), \\ uv = w &\Leftrightarrow [w]_\alpha = [u]_\alpha [v]_\alpha \quad (\alpha \in [0, 1]). \end{aligned}$$

Definition 1.3 (see [7])**.** Let W be the set of all closed bounded intervals $A = [A_1, A_2]$. If we define, for A and $B = [B_1, B_2]$ from W, the relation

$$d(A, B) = \max\{|A_1 - B_1|, |A_2 - B_2|\},$$

then (W, d) is a complete metric space. Bede [2] defined on E^1 the Hausdorff metric

$$D(u, v) = \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha).$$

Lemma 1.4 (see [2])**.** *Let $x, y, z, u \in E^1$ and $k \in \mathbb{R}$. Then*

(i) (E^1, D) *is a complete metric space;*
(ii) $D(kx, ky) = |k|\ D(x, y)$;
(iii) $D(x + y, z + y) = D(x, z)$;
(iv) $D(x + y, z + u) \leq D(x, z) + D(y, u)$.

Matloka [15] introduced bounded and convergent sequences of fuzzy numbers and studied their properties. In [20], Pringsheim gave a notion of convergence of double sequences. Savaş [21] extended this notion of Pringsheim's convergence to double sequences of fuzzy numbers.

Definition 1.5 (see [15, 21])**.** A double sequence $x = (x_{jk})$ of fuzzy numbers is a function $x : \mathbb{N} \times \mathbb{N} \to E^1$. The fuzzy number x_{jk} denotes the value of the function at a point $(j, k) \in \mathbb{N} \times \mathbb{N}$ and is called the (j, k)-th term of the double sequence x. By ${}_2W^F$ we denote the set of all double sequences of fuzzy numbers.

Definition 1.6 (see [21])**.** Consider the sequence $(x_{jk}) \in {}_2W^F$. If for every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ and $L \in E^1$ such that $D(x_{jk}, L) < \epsilon$ for all $j, k > n_0$, then we say that the sequence (x_{jk}) converges in the sense of Pringsheim to the limit L and write $P\text{-}\lim_{j,k} x_{jk} = L$.

Definition 1.7 (see [22])**.** A sequence $x = (x_{jk}) \in {}_2W^F$ is said to be statistically convergent to $L \in E^1$ if, for every $\epsilon > 0$, $\delta_2(K(m, n, \epsilon)) = 0$, where $K(m, n, \epsilon) = \{(j, k) : j \leq m, k \leq n : D(x_{jk}, L) \geq \epsilon\}$. In this case we write $st_2 - \lim x_{jk} = L$.

Precisely, $st_2 - \lim x_{jk} = L$ if

$$P\text{-}\lim_{m,n} \tfrac{1}{mn} |(j, k); j \leq m, k \leq n : D(x_{jk}, L) \geq \epsilon| = 0.$$

2. Main results and discussions

Let $p := (p_j)_{j=0}^{\infty}$ and $q := (q_j)_{k=0}^{\infty}$ be sequences of non-negative numbers such that $p_0,\ q_0 > 0$,

$$P_m := \sum_{j=0}^{m} p_m \to \infty \text{ as } m \to \infty,$$

and

$$Q_n := \sum_{k=0}^{n} q_n \to \infty \text{ as } n \to \infty.$$

The weighted mean t_{mn} of $x = (x_{jk}) \in {}_2W^F$ is defined as

$$t_{mn} := \tfrac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k x_{jk}, \quad m, n \geq 0.$$

Definition 2.1. If t_{mn} converges to L as $\min(m,n) \to \infty$, then we say that the sequence $x = (x_{jk}) \in {}_2W^F$ is $(\overline{N}^2, p, q)^F$-summable to L.

Definition 2.2. A sequence $x = (x_{jk}) \in {}_2W^F$ is called $(\overline{N}^2, p, q)^F$-statistically summable to L if $st_2\text{-}\lim t_{mn} = L$.

Definition 2.3. Let $K \subseteq \mathbb{N}\times\mathbb{N}$ and $\epsilon > 0$. We define the double weighted ϵ-density of K by

$$\delta_{\overline{N}_2}(K,\epsilon) := \lim_{m,n} \tfrac{1}{P_m Q_n} |K_{P_m Q_n}(\epsilon)|,$$

provided the limit exists, where

$$K_{P_m Q_n}(\epsilon) := \{(j,k); j \leq P_m, k \leq Q_n : p_j q_k D(x_{jk}, L) \geq \epsilon\},$$

$\liminf p_j > 0$, $\liminf q_j > 0$. A sequence $x = (x_{jk}) \in {}_2W^F$ is said to be weighted statistically (or $S_{\overline{N}_2}{}^F$-statistically) convergent to L if, for every $\epsilon > 0$, $\delta_{\overline{N}_2}(K,\epsilon) = 0$.

Theorem 2.4. *Let $p_j q_k D(x_{jk}, L) \leq M$ for all $j,k \in \mathbb{N}$. If a sequence $x = (x_{jk}) \in {}_2W^F$ is $S_{\overline{N}_2}{}^F$-statistically convergent to L, then it is $(\overline{N}^2, p, q)^F$-statistically summable to L.*

Proof. We have

$$\begin{aligned}
D\left(t_{mn}, L\right) &= D\left(\tfrac{1}{P_m Q_n}\sum_{j=0}^{m}\sum_{k=0}^{n} p_j q_k x_{jk}, L\right) \\
&\leq \tfrac{1}{P_m Q_n}\sum_{j=0}^{m}\sum_{k=0}^{n} p_j q_k D(x_{jk}, L) \\
&= \tfrac{1}{P_m Q_n} \sum_{\substack{j=0 \\ (j,k)\in K_{P_m Q_n}(\epsilon)}}^{m}\sum_{k=0}^{n} p_j q_k D(x_{jk}, L) + \tfrac{1}{P_m Q_n} \sum_{\substack{j=0 \\ (j,k)\notin K_{P_m Q_n}(\epsilon)}}^{m}\sum_{k=0}^{n} p_j q_k D(x_{jk}, L) \\
&\leq \left(\sup_{j,k} p_j q_k D(x_{jk}, L)\right) |K_{P_m Q_n}(\epsilon)| + \tfrac{1}{P_m Q_n} \sum_{\substack{j=0 \\ (j,k)\notin K_{P_m Q_n}(\epsilon)}}^{m}\sum_{k=0}^{n} \epsilon \\
&\leq \tfrac{1}{P_m Q_n} M |K_{P_m Q_n}(\epsilon)| + \epsilon \to 0 + \epsilon \text{ as } m,n \to \infty.
\end{aligned}$$

This implies that $t_{mn} \to L$. $\square$

Remark 2.5. If we set $p_j = 1,\ q_k = 1$ $(j,\, k = 0, 1, 2, \dots)$ in Theorem 2.4 then we get Theorem 4.1 of [19].

It follows from Theorem 2.6 of [6] and Example 4.2 of [19] that the converse implication of Theorem 2.4 does not hold in general. We find some conditions under which the converse also holds. We give two-sided Tauberian conditions, each of which is necessary and sufficient for statistical convergence to be followed from statistical summability by weighted means.

Definition 2.6. We define the sums

$$\tau_{mn}^{>} := \tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k x_{jk} \quad (\lambda > 1)$$

and

$$\tau_{mn}^{<} := \tfrac{1}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k x_{jk} \quad (0 < \lambda < 1)$$

for sufficiently large non-negative integers m, n. Here λ_n denotes the integer part $[\lambda n]$ of the product λn.

Fridy and Orhan [12] introduced the idea of statistical limit inferior and statistical limit superior of a sequence of real numbers. Móricz and Orhan [16] proved the following lemma.

Lemma 2.7 (see [16])**.** *If $\{R_m\}$ is a non-decreasing sequence of positive numbers, then the conditions*

$$st\text{-}\liminf_{m\to\infty} \tfrac{R_{\lambda_m}}{R_m} > 1 \textit{ for every } \lambda > 1 \tag{1}$$

and

$$st\text{-}\liminf_{m\to\infty} \tfrac{R_m}{R_{\lambda_m}} > 1 \textit{ for every } 0 < \lambda < 1 \tag{2}$$

are equivalent.

Lemma 2.8. *Let $p = (p_j)_{j=0}^{\infty}$ and $q = (q_k)_{k=0}^{\infty}$ be sequences of non-negative numbers such that $p_0,\ q_0 > 0$. If a sequence $x = (x_{mn}) \in {}_2W^F$ is $(\overline{N}^2, p, q)^F$-statistically summable to L, then, for $\lambda > 0$,*

$$st_2\text{-}\lim t_{\lambda_m\lambda_n} = L, \quad st_2\text{-}\lim t_{\lambda_m n} = L, \quad \textit{and } st_2\text{-}\lim t_{m\lambda_n} = L.$$

Proof. We only show $st_2\text{-}\lim t_{\lambda_m\lambda_n} = L$, the proofs of other cases are similar. For $\lambda = 1$, it is trivial.

Suppose $\lambda > 1$. For all $M, N \geq 1$ and $\epsilon > 0$,

$$\{m \leq M, n \leq N : D(t_{\lambda_m\lambda_n}, L) \geq \epsilon\} \subseteq \{m \leq \lambda_m, n \leq \lambda_n : D(t_{\lambda_m\lambda_n}, L) \geq \epsilon\},$$

whence we find

$$\begin{aligned} &\tfrac{1}{MN}|\{m \leq M, n \leq N : D(t_{\lambda_m\lambda_n}, L) \geq \epsilon\}| \\ &\quad \leq \tfrac{\lambda.\lambda}{\lambda_M\lambda_N}|\{m \leq \lambda_M, n \leq \lambda_N : D(t_{mn}, L) \geq \epsilon\}| \to 0 \text{ as } \min(M,N) \to \infty. \end{aligned}$$

The last step follows from the fact that x is $(\overline{N}^2, p, q)^F$-statistically summable to L.

For $0 < \lambda < 1$, the sequence $(\lambda_j)_{j=1}^{\infty}$ is non-decreasing. If integers j and s are such that $m = \lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+s-1} < \lambda_{j+s}$, then

$$m \le \lambda j < \lambda(j+1) < \cdots < \lambda(j+s-1) < m+1 \le \lambda(j+s).$$

This implies that

$$m + \lambda(s-1) \le \lambda j + \lambda(s-1) = \lambda(j+s-1) = m+1$$

and so $s < 1 + \lambda^{-1}$. Here the pairs (j,k) with $\lambda_j = m$ and $\lambda_k = n$ occur at most $(1+\lambda^{-1})^2$ times. Moreover, we have $\lambda_N/N \le 2\lambda$ for $N \ge \max\{\lambda^{-1} - 2, 0\}$. Consequently,

$$\begin{aligned}
&\tfrac{1}{MN} \left|\{m \le M, n \le N : D(t_{\lambda_m\lambda_n}, L) \ge \epsilon\}\right| \\
&\quad \le \tfrac{(1+\lambda^{-1})^2}{MN} \left|\{m \le \lambda_M, n \le \lambda_N : D(t_{mn}, L) \ge \epsilon\}\right| \\
&\quad \le \tfrac{(1+\lambda^{-1})^2(2\lambda)^2}{\lambda_M\lambda_N} \left|\{m \le \lambda_M, n \le \lambda_N : D(t_{mn}, L) \ge \epsilon\}\right| \to 0
\end{aligned}$$

as $\min(M,N) \to \infty$. This step completes the proof. □

Lemma 2.9. *Let the sequences p and q be the same as in Lemma 2.8.*

(1) *If $\lambda > 1$, $\lambda_m > m$, and $\lambda_n > n$, then*

$$\begin{aligned}
D(\tau^{>}_{mn}, t_{mn}) \le{}& \tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \left(D(t_{\lambda_m\lambda_n}, t_{\lambda_m n}) + D(t_{mn}, t_{m\lambda_n})\right) \\
&+ \tfrac{P_{\lambda_m}}{P_{\lambda_m}-P_m} D(t_{\lambda_m n}, t_{mn}) + \tfrac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n} D(t_{m\lambda_n}, t_{mn}).
\end{aligned} \tag{3}$$

(2) *If $0 < \lambda < 1$, $\lambda_m < m$, and $\lambda_n < n$, then*

$$\begin{aligned}
D(\tau^{<}_{mn}, t_{mn}) \le{}& \tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})} \left(D(t_{\lambda_m\lambda_n}, t_{\lambda_m n}) + D(t_{mn}, t_{m\lambda_n})\right) \\
&+ \tfrac{P_{\lambda_m}}{P_m-P_{\lambda_m}} D(t_{\lambda_m n}, t_{mn}) + \tfrac{Q_{\lambda_n}}{Q_n-Q_{\lambda_n}} D(t_{m\lambda_n}, t_{mn}).
\end{aligned} \tag{4}$$

Proof. (1) Let $\lambda > 1$, $\lambda_m > m$, and $\lambda_n > n$. By definition of $\tau^{>}_{mn}$ and relations (ii), (iv) of Lemma 1.4 we have

$$D\left(\tau^{>}_{mn}, t_{mn}\right) = D\left(\tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n} p_j q_k x_{jk}, t_{mn}\right)$$

$$\begin{aligned}
= D\Bigg(&\tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n} p_j q_k x_{jk} \\
&+ \tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \left(\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n} + \sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n} + 2\sum_{j=0}^{m}\sum_{k=0}^{n}\right) p_j q_k x_{jk}, \\
&t_{mn} + \tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \left(\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n} + \sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n} + 2\sum_{j=0}^{m}\sum_{k=0}^{n}\right) p_j q_k x_{jk}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= D\Bigg(\tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=0}^{\lambda_m}\sum_{k=0}^{\lambda_n} p_j q_k x_{jk}\\
&\quad +\tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=0}^{m}\sum_{k=0}^{n} p_j q_k x_{jk},\\
&\quad t_{mn}+\tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=0}^{m}\sum_{k=0}^{\lambda_n} p_j q_k x_{jk}\\
&\quad +\tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=0}^{\lambda_m}\sum_{k=0}^{n} p_j q_k x_{jk}\Bigg)\\
&= D\Big(\tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{\lambda_m\lambda_n}+\tfrac{P_mQ_n}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{mn},\\
&\quad t_{mn}+\tfrac{P_{\lambda_m}Q_n}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{\lambda_m n}+\tfrac{P_mQ_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{m\lambda_n}\Big)\\
&= D\Big(\tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{\lambda_m\lambda_n}+\tfrac{P_mQ_n}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{mn}\\
&\quad +\tfrac{P_{\lambda_m}}{(P_{\lambda_m}-P_m)}t_{\lambda_m n}+\tfrac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)}t_{m\lambda_n},\ t_{mn}+\tfrac{P_{\lambda_m}Q_n}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{\lambda_m n}\\
&\quad +\tfrac{P_{\lambda_m}}{(P_{\lambda_m}-P_m)}t_{\lambda_m n}+\tfrac{P_mQ_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{m\lambda_n}+\tfrac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)}t_{m\lambda_n}\Big)\\
&= D\Big(\tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{\lambda_m\lambda_n}+\tfrac{P_mQ_n}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{mn}+\tfrac{P_{\lambda_m}}{(P_{\lambda_m}-P_m)}t_{\lambda_m n}\\
&\quad +\tfrac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)}t_{m\lambda_n},\ t_{mn}+\tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\left(t_{\lambda_m n}+t_{m\lambda_n}\right)\Big)\\
&\le \tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}D\left(t_{\lambda_m\lambda_n},t_{\lambda_m n}\right)+D\Big(\tfrac{P_mQ_n}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{mn}\\
&\quad +\tfrac{P_{\lambda_m}}{(P_{\lambda_m}-P_m)}t_{\lambda_m n}+\tfrac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)}t_{m\lambda_n}+\tfrac{P_m}{(P_{\lambda_n}-P_n)}t_{mn}+\tfrac{Q_n}{(Q_{\lambda_n}-Q_n)}t_{mn}+t_{mn},\\
&\quad t_{mn}+\tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{m\lambda_n}+\tfrac{P_m}{(P_{\lambda_n}-P_n)}t_{mn}+\tfrac{Q_n}{(Q_{\lambda_n}-Q_n)}t_{mn}+t_{mn}\Big)\\
&= \tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}D\left(t_{\lambda_m\lambda_n},t_{\lambda_m n}\right)+D\Big(\tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{mn}\\
&\quad +\tfrac{P_{\lambda_m}}{(P_{\lambda_m}-P_m)}t_{\lambda_m n}+\tfrac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)}t_{m\lambda_n},\\
&\quad \tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}t_{m\lambda_n}+\tfrac{P_{\lambda_m}}{(P_{\lambda_n}-P_n)}t_{mn}+\tfrac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)}t_{mn}\Big)\\
&\le \tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\left(D(t_{\lambda_m\lambda_n},t_{\lambda_m n})+D(t_{mn},t_{m\lambda_n})\right)\\
&\quad +\tfrac{P_{\lambda_m}}{(P_{\lambda_m}-P_m)}D\left(t_{\lambda_m n},t_{mn}\right)+\tfrac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)}D\left(t_{m\lambda_n},t_{mn}\right).
\end{aligned}
$$

(2) Let $0 < \lambda < 1$, $\lambda_m < m$ and $\lambda_n < n$. By definition of $\tau^{<}_{mn}$ and relations (ii), (iv) of Lemma 1.4 we have

$$D\left(\tau^{<}_{mn}, t_{mn}\right) = D\left(\frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k x_{jk}, t_{mn}\right)$$

$$= D\Bigg(\frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k x_{jk}$$

$$+ \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \left(\sum_{j=0}^{\lambda_m} \sum_{k=\lambda_n+1}^{n} + \sum_{j=\lambda_m+1}^{m} \sum_{k=0}^{\lambda_n} + 2\sum_{j=0}^{\lambda_m}\sum_{k=0}^{\lambda_n}\right) p_j q_k x_{jk},$$

$$t_{mn} + \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \left(\sum_{j=0}^{\lambda_m} \sum_{k=\lambda_n+1}^{n} + \sum_{j=\lambda_m+1}^{m} \sum_{k=0}^{\lambda_n} + 2\sum_{j=0}^{\lambda_m}\sum_{k=0}^{\lambda_n}\right) p_j q_k x_{jk}\Bigg)$$

$$= D\Bigg(\frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=0}^{\lambda_m}\sum_{k=0}^{\lambda_n} p_j q_k x_{jk}$$

$$+ \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=0}^{m}\sum_{k=0}^{n} p_j q_k x_{jk},$$

$$t_{mn} + \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=0}^{m}\sum_{k=0}^{\lambda_n} p_j q_k x_{jk}$$

$$+ \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=0}^{\lambda_m}\sum_{k=0}^{n} p_j q_k x_{jk}\Bigg)$$

$$= D\Big(\frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} t_{\lambda_m \lambda_n} + \frac{P_m Q_n}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} t_{mn},$$

$$t_{mn} + \frac{P_{\lambda_m} Q_n}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} t_{\lambda_m n} + \frac{P_m Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} t_{m \lambda_n}\Big)$$

$$= D\Big(\frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} t_{\lambda_m \lambda_n} + \frac{P_m Q_n}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} t_{mn},$$

$$t_{mn} + \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} t_{\lambda_m n} + \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} t_{m \lambda_n}$$

$$+ \frac{P_{\lambda_m}}{(P_m - P_{\lambda_m})} t_{\lambda_m n} + \frac{Q_{\lambda_n}}{(Q_n - Q_{\lambda_n})} t_{m \lambda_n}\Big)$$

$$\leq \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} D\left(t_{\lambda_m \lambda_n}, t_{\lambda_m n}\right) + D\Big(\frac{P_m Q_n}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} t_{mn},$$

$$t_{mn} + \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} t_{m \lambda_n} + \frac{P_{\lambda_m}}{(P_m - P_{\lambda_m})} t_{\lambda_m n} + \frac{Q_{\lambda_n}}{(Q_n - Q_{\lambda_n})} t_{m \lambda_n}\Big)$$

$$= \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} D\left(t_{\lambda_m \lambda_n}, t_{\lambda_m n}\right) + D\Big(\frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} t_{mn}$$

$$
\begin{aligned}
&+\tfrac{P_{\lambda_m}}{(P_m-P_{\lambda_m})}t_{mn}+\tfrac{Q_{\lambda_n}}{(Q_n-Q_{\lambda_n})}t_{mn}+t_{mn},\, t_{mn}+\tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}t_{m\lambda_n}\\
&+\tfrac{P_{\lambda_m}}{(P_m-P_{\lambda_m})}t_{\lambda_m n}+\tfrac{Q_{\lambda_n}}{(Q_n-Q_{\lambda_n})}t_{m\lambda_n}\Big)\\
\le\ & \tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}D\left(t_{\lambda_m\lambda_n},t_{\lambda_m n}\right)+\tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}D\left(t_{mn},t_{m\lambda_n}\right)\\
&+\tfrac{P_{\lambda_m}}{(P_m-P_{\lambda_m})}D\left(t_{mn},t_{\lambda_m n}\right)+\tfrac{Q_{\lambda_n}}{(Q_n-Q_{\lambda_n})}D\left(t_{mn},t_{m\lambda_n}\right).
\end{aligned}
$$

Lemma is proved. □

Theorem 2.10. *Let $p=(p_j)_{j=0}^{\infty}$ and $q=(q_k)_{k=0}^{\infty}$ be sequences of non-negative numbers such that $p_0, q_0>0$. Suppose that both sequences (P_m) and (Q_n) satisfy either (1) or (2). If a sequence $x=(x_{mn})\in {}_2W^F$ is $(\overline{N}^2,p,q)^F$-statistically summable to L, then, for $\lambda>1$, $st_2\text{-}\lim_{m,n\to\infty}\tau^{>}_{mn}=L$, i.e.,*

$$
st_2\text{-}\lim_{m,n\to\infty}\tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_jq_kx_{jk}=L, \tag{5}
$$

and, for $0<\lambda<1$, $st_2\text{-}\lim_{m,n\to\infty}\tau^{<}_{mn}=L$, i.e.,

$$
st_2\text{-}\lim_{m,n\to\infty}\tfrac{1}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}p_jq_kx_{jk}=L. \tag{6}
$$

Proof. Suppose $\lambda>1$. If $P_{\lambda_m}>P_m$ and $Q_{\lambda_n}>Q_n$, then

$$
D(\tau^{>}_{mn},L)=D(\tau^{>}_{mn}+t_mn,t_{mn}+L)\le D(\tau^{>}_{mn},t_{mn})+D(t_{mn},L),
$$

and using (3), we get

$$
\begin{aligned}
D(\tau^{>}_{mn},L)\le\ & \tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}D(t_{\lambda_m\lambda_n},t_{\lambda_m n})\\
&+\tfrac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}D(t_{mn},t_{m\lambda_n})-\tfrac{P_{\lambda_m}}{P_{\lambda_m}-P_m}D(t_{\lambda_m n},t_{mn})\\
&-\tfrac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n}D(t_{m\lambda_n},t_{mn})+D(t_{mn},L).
\end{aligned}
$$

By (1) we obtain

$$
\begin{aligned}
st\text{-}\limsup_{m\to\infty}\tfrac{P_{\lambda_m}}{P_{\lambda_m}-P_m}&=\left\{st\text{-}\liminf_{m\to\infty}\left(1-\tfrac{P_m}{P_{\lambda_m}}\right)\right\}^{-1}\\
=\left\{1-\left(st\text{-}\limsup_{m\to\infty}\tfrac{P_m}{P_{\lambda_m}}\right)\right\}^{-1}&=\left\{1-\tfrac{1}{st\text{-}\liminf_{m\to\infty}\frac{P_{\lambda_m}}{P_m}}\right\}^{-1}<\infty.
\end{aligned}
$$

Similarly we have

$$
st\text{-}\limsup_{m\to\infty}\tfrac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n}<\infty.
$$

Thus, for any $\epsilon>0$, we conclude that

$$
\left\{m\le M,n\le N: D(\tau^{>}_{mn},L)\ge\epsilon\right\}
$$

$$
\begin{aligned}
&\subseteq \left\{m \le M, n \le N : \tfrac{P_{\lambda_m} Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} D(t_{\lambda_m\lambda_n}, t_{\lambda_m n}) \ge \tfrac{\epsilon}{5}\right\}\\
&\quad \bigcup \left\{m \le M, n \le N : \tfrac{P_{\lambda_m} Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} D(t_{mn}, t_{m\lambda_n}) \ge \tfrac{\epsilon}{5}\right\}\\
&\quad \bigcup \left\{m \le M, n \le N : \tfrac{P_{\lambda_m}}{P_{\lambda_m}-P_m} D(t_{\lambda_m n}, t_{mn}) \ge \tfrac{\epsilon}{5}\right\}\\
&\quad \bigcup \left\{m \le M, n \le N : \tfrac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n} D(t_{m\lambda_n}, t_{mn}) \ge \tfrac{\epsilon}{5}\right\}\\
&\quad \bigcup \left\{m \le M, n \le N : D(t_{mn}, L) \ge \tfrac{\epsilon}{5})\right\}.
\end{aligned}
$$

Finally, using Lemma 2.8 and the fact that $x = (x_{mn})$ is $(\overline{N}^2, p, q, 1, 1)^F$-statistically summable to L, we get (5).

If $0 < \lambda < 1$, $P_{\lambda_m} < P_m$, $Q_{\lambda_n} > Q_n$, then, using the inequality (4) and proceeding in a similar manner as above, we obtain (6).

□

Theorem 2.11. *Let the sequences p, q, and (P_m), (Q_n) be the same as in Theorem 2.10. If a double sequence of fuzzy numbers $x = (x_{mn})$ is $(\overline{N}^2, p, q)^F$-statistically summable to L, then x is statistically convergent to L, i.e.,*

$$st_2\text{-}\lim x_{mn} = L, \tag{7}$$

if and only if one of the following statements hold for every $\epsilon > 0$:

$$
\begin{aligned}
&\inf_{\lambda>1} \limsup_{M,N\to\infty} \tfrac{1}{MN} \big|\{m \le M, n \le N :\\
&\qquad D\left(\tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k x_{jk}, x_{mn}\right) \ge \epsilon\Bigg\}\Bigg| = 0
\end{aligned}
\tag{8}
$$

or

$$
\begin{aligned}
&\inf_{0<\lambda<1} \limsup_{M,N\to\infty} \tfrac{1}{MN} \big|\{m \le M, n \le N :\\
&\qquad D\left(\tfrac{1}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k x_{jk}, x_{mn}\right) \ge \epsilon\Bigg\}\Bigg| = 0.
\end{aligned}
\tag{9}
$$

Proof. Let $x = (x_{mn})$ be a double sequence of fuzzy numbers which is $(\bar{N}^2, p, q)^F$-statistically summable to L.

Necessity. Suppose that x is statistically convergent to L. Then, using relations (5) and (7), we get (8). Similarly, (9) follows from (6) and (7).

Sufficiency. Suppose that one of the conditions (8) and (9) holds. To prove $st_2\text{-}\lim\limits_{m,n\to\infty} x_{mn} = L$, it is enough to prove that

$$st_2\text{-}\lim D(x_{mn}, t_{mn}) = 0. \tag{10}$$

First, we suppose that (8) holds. Since in the case $\lambda > 1$

$$\begin{aligned}
D(t_{mn}, x_{mn}) &\le D\left(t_{mn}, \tau^{>}_{mn}\right) + D\left(\tau^{>}_{mn}, x_{mn}\right) \\
&\le \tfrac{P_{\lambda_m} Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} D\left(t_{\lambda_m \lambda_n}, t_{\lambda_m n}\right) + \tfrac{P_{\lambda_m} Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} D\left(t_{mn}, t_{m\lambda_n}\right) \\
&\quad + \tfrac{P_{\lambda_m}}{(P_{\lambda_m}-P_m)} D\left(t_{\lambda_m n}, t_{mn}\right) + \tfrac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)} D\left(t_{m\lambda_n}, t_{mn}\right) \\
&\quad + D\left(\tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k x_{jk}, x_{mn}\right),
\end{aligned}$$

for any $\epsilon > 0$ we have

$$\begin{aligned}
&\{m \le M, n \le N : D(t_{mn}, x_{mn}) \ge \epsilon\} \\
&\subseteq \left\{m \le M, n \le N : \tfrac{P_{\lambda_m} Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} D\left(t_{\lambda_m \lambda_n}, t_{\lambda_m n}\right) \ge \tfrac{\epsilon}{5}\right\} \\
&\quad \cup \left\{m \le M, n \le N : \tfrac{P_{\lambda_m} Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} D\left(t_{mn}, t_{m\lambda_n}\right) \ge \tfrac{\epsilon}{5}\right\} \\
&\quad \cup \left\{m \le M, n \le N : \tfrac{P_{\lambda_m}}{(P_{\lambda_m}-P_m)} D\left(t_{\lambda_m n}, t_{mn}\right) \ge \tfrac{\epsilon}{5}\right\} \\
&\quad \cup \left\{m \le M, n \le N : \tfrac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)} D\left(t_{m\lambda_n}, t_{mn}\right) \ge \tfrac{\epsilon}{5}\right\} \\
&\quad \cup \left\{m \le M, n \le N : D\left(\tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k x_{jk}, x_{mn}\right) \ge \tfrac{\epsilon}{5}\right\}.
\end{aligned}$$

By (8), for any given $\delta > 0$, there exists $\lambda > 1$ such that

$$\begin{aligned}
&\limsup_{M,N\to\infty} \tfrac{1}{MN} \Big|\Big\{m \le M, n \le N : \\
&\qquad D\left(\tfrac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k x_{jk}, x_{mn}\right) \ge \tfrac{\epsilon}{5}\Big\}\Big| \le \delta.
\end{aligned}$$

From Lemma 2.8 it follows that

$$\limsup_{M,N\to\infty} \tfrac{1}{MN} \left| \left\{m \le M, n \le N : \tfrac{P_{\lambda_m} Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} D\left(t_{\lambda_m \lambda_n}, t_{\lambda_m n}\right) \ge \tfrac{\epsilon}{5}\right\}\right| = 0,$$

$$\limsup_{M,N\to\infty} \tfrac{1}{MN} \left| \left\{m \le M, n \le N : \tfrac{P_{\lambda_m} Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} D\left(t_{mn}, t_{m\lambda_n}\right) \ge \tfrac{\epsilon}{5}\right\}\right| = 0,$$

$$\limsup_{M,N\to\infty} \tfrac{1}{MN} \left| \left\{m \le M, n \le N : \tfrac{P_{\lambda_m}}{(P_{\lambda_m}-P_m)} D\left(t_{\lambda_m n}, t_{mn}\right) \ge \tfrac{\epsilon}{5}\right\}\right| = 0,$$

$$\limsup_{M,N\to\infty} \tfrac{1}{MN} \left| \left\{m \le M, n \le N : \tfrac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)} D\left(t_{m\lambda_n}, t_{mn}\right) \ge \tfrac{\epsilon}{5}\right\}\right| = 0.$$

Thus we have

$$\lim_{M,N\to\infty} \sup \tfrac{1}{MN} \left| \{m \le M, n \le N : D(x_{mn}, t_{mn}) \ge \epsilon\}\right| \le \delta.$$

Since δ is arbitrary, we can say that, for every $\epsilon > 0$,

$$\lim_{M,N\to\infty} \frac{1}{MN} \left| \{m \le M, n \le N : D(x_{mn}, t_{mn}) \ge \epsilon\} \right| = 0,$$

i.e., (10) is true.

Secondly, if (9) is satisfied, then (10) may be proved in a similar way, considering the case $0 < \lambda < 1$ with $P_m - P_{\lambda_m}$ and $Q_n - Q_{\lambda_n}$ instead of $P_{\lambda_m} - P_m$ and $Q_{\lambda_n} - Q_n$, respectively. □

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