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$\mathcal{I}_2$-Relative uniform convergence and Korovkin type approximation

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Abstract. In the present paper, an interesting type of convergence named ideal relative uniform convergence for double sequences of functions has been introduced for the first time. Then, the Korovkin type approximation theorem via this new type of convergence has been proved. An example to show that the new type of convergence is stronger than the convergence considered before has been given. Finally, the rate of $\mathcal{I}_2$-relative uniform convergence has been computed.

1. Introduction

A compact subset of real numbers has been named S and the space of all continuous real-valued functions on S has been named $C(S)$. Let $\{L_n\}$ be a sequence of positive linear operators that maps $C(S)$ to itself. Suppose that the sequence $\{L_n(f_r)\}$ converges to f_r uniformly on S for the functions $f_r : x \to x^r$, $r = 0, 1, 2$. Then, Korovkin [20] established that this function sequence converges to f uniformly on S for every $f \in C(S)$. Afterwards Korovkin's result has been extended in many directions (see, e.g., [1, 8, 10, 13, 14, 18, 26]). Recently, Demirci and Orhan [6] have presented statistically relatively uniform convergence for single sequences and more recently, Okçu Şahin and Dirik [25] have suggested this notion for double sequences (see also [7]). The authors prove a Korovkin type theorem on $C(S)$ via these new types of convergence, and many researchers have used these notions and their further generalizations for proving Korovkin type approximation theorems on different spaces (see for instance [5, 9, 12, 19, 22, 31]). In this paper, we define the notion of $\mathcal{I}_2$-relative uniform convergence that is a new

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and interesting type of convergence. We use this notion to prove a Korovkin type theorem and we present an application that shows our theorem being a non-trivial generalization of the classical and the ideal cases of the Korovkin's results. Finally, we study the rate of convergence via modulus of continuity.

2. Preliminaries

We begin by well known and important convergences for double sequences: Pringsheim convergence and statistical convergence.

Let $\mathbb{N}$ be the set of all natural numbers and $x = \{x_{mn}\}$ be a double sequence. Then x is said to be convergent in Pringsheim's sense if and only if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{mn} - l| < \varepsilon$ whenever $m, n > N$. Here l is the P-limit of x, it is denoted by $P - \lim\limits_{m,n} x_{mn} = l$, and we call such an x Pringsheim convergent or, more simply, "*P-convergent*" (see [27]). Also, $x = \{x_{mn}\}$ is said to be bounded if and only if there exists a positive number N such that $|x_{mn}| \leq N$ for all $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. As it is known, in contrast to the case of a single sequence, a convergent double sequence need not be bounded.

In 1951, Fast [17] and Steinhaus [28] gave the now well known statistical convergence of single sequences, independently. Later on, this concept was extended to double sequences by Moricz [24]. Let $D \subset \mathbb{N}^2$ be a two-dimensional subset of positive integers and $|D|$ denotes the cardinality of D, then the double natural density of D is given by

$$\delta_2(D) := P - \lim_{i,j} \frac{|\{m \leq i,\ n \leq j : (m,n) \in D\}|}{ij}$$

whenever the limit exists. The number sequence $x = \{x_{mn}\}$ is *statistically convergent* to l provided that for every $\varepsilon > 0$ the set

$$D := D_{ij}(\varepsilon) := \{m \leq i,\ n \leq j : \ |x_{mn} - l| \geq \varepsilon\}$$

has natural density zero; in that case we write $st_2 - \lim\limits_{m,n} x_{mn} = l$. Obviously, if a double sequence is P-convergent then it converges to the same value statistically, but a statistically convergent double sequence may not be P-convergent. In addition, please note that as in the case of P-convergence, a statistically convergent double sequence need not be bounded.

The concept of uniform convergence of a sequence of functions relative to a scale function was first given by Moore in [23]. Then, Chittenden [3] gave a definition, which is equivalent to the definition given by Moore for single sequences and Okçu Şahin and Dirik [25] extended this idea to double sequences as follows:

Definition 1 (see [25]). A double function sequence $\{f_{m,n}\}$ defined on any compact subset of the real two-dimensional space converges to the limit function f relatively uniformly if there exists a function $\sigma(x, y)$ defined on any compact subset of the real two-dimensional space (which in the literature is called a scale function) such that for every $\varepsilon > 0$ there is an integer n_ε such that for $m, n > n_\varepsilon$ the inequality

$$|f_{mn}(x,y) - f(x,y)| < \varepsilon\, |\sigma(x,y)|$$

holds uniformly in (x,y). The double sequence $\{f_{m,n}\}$ is said to be convergent *uniformly relative to the scale function* σ, briefly, relatively uniformly convergent.

Kostyrko et al. [21] have defined $\mathcal{I}$-convergence using the ideal $\mathcal{I}$. This type of convergence can be seen as a general form of statistical convergence. A class $\mathcal{I}$ of subsets of X, a non-empty set, is called an *ideal* in X if and only if (i) $\emptyset \in \mathcal{I}$, (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and (iii) for each $A \in \mathcal{I}$ and $B \subset A$ we have $B \in \mathcal{I}$. If $\{x\} \in \mathcal{I}$ for each $x \in X$ then an ideal is called *admissible.* If $\mathcal{I}$ is a *non-trivial ideal* in X (i.e. $X \notin \mathcal{I}$, $\mathcal{I} \neq \{\emptyset\}$) then the family of sets $\mathcal{F} = \{U \subset X : (\exists A \in \mathcal{I})\,(U = X \setminus A)\}$ is a *filter* in X and we call such a filter the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N}^2$ is called s*trongly admissible* if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$. It is obvious that a strongly admissible ideal is admissible too. Let

$$\mathcal{I}_2^0 = \left\{B \subset \mathbb{N}^2 : (\exists m(B) \in \mathbb{N})\,(i, j \geq m(B) \Rightarrow (i,j) \notin B)\right\},$$

then $\mathcal{I}_2^0$ is a non-trivial strongly admissible ideal ([4]) and clearly $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

For the rest of the paper, we use $\mathcal{I}_2$ as a non-trivial strongly admissible ideal in $\mathbb{N}^2$.

Now, we recall the ideal convergences for the sequences of functions and then we introduce our new type of convergence.

Definition 2 (see [15])**.** A double sequence $\{f_{m,n}\}$ is said to be *ideal pointwise convergent* to a function f on a set $S^2 \subset \mathbb{R}^2$, written in short as $\mathcal{I}_2 - f_{mn} \to f$ on S^2, if for every $\varepsilon > 0$ and for each $(x,y) \in S^2$,

$$\left\{(m,n) \in \mathbb{N}^2 : |f_{mn}(x,y) - f(x,y)| \geq \varepsilon\right\} \in \mathcal{I}_2. \tag{1}$$

Definition 3 (see [16])**.** A double sequence $\{f_{m,n}\}$ is said to be *ideal uniformly convergent* to a function f on a set $S^2 \subset \mathbb{R}^2$, written in short as $\mathcal{I}_2 - f_{mn} \rightrightarrows f$ on S^2, if for every $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \sup_{(x,y) \in S^2} |f_{mn}(x,y) - f(x,y)| \geq \varepsilon\right\} \in \mathcal{I}_2. \tag{2}$$

Definition 4. A double sequence $\{f_{m,n}\}$ is said to be *ideal relatively uniformly convergent* to a function f on a set $S^2 \subset \mathbb{R}^2$ if there exists a function $\sigma(x,y)$, called a scale function, with $|\sigma(x,y)| \neq 0$ such that

$$\left\{(m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in S^2} \left| \frac{f_{mn}(x,y) - f(x,y)}{\sigma(x,y)} \right| \geq \varepsilon \right\} \in \mathcal{I}_2. \tag{3}$$

We denote this by $\mathcal{I}_2 - f_{mn} \rightrightarrows f \quad (S^2;\sigma)$.

It is worth noting that, if we take $\mathcal{I}_2 = \mathcal{I}_2^0$ and $\mathcal{I}_2 = \mathcal{I}_2^\delta$, the set of all subsets of $\mathbb{N}^2$ with double natural density zero, then we get the concepts of relative uniform convergence and statistical relative uniform convergence of the double sequence of functions, respectively ([25]).

In view of the above definitions, we immediately have the following result.

Lemma 1. *$f_{mn} \rightrightarrows f$ on S^2 implies $\mathcal{I}_2 - f_{mn} \rightrightarrows f$ on S^2, which also implies $\mathcal{I}_2 - f_{mn} \rightrightarrows f \quad (S^2;\sigma)$.*

However, an example can be provided to show that the converse of Lemma 1 is not always true. The following is such an example.

Example 1. Let $\mathcal{I}_2 = \mathcal{I}_2^\delta$ and $B \in \mathcal{I}_2^\delta$ be an infinite set. For each $(m,n) \in \mathbb{N}^2$, define $g_{mn} : [0,1]^2 \to \mathbb{R}$ by

$$g_{mn}(x,y) = \begin{cases} mn^2xy^2, & (m,n) \in B, \\ \frac{3mn^2xy^2}{2+m^2n^3x^2y^4}, & (m,n) \notin B. \end{cases} \tag{4}$$

Then it is seen to be $\mathcal{I}_2^\delta - g_{mn} \rightrightarrows g = 0 \ ([0,1]^2;\sigma)$,

$$\sigma(x,y) = \begin{cases} 1, & x = 0 \text{ or } y = 0, \\ \frac{1}{xy^2}, & (x,y) \in (0,1] \times (0,1], \end{cases} \tag{5}$$

however $\{g_{mn}\}$ is neither ideally (or statistically) uniform convergent nor classically uniform convergent to the function $g = 0$ on the interval $[0,1]^2$.

3. Ideal relative Korovkin type approximation

In this section, we handle the notion of $\mathcal{I}_2$-relative uniform convergence to prove a Korovkin type approximation theorem for a double sequence of positive linear operators defined on $C(S^2)$. Note that $C(S^2)$ is a Banach space with the norm $\|.\|$ defined by $\|f\| := \sup_{(x,y)\in S^2} |f(x,y)|$, $f \in C(S^2)$. Let L be a linear operator from $C(S^2)$ into itself. We denote the value of $L(f)$ at a point $(x,y) \in S^2$ by $L(f(s,t);x,y)$ or $L(f;x,y)$. In addition, throughout this section, we use the test functions $e_0(x,y) = 1$, $e_1(x,y) = x$, $e_2(x,y) = y$ and $e_3(x,y) = x^2 + y^2$.

Before we continue, let us remind the classical and statistical forms of a Korovkin's type theorem.

Theorem 1 (see [30]). *Let $\{L_{mn}\}$ be a double sequence of positive linear operators acting from $C\left(S^2\right)$ into itself. Then, for all $f \in C\left(S^2\right)$,*

$$L_{mn}(f) \rightrightarrows f \quad on\ S^2$$

if and only if

$$L_{mn}(e_r) \rightrightarrows e_r \ on\ S^2,\ r = 0, 1, 2, 3.$$

Theorem 2 (see [11]). *Let $\{L_{mn}\}$ be a double sequence of positive linear operators acting from $C\left(S^2\right)$ into itself. Then, for all $f \in C\left(S^2\right)$,*

$$st_2 - L_{mn}(f) \rightrightarrows f \quad on\ S^2$$

if and only if

$$st_2 - L_{mn}(e_r) \rightrightarrows e_r \ on\ S^2,\ r = 0, 1, 2, 3.$$

Now, we give the main approximation result of this section.

Theorem 3. *Let $\{L_{mn}\}$ be a double sequence of positive linear operators acting from $C\left(S^2\right)$ into itself and let σ, σ_r be the scale functions (possibly unbounded). Then, for all $f \in C\left(S^2\right)$,*

$$\mathcal{I}_2 - L_{mn}(f) \rightrightarrows f \quad (S^2; \sigma) \tag{6}$$

if and only if

$$\mathcal{I}_2 - L_{mn}(e_r) \rightrightarrows e_r \ (S^2; \sigma_r),\ r = 0, 1, 2, 3. \tag{7}$$

Proof. In view of the hypotheses, since $e_r \in C\left(S^2\right)$ for each $r = 0, 1, 2, 3$, the condition (7) follows from the condition (6). The chief point is in giving the proof of the converse part. In the first step, by the continuity of f on S^2, f is bounded on S^2 and we can write

$$|f(x, y)| \leq K_f,$$

where $K_f = \|f\|$. Also, since f is continuous on S^2, we write that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(x, y) - f(s, t)| < \varepsilon$ for all $(x, y) \in S^2$ satisfying $|x - s| < \delta$ and $|y - t| < \delta$. Also, we get for all $(x, y), (s, t) \in S^2$ satisfying $|x - s| > \delta$ and $|y - t| > \delta$ that

$$|f(x, y) - f(s, t)| \leq \frac{2K_f}{\delta^2}\left\{(x - s)^2 + (y - t)^2\right\}.$$

Hence, we get for $(x, y), (s, t) \in S^2$ that

$$|f(x, y) - f(s, t)| < \varepsilon + \frac{2K_f}{\delta^2}\left\{(x - s)^2 + (y - t)^2\right\}.$$

Since L_{mn} is linear and positive, we have that

$$\begin{aligned} |L_{mn}(f; x, y) - f(x, y)| \ &\leq \ L_{mn}(|f(x, y) - f(s, t)|; x, y) \\ &\quad + |f(x, y)|\, |L_{mn}(e_0; x, y) - e_0(x, y)| \end{aligned}$$

$$\begin{aligned}&\le L_{mn}(\varepsilon+\frac{2K_f}{\delta^2}\left\{(x-s)^2+(y-t)^2\right\};x,y)\\&+K_f\left|L_{mn}\left(e_0;x,y\right)-e_0(x,y)\right|\end{aligned}$$

holds for every $(x,y)\in S^2$ and $m,n\in\mathbb{N}$. Also, using the square of difference and considering the linearity and positivity of L_{mn} again, the following inequality is obtained from the above inequality:

$$\begin{aligned}\left|L_{mn}(f;x,y)-f(x,y)\right| \le\ & \varepsilon+\left\{\varepsilon+K_f+\frac{4K_f}{\delta^2}E^2\right\}\left|L_{mn}\left(e_0;x,y\right)-e_0(x,y)\right|\\&+\frac{4K_f}{\delta^2}E\left|L_{mn}\left(e_1;x,y\right)-e_1(x,y)\right|\\&+\frac{4K_f}{\delta^2}E\left|L_{mn}\left(e_2;x,y\right)-e_2(x,y)\right|\\&+\frac{2K_f}{\delta^2}\left|L_{mn}\left(e_3;x,y\right)-e_3(x,y)\right|,\end{aligned}$$

where $E:=\max\{|x|,|y|\}$. Now choose $\sigma(x,y)=\max\{|\sigma_r(x,y)|;r=0,1,2,3\}$ and multiply both sides of the above inequality by $\frac{1}{|\sigma(x,y)|}$, the last inequality implies that

$$\begin{aligned}\left|\frac{L_{mn}(f;x,y)-f(x,y)}{\sigma(x,y)}\right| \le\ & \frac{\varepsilon}{|\sigma(x,y)|}+K\left\{\left|\frac{L_{mn}\left(e_0;x,y\right)-e_0(x,y)}{\sigma_0(x,y)}\right|\right.\\&+\left|\frac{L_{mn}\left(e_1;x,y\right)-e_1(x,y)}{\sigma_1(x,y)}\right|\\&+\left|\frac{L_{mn}\left(e_2;x,y\right)-e_2(x,y)}{\sigma_2(x,y)}\right|\\&\left.+\left|\frac{L_{mn}\left(e_3;x,y\right)-e_3(x,y)}{\sigma_3(x,y)}\right|\right\},\end{aligned}$$

where $K:=\max\left\{\varepsilon+K_f+\frac{4K_f}{\delta^2}E^2,\frac{4K_f}{\delta^2}E,\frac{2K_f}{\delta^2}\right\}$. Thus, taking supremum over $(x,y)\in S^2$, we have

$$\begin{aligned}&\sup_{(x,y)\in S^2}\left|\frac{L_{mn}(f;x,y)-f(x,y)}{\sigma(x,y)}\right|\\&\le \sup_{(x,y)\in S^2}\frac{\varepsilon}{|\sigma(x,y)|}+K\left\{\sup_{(x,y)\in S^2}\left|\frac{L_{mn}\left(e_0;x,y\right)-e_0(x,y)}{\sigma_0(x,y)}\right|\right.\\&\quad+\sup_{(x,y)\in S^2}\left|\frac{L_{mn}\left(e_1;x,y\right)-e_1(x,y)}{\sigma_1(x,y)}\right|+\sup_{(x,y)\in S^2}\left|\frac{L_{mn}\left(e_2;x,y\right)-e_2(x,y)}{\sigma_2(x,y)}\right|\\&\quad\left.+\sup_{(x,y)\in S^2}\left|\frac{L_{mn}\left(e_3;x,y\right)-e_3(x,y)}{\sigma_3(x,y)}\right|\right\}.\end{aligned}\tag{8}$$

Now, let $r>0$ be given. Choose $\varepsilon>0$ such that $\sup\limits_{(x,y)\in S^2}\frac{\varepsilon}{|\sigma(x,y)|}<r$. Then

$$D:=\left\{(m,n)\in\mathbb{N}^2:\sup_{(x,y)\in S^2}\left|\frac{L_{mn}(f;x,y)-f(x,y)}{\sigma\left(x,y\right)}\right|\geq r\right\}$$

and

$$D_r:=\left\{(m,n)\in\mathbb{N}^2:\sup_{(x,y)\in S^2}\left|\frac{L_{mn}(e_r;x,y)-e_r(x,y)}{\sigma_r\left(x,y\right)}\right|\geq\frac{r-\sup\limits_{(x,y)\in S^2}\frac{\varepsilon}{|\sigma(x,y)|}}{3K}\right\},$$

$r=0,1,2,3$. In view of (8), clearly $D\subset\bigcup\limits_{r=0}^{3}D_r$ and by (7), $D_r\in\mathcal{I}_2$ for $r=0,1,2,3$. Hence, by the definition of an ideal $\bigcup\limits_{r=0}^{3}D_r\in\mathcal{I}_2$, $D\in\mathcal{I}_2$. Therefore we get

$$\mathcal{I}_2-L_{mn}\left(f\right)\rightrightarrows f\ \ \left(S^2;\sigma\right).$$

Thus, we reach our required result. □

If the scale function is replaced by a nonzero constant, the following result immediately follows from our Theorem 3.

Corollary 1 (see [2])**.** *Let $\{L_{mn}\}$ be a double sequence of positive linear operators acting from $C\left(S^2\right)$ into itself. Then, for all $f\in C\left(S^2\right)$,*

$$\mathcal{I}_2-L_{mn}\left(f\right)\rightrightarrows f\quad on\ S^2$$

if and only if

$$\mathcal{I}_2-L_{mn}\left(e_r\right)\rightrightarrows e_r\ \ on\ S^2,\ \ r=0,1,2,3.$$

Now, we give an example that shows that our main theorem is a non-trivial generalization of the classical and the ideal cases of the Korovkin results.

Example 2. Consider the following Bernstein operators (see [29]) given by

$$B_{mn}\left(f;x,y\right)=\sum_{k=0}^{m}\sum_{l=0}^{n}f\left(\frac{k}{m},\frac{l}{n}\right)\binom{m}{k}\binom{n}{l}x^k\left(1-x\right)^{m-k}y^l\left(1-y\right)^{n-l},\tag{9}$$

where $(x,y)\in S^2=[0,1]^2$; $f\in C\left(S^2\right)$. *Using these polynomials, we introduce the following positive linear operators on $C\left(S^2\right)$:*

$$L_{mn}\left(f;x,y\right)=\left(1+g_{mn}\left(x,y\right)\right)B_{mn}\left(f;x,y\right),\tag{10}$$

where $g_{mn}\left(x,y\right)$ is given by (4). Now, observe that

$$\begin{aligned}L_{mn}\left(e_0;x,y\right)&=\left(1+g_{mn}\left(x,y\right)\right)e_0\left(x,y\right),\\ L_{mn}\left(e_1;x,y\right)&=\left(1+g_{mn}\left(x,y\right)\right)e_1\left(x,y\right),\end{aligned}$$

$$\begin{aligned} L_{mn}\left(e_2;x,y\right) &= \left(1+g_{mn}\left(x,y\right)\right)e_2\left(x,y\right), \\ L_{mn}\left(e_3;x,y\right) &= \left(1+g_{mn}\left(x,y\right)\right)\left[e_3\left(x,y\right)+\frac{x-x^2}{m}+\frac{y-y^2}{n}\right]. \end{aligned}$$

In the view of $\mathcal{I}_2^{\delta}-g_{mn}\rightrightarrows g=0\ (S^2;\sigma)$, $\sigma\left(x,y\right)$ is given by (5), we conclude that

$$\mathcal{I}_2^{\delta}-L_{mn}\left(e_r\right)\rightrightarrows e_r\ (S^2;\sigma) \text{ for each } r=0,1,2,3.$$

Thus, by Theorem 3, we immediately see that

$$\mathcal{I}_2^{\delta}-L_{mn}\left(f\right)\rightrightarrows f\ \ (S^2;\sigma) \text{ for all } f\in C\left(S^2\right).$$

Unfortunately, since $\{g_{mn}\}$ is neither ideally (or statistically) uniform convergent nor classically uniform convergent to the function $g=0$ on the interval S^2, we see that Theorem 1, Theorem 2 and Corollary 1 do not work for our operators defined by (10). Consequently, it is shown that our version is more general than those given before.

4. Rate of $\mathcal{I}_2$-relative uniform convergence

This section is devoted to computing the rate of $\mathcal{I}_2$-relative uniform convergence. Let us remind that the modulus of continuity of a function $f\in C(S^2)$ is defined by

$$\omega(f,\delta)=\sup_{\sqrt{(x-s)^2+(y-t)^2}\le\delta}|f(x,y)-f(s,t)| \quad (\delta>0),\ f\in C(S^2).$$

We start with the following definitions.

Definition 5. Let $\{a_{mn}\}$ be a positive non-increasing double sequence. We say that a sequence $\{f_{mn}\}$ is $\mathcal{I}_2$-relatively uniform convergent to f with the rate of $o\left(a_{mn}\right)$ if for every $\varepsilon>0$,

$$\left\{(m,n)\in\mathbb{N}^2:\sup_{(x,y)\in S^2}\left|\frac{f_{mn}(x,y)-f(x,y)}{\sigma(x,y)}\right|\ge\varepsilon a_{mn}\right\}\in\mathcal{I}_2,$$

and this is written in short as

$$\mathcal{I}_2-(f_{mn}-f)=o(a_{mn})\ (S^2;\sigma).$$

Definition 6. Let $\{a_{mn}\}$ be a positive non-increasing double sequence. We say that a sequence $\{f_{mn}\}$ is $\mathcal{I}_2$-relatively uniform bounded with the rate of $O\left(a_{mn}\right)$ if there exists a positive number K such that

$$\left\{(m,n)\in\mathbb{N}^2:\sup_{(x,y)\in S^2}\left|\frac{f_{mn}(x,y)}{\sigma(x,y)}\right|\ge Ka_{mn}\right\}\in\mathcal{I}_2,$$

and this is written in short as

$$\mathcal{I}_2-(f_{mn})=O(a_{mn})\ (S^2;\sigma).$$

Lemma 2. *Let $\{f_{mn}\}$ and $\{g_{mn}\}$ be double sequences of functions belonging to $C(S^2)$. Assume that $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ are positive non-increasing double sequences such that $\mathcal{I}_2 - (f_{mn} - f) = o(\alpha_{mn})$ $(S^2; \sigma_0)$ and $\mathcal{I}_2 - (g_{mn} - g) = o(\beta_{mn})$ $(S^2; \sigma_1)$, $|\sigma_r(x,y)| > 0$ and $\sigma_r(x,y)$ is unbounded $r = 0,1$. Then the following statements hold:*

(i) $\mathcal{I}_2 - (f_{mn} + g_{mn}) - (f+g) = o(\max\{\alpha_{mn}, \beta_{mn}\})$ $(S^2; \max\{|\sigma_r(x,y)|\})$

(ii) $\mathcal{I}_2 - (f_{mn} - f)(g_{mn} - g) = o(\alpha_{mn}\beta_{mn})$ $(S^2; \sigma_0(x,y)\,\sigma_1(x,y))$,

(iii) $\mathcal{I}_2 - (\lambda(f_{mn} - f)) = o(\alpha_{mn})$ $(S^2; \sigma_0(x,y))$ *for any real number* λ,

(iv) $\mathcal{I}_2 - \sqrt{|f_{mn} - f|} = o(\alpha_{mn})$ $(S^2; \sqrt{|\sigma_0(x,y)|})$.

Furthermore, actually, a similar statement holds true when "o" is replaced by "O".

Proof. (i) Assume that $\mathcal{I}_2 - (f_{mn} - f) = o(\alpha_{mn})$ $(S^2; \sigma_0)$ and $\mathcal{I}_2 - (g_{mn} - g) = o(\beta_{mn})$ $(S^2; \sigma_1)$. Also, for every $\varepsilon > 0$ define

$$H := \left\{(m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in S^2} \left| \frac{(f_n + g_n)(x,y) - (f+g)(x,y)}{\sigma(x,y)} \right| \geq \varepsilon\gamma_{mn} \right\},$$

$$H_0 := \left\{(m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in S^2} \left| \frac{f_n(x,y) - f(x,y)}{\sigma_0(x,y)} \right| \geq \frac{\varepsilon}{2}\alpha_{mn} \right\},$$

$$H_1 := \left\{(m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in S^2} \left| \frac{g_n(x,y) - g(x,y)}{\sigma_1(x,y)} \right| \geq \frac{\varepsilon}{2}\beta_{mn} \right\},$$

where $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$ for every $(m,n) \in \mathbb{N}^2$ and $\sigma(x,y) = \max\{|\sigma_r(x,y)|; r = 0,1\}$. Then observe that $H \subset H_0 \cup H_1$. By assumption, $H_0, H_1 \in \mathcal{I}_2$, so that by the definition of an ideal, $H_0 \cup H_1 \in \mathcal{I}_2$, $H \in \mathcal{I}_2$ which completes the proof of (i). The proofs of (ii), (iii) and (iv) are similar to the proof of (i). Hence, we omit them. □

Lemma 3. *Let $\{f_{mn}\}$ and $\{g_{mn}\}$ be function sequences belonging to $C(S^2)$ satisfying $0 \leq f_{mn} \leq g_{mn}$. Assume that $\{\alpha_{mn}\}$ is a positive non-increasing sequence such that $\mathcal{I}_2 - g_{mn} = o(\alpha_{mn})$ $(S^2; \sigma)$, then $\mathcal{I}_2 - f_{mn} = o(\alpha_{mn})$ $(S^2; \sigma)$, $|\sigma(x,y)| > 0$ and $\sigma(x,y)$ is unbounded. Moreover, the result holds when "o" is replaced by "O".*

Proof. Since $0 \leq f_{mn}(x,y) \leq g_{mn}(x,y)$ for every $(x,y) \in S^2$ and any $(m,n) \in \mathbb{N}^2$, we have for every $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in S^2} \left| \frac{f_{mn}(x,y)}{\sigma(x,y)} \right| \geq \varepsilon\alpha_{mn} \right\}$$
$$\subset \left\{(m,n) \in \mathbb{N}^2 : \sup_{(x,y)\in S^2} \left| \frac{g_{mn}(x,y)}{\sigma(x,y)} \right| \geq \varepsilon\alpha_{mn} \right\}.$$

Using that $\mathcal{I}_2 - g_{mn} = o(\alpha_{mn})$ $(S^2; \sigma)$, we obtain the result. □

Now, we have the following result.

Theorem 4. *Let $\{L_{mn}\}$ be a double sequence of positive linear operators acting from $C\left(S^2\right)$ into itself. Also, let $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ be positive non-increasing double sequences. Assume that the following conditions hold:*

(a) $\mathcal{I}_2 - (L_{mn}(e_0) - e_0) = o(\alpha_{mn})\ (S^2; \sigma_0)$,

(b) $\mathcal{I}_2 - \omega(f, \delta_{mn}) = o(\beta_{mn})\ (S^2; \sigma_1)$, *where* $\delta_{mn}(x,y) = \sqrt{L_{mn}(\varphi_{(x,y)}; x, y)}$ *with* $\varphi_{(x,y)}(s,t) = (x-s)^2 + (y-t)^2$.

Then we have, for all $f \in C(S^2)$,

$$\mathcal{I}_2 - (L_{mn}(f) - f) = o(\gamma_{mn})\ (S^2; \sigma),$$

where $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}, \alpha_{mn}\beta_{mn}\}$ and $\sigma(x,y) = \max\{|\sigma_0(x,y)|, \ |\sigma_1(x,y)|, \ |\sigma_0(x,y)\sigma_1(x,y)|\}$, $|\sigma_r(x,y)| > 0$ and $\sigma_r(x,y)$ is unbounded, $r = 0, 1$. It can be readily seen that, a similar result holds when "o" is replaced by "O".

Proof. Let $f \in C(S^2)$ and $(x,y) \in S^2$. Since L_{mn} is linear, positive and also, using the property

$$\omega(f, \sqrt{\varphi_{(x,y)}(s,t)}) \leq \left(1 + \frac{\varphi_{(x,y)}(s,t)}{\delta^2}\right)\omega(f,\delta)$$

of the modulus of continuity, we get, for any δ, that

$$\begin{aligned}
& |L_{mn}(f;x,y) - f(x,y)| \\
\leq\ & L_{mn}(|f(x,y) - f(s,t)|; x,y) + |f(x,y)|\,|L_{mn}(e_0;x,y) - e_0(x,y)| \\
\leq\ & L_{mn}(\omega(f, \sqrt{\varphi_{(x,y)}(s,t)}); x,y) + |f(x,y)|\,|L_{mn}(e_0;x,y) - e_0(x,y)| \\
\leq\ & \omega(f,\delta)L_{mn}(\left(1 + \frac{\varphi_{(x,y)}(s,t)}{\delta^2}\right); x,y) \\
& + |f(x,y)|\,|L_{mn}(e_0;x,y) - e_0(x,y)| \\
\leq\ & \omega(f,\delta)\left\{L_{mn}(e_0;x,y) + \frac{1}{\delta^2}L_{mn}\left(\varphi_{(x,y)}(s,t); x,y\right)\right\} \\
& + |f(x,y)|\,|L_{mn}(e_0;x,y) - e_0(x,y)|.
\end{aligned}$$

Setting $\delta := \delta_{mn}(x,y) = \sqrt{L_{mn}(\varphi_{(x,y)}; x,y)}$ we may write that

$$\begin{aligned}
|L_{mn}(f;x,y) - f(x,y)| \leq\ & 2\omega(f,\delta_{mn}) + \omega(f,\delta_{mn})\,|L_{mn}(e_0;x,y) - e_0(x,y)| \\
& + M\,|L_{mn}(e_0;x,y) - e_0(x,y)|,
\end{aligned}$$

where $M = \|f\|$. In view of the above inequality, we may write

$$\sup_{(x,y)\in S^2}\left|\frac{L_{mn}(f;x,y) - f(x,y)}{\sigma(x,y)}\right|$$

$$
\begin{aligned}
\leq\ & 2 \sup_{(x,y)\in S^2} \frac{\omega(f,\delta_{mn})}{|\sigma_1(x,y)|} \\
& + \sup_{(x,y)\in S^2} \frac{\omega(f,\delta_{mn})}{|\sigma_1(x,y)|} \sup_{(x,y)\in S^2} \left| \frac{L_{mn}(e_0;x,y)-e_0(x,y)}{\sigma_0(x,y)} \right| \\
& + M \sup_{(x,y)\in S^2} \left| \frac{L_{mn}(e_0;x,y)-e_0(x,y)}{\sigma_0(x,y)} \right| .
\end{aligned}
$$

Hence, using the conditions (a) and (b), Lemma 2 and Lemma 3 the proof is complete. □

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