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Equalities between the BLUEs and BLUPs under the partitioned linear fixed model and the corresponding mixed model

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Abstract. In this article we consider the partitioned fixed linear model $\mathscr{F}: \mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}$ and the corresponding mixed model $\mathscr{M}: \mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is a random error vector and $\mathbf{u}$ is a random effect vector. In 2006, Isotalo, Möls, and Puntanen found conditions under which an arbitrary representation of the best linear unbiased estimator (BLUE) of an estimable parametric function of $\boldsymbol{\beta}_1$ in the fixed model $\mathscr{F}$ remains BLUE in the mixed model $\mathscr{M}$. In this paper we extend the results concerning further equalities arising from models $\mathscr{F}$ and $\mathscr{M}$.

1. Introduction

Let the partitioned linear fixed effects model be

$$\mathscr{F} = \{\mathbf{y},\, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2,\, \mathbf{V}\} = \{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \mathbf{V}\}\,,$$

i.e., the n-dimensional observable random vector $\mathbf{y}$ is of the form

$$\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}\,, \quad \operatorname{cov}(\boldsymbol{\varepsilon}) = \mathbf{V},\, \mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}\,,$$

where $\mathbf{X}_1 \in \mathbb{R}^{n\times p_1}$ and $\mathbf{X}_2 \in \mathbb{R}^{n\times p_2}$ are known matrices, $p_1 + p_2 = p$, $\boldsymbol{\beta}_i \in \mathbb{R}^{p_i}$, $i = 1, 2$, are vectors of unknown fixed effects. The covariance matrix $\mathbf{V}$ of the random error vector $\boldsymbol{\varepsilon}$ is assumed to be known.

Consider the linear mixed model $\mathscr{M}$ which is obtained from $\mathscr{F}$ by replacing the fixed vector $\boldsymbol{\beta}_2$ with the random effect vector $\mathbf{u}$:

$$\mathscr{M}: \mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} + \boldsymbol{\varepsilon}, \quad \operatorname{cov}(\boldsymbol{\varepsilon}) = \mathbf{V},\, \mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}\,,$$

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where $\mathbf{X}_1$ and $\mathbf{X}_2$ are as in $\mathscr{F}$, $\boldsymbol{\beta}_1$ is a vector of unknown fixed effects, $\mathbf{u}$ is an unobservable vector of random effects with $\mathrm{E}(\mathbf{u}) = \mathbf{0}$, $\mathrm{cov}(\mathbf{u}) = \mathbf{D}$, $\mathrm{cov}(\boldsymbol{\varepsilon}, \mathbf{u}) = \mathbf{0}$; $\mathbf{V}$ and $\mathbf{D}$ are assumed to be known. In this situation we have

$$\mathrm{cov}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathbf{u}\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}\end{pmatrix}, \quad \mathrm{cov}\begin{pmatrix}\mathbf{y}\\ \mathbf{u}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\Sigma} & \mathbf{X}_2\mathbf{D}\\ \mathbf{D}\mathbf{X}_2' & \mathbf{D}\end{pmatrix},$$

$$\mathrm{cov}(\mathbf{y}) = \mathrm{cov}(\mathbf{X}_2\mathbf{u} + \boldsymbol{\varepsilon}) = \boldsymbol{\Sigma} = \mathbf{X}_2\mathbf{D}\mathbf{X}_2' + \mathbf{V}.$$

Notice that under $\mathscr{F}$ we have $\mathrm{cov}(\mathbf{y}) = \mathbf{V}$ but under $\mathscr{M}$, $\mathrm{cov}(\mathbf{y}) = \boldsymbol{\Sigma}$.

As for notation, $\mathrm{r}(\mathbf{A})$, $\mathbf{A}^-$, $\mathbf{A}^+$, $\mathscr{C}(\mathbf{A})$, and $\mathscr{C}(\mathbf{A})^\perp$, denote, respectively, the rank, a generalized inverse, the (unique) Moore–Penrose inverse, the column space, and the orthogonal complement of $\mathscr{C}(\mathbf{A})$. By $\mathbf{A}^\perp$ we denote any matrix satisfying $\mathscr{C}(\mathbf{A}^\perp) = \mathscr{C}(\mathbf{A})^\perp$. Furthermore, we will write $\mathbf{P}_\mathbf{A} = \mathbf{A}\mathbf{A}^+ = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ to denote the orthogonal projector onto $\mathscr{C}(\mathbf{A})$. The orthogonal projector onto $\mathscr{C}(\mathbf{A})^\perp$ is denoted as $\mathbf{Q}_\mathbf{A} = \mathbf{I}_a - \mathbf{P}_\mathbf{A}$, where $\mathbf{I}_a$ refers to the $a \times a$ identity matrix and a is the number of rows of $\mathbf{A}$. We use the short notations

$$\mathbf{M} = \mathbf{I}_n - \mathbf{P}_\mathbf{X} \in \{\mathbf{X}^\perp\}, \quad \mathbf{M}_i = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}_i} \in \{\mathbf{X}_i^\perp\},\ i = 1, 2.$$

Let $\mathbf{K} \in \mathbb{R}^{k\times p}$. Then a linear statistic $\mathbf{Ay}$ is said to be a linear unbiased estimator (LUE) for $\mathbf{K}\boldsymbol{\beta}$ in $\mathscr{F}$ if its expectation is equal to $\mathbf{K}\boldsymbol{\beta}$, which happens if and only if $\mathbf{K}' = \mathbf{X}'\mathbf{A}'$; then $\mathbf{K}\boldsymbol{\beta}$ is said to be estimable. The LUE $\mathbf{Ay}$ is the best linear unbiased estimator, BLUE, of estimable $\mathbf{K}\boldsymbol{\beta}$ if $\mathbf{Ay}$ has the smallest covariance matrix in the Löwner sense among all LUEs of $\mathbf{K}\boldsymbol{\beta}$:

$$\mathrm{cov}(\mathbf{Ay}) \leq_\mathrm{L} \mathrm{cov}(\mathbf{A}_\#\mathbf{y}) \quad \text{for all } \mathbf{A}_\# \in \mathbb{R}^{k\times n} : \mathbf{A}_\#\mathbf{X} = \mathbf{K}.$$

Correspondingly, the linear predictor $\mathbf{By}$ is said to be unbiased (LUP) for a q-dimensional random vector $\mathbf{g} = \mathbf{K}_1\boldsymbol{\beta}_1 + \mathbf{Ju}$ under $\mathscr{M}$ if the expected prediction error is zero, i.e., $\mathrm{E}(\mathbf{g} - \mathbf{By}) = \mathbf{0}$ for all $\boldsymbol{\beta}_1$; here $\mathbf{K}_1 \in \mathbb{R}^{q\times p_1}$ and $\mathbf{J} \in \mathbb{R}^{q\times p_2}$. Now a LUP $\mathbf{By}$ is the best linear unbiased predictor, BLUP for $\mathbf{g}$ if it minimizes the covariance matrix of the prediction error among all LUPs, i.e., we have the Löwner ordering

$$\mathrm{cov}(\mathbf{g} - \mathbf{By}) \leq_\mathrm{L} \mathrm{cov}(\mathbf{g} - \mathbf{B}_\#\mathbf{y}) \quad \text{for all } \mathbf{B}_\# \in \mathbb{R}^{q\times n} : \mathbf{B}_\#\mathbf{X}_1 = \mathbf{K}_1.$$

Suppose we are interested in comparing the BLUE of $\mathbf{K}_1\boldsymbol{\beta}_1$ under $\mathscr{F}$ and $\mathscr{M}$. To do this we have to assume that $\mathbf{K}_1\boldsymbol{\beta}_1$ is estimable in both models. By Groß and Puntanen [2, Lemma 1], $\mathbf{K}_1\boldsymbol{\beta}_1$ is estimable under $\mathscr{F}$ if and only if $\mathscr{C}(\mathbf{K}_1') \subseteq \mathscr{C}(\mathbf{X}_1'\mathbf{M}_2)$, i.e., $\mathbf{K}_1 = \mathbf{L}\mathbf{M}_2\mathbf{X}_1$ for some matrix $\mathbf{L}$. Thus if we wish to consider the estimation of all estimable parametric functions of $\boldsymbol{\beta}_1$ under $\mathscr{F}$, then it is equivalent to consider $\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$. In other words, the reason to concentrate on estimating $\boldsymbol{\theta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$ is that the properties obtained are valid for all parametric functions of the type $\mathbf{K}_1\boldsymbol{\beta}_1$ that are estimable under the partitioned model $\mathscr{F}$.

Clearly if $\mathbf{K}_1\boldsymbol{\beta}_1$ is estimable under $\mathscr{F}$ then it is estimable under $\mathscr{M}$.

It is well known that $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is estimable in $\mathscr{F}$ if and only if

$$\mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2) = \{\mathbf{0}\}\,. \tag{1}$$

This follows from the requirement $\mathscr{C}(\mathbf{X}_1') \subseteq \mathscr{C}(\mathbf{X}_1'\mathbf{M}_2)$, i.e., $\mathscr{C}(\mathbf{X}_1') = \mathscr{C}(\mathbf{X}_1'\mathbf{M}_2)$, which holds if and only if (1) holds.

For Lemma 1.1, characterizing the BLUE, see, e.g., Rao [20, p. 282], and the BLUP, see, e.g., Christensen [1, p. 294], and [12, p. 1015]. For further references, see Haslett et al. [3, 4]. For the general reviews of the BLUP-properties, see, e.g., Tian [23, 24].

Lemma 1.1. *Consider the models $\mathscr{F}$ and $\mathscr{M}$, and denote $\boldsymbol{\Sigma} = \mathbf{X}_2\mathbf{D}\mathbf{X}_2' + \mathbf{V}$. Then the following statements hold.*

(a) $\mathbf{A}_1\mathbf{y}$ *is the* BLUE *for estimable* $\mathbf{K}\boldsymbol{\beta}$ *under* $\mathscr{F}$ *if and only if*

$$\mathbf{A}_1(\mathbf{X} : \mathbf{V}\mathbf{M}) = (\mathbf{K} : \mathbf{0})\,,\ \textit{i.e.,}\ \mathbf{A}_1 \in \{\mathbf{P}_{\mathbf{K}\boldsymbol{\beta}\,|\mathscr{F}}\}\,.$$

(b) $\mathbf{A}_2\mathbf{y}$ *is the* BLUE *for estimable* $\mathbf{K}_1\boldsymbol{\beta}_1$ *under* $\mathscr{M}$ *if and only if*

$$\mathbf{A}_2(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1) = (\mathbf{K}_1 : \mathbf{0})\,,\ \textit{i.e.,}\ \mathbf{A}_2 \in \{\mathbf{P}_{\mathbf{K}_1\boldsymbol{\beta}_1\,|\mathscr{M}}\}\,.$$

(c) $\mathbf{A}_3\mathbf{y}$ *is the* BLUP *for* $\mathbf{J}\mathbf{u}$ *under* $\mathscr{M}$ *if and only if*

$$\mathbf{A}_3(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1) = (\mathbf{0} : \mathbf{J}\mathbf{D}\mathbf{J}'\mathbf{M}_1)\,,\ \textit{i.e.,}\ \mathbf{A}_3 \in \{\mathbf{P}_{\mathbf{J}\mathbf{u}\,|\mathscr{M}}\}\,.$$

(d) $\mathbf{A}_4\mathbf{y}$ *is the* BLUP *for* $\mathbf{g} = \mathbf{K}_1\boldsymbol{\beta}_1 + \mathbf{J}\mathbf{u}$ *under* $\mathscr{M}$ *if and only if*

$$\mathbf{A}_4(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1) = (\mathbf{K}_1 : \mathbf{J}\mathbf{D}\mathbf{J}'\mathbf{M}_1)\,,\ \textit{i.e.,}\ \mathbf{A}_4 \in \{\mathbf{P}_{\mathbf{g}\,|\mathscr{M}}\}\,.$$

Remark 1.1. Notice the difference between the notations like

$$\mathbf{P}_{\mathbf{A}} = \mathbf{A}\mathbf{A}^+, \quad \{\mathbf{P}_{\mathbf{K}_1\boldsymbol{\beta}_1\,|\mathscr{M}}\}\,.$$

Above $\mathbf{P}_{\mathbf{A}}$ is the (unique) orthogonal projector onto $\mathscr{C}(\mathbf{A})$, while $\{\mathbf{P}_{\mathbf{K}_1\boldsymbol{\beta}_1\,|\mathscr{M}}\}$ is a set of matrices $\mathbf{A}_2$ satisfying $\mathbf{A}_2(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1) = (\mathbf{K}_1 : \mathbf{0})$. □

If $\mathbf{A}_2 \in \{\mathbf{P}_{\mathbf{K}_1\boldsymbol{\beta}_1\,|\mathscr{M}}\}$ and $\mathbf{A}_3 \in \{\mathbf{P}_{\mathbf{J}\mathbf{u}\,|\mathscr{M}}\}$, i.e.,

$$\begin{pmatrix}\mathbf{A}_2\\ \mathbf{A}_3\end{pmatrix}(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1) = \begin{pmatrix}\mathbf{K}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{J}\mathbf{D}\mathbf{J}'\mathbf{M}_1\end{pmatrix}, \tag{2}$$

then premultiplying (2) by $(\mathbf{I}_q : \mathbf{I}_q)$ we immediately see that

$$\mathbf{A}_2 + \mathbf{A}_3 \in \{\mathbf{P}_{\mathbf{K}_1\boldsymbol{\beta}_1+\mathbf{J}\mathbf{u}\,|\mathscr{M}}\}\,,$$

i.e., under $\mathscr{M}$ we have

$$\mathrm{BLUP}(\mathbf{K}_1\boldsymbol{\beta}_1 + \mathbf{J}\mathbf{u}) = \mathrm{BLUE}(\mathbf{K}_1\boldsymbol{\beta}_1) + \mathrm{BLUP}(\mathbf{J}\mathbf{u})\,. \tag{3}$$

It is well known, see, e.g., Rao [20], that

$$\mathbf{G} = \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+, \tag{4}$$

where

$$\mathbf{W} = \mathbf{X}_1\mathbf{X}_1' + \mathbf{X}_2\mathbf{X}_2' + \mathbf{V} = \mathbf{X}\mathbf{X}' + \mathbf{V} \tag{5}$$

is one solution to the equation $\mathbf{A}(\mathbf{X} : \mathbf{V}\mathbf{M}) = (\mathbf{X} : \mathbf{0})$; recall that $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ is always estimable in $\mathscr{F}$. The matrix $\mathbf{G}$ is unique for the choice of generalized inverses marked as "$^-$" but to obtain uniqueness for $\mathbf{G}$ (which somewhat simplifies our considerations) we have to choose the Moore–Penrose inverse $\mathbf{W}^+$ in the end of the expression (4).

Below are some solutions to equations appearing in Lemma 1.1 (for references, see, e.g. [19, Ch. 10]):

$$\begin{aligned}
\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{F}} &= \mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_2 && \in \{\mathbf{P}_{\boldsymbol{\mu}_1\,|\mathscr{F}}\}\,,\\
\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{F}} &= \mathbf{M}_2\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{F}} && \in \{\mathbf{P}_{\boldsymbol{\theta}_1\,|\mathscr{F}}\}\,,\\
\mathbf{G}_{\boldsymbol{\theta}_2\,|\mathscr{F}} &= \mathbf{M}_1\mathbf{X}_2(\mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{X}_2)^-\mathbf{X}_2'\dot{\mathbf{M}}_1 && \in \{\mathbf{P}_{\boldsymbol{\theta}_2\,|\mathscr{F}}\}\,,\\
\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}} &= \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_m^-\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{W}_m^+ && \in \{\mathbf{P}_{\boldsymbol{\mu}_1\,|\mathscr{M}}\}\,,\\
\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}} &= \mathbf{M}_2\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}} && \in \{\mathbf{P}_{\boldsymbol{\theta}_1\,|\mathscr{M}}\}\,,\\
\mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}} &= \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1(\mathbf{M}_1\boldsymbol{\Sigma}\mathbf{M}_1)^+\mathbf{M}_1 && \in \{\mathbf{P}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}}\}\,,\\
\mathbf{G}_{\mathbf{M}_1\mathbf{X}_2\mathbf{u}\,|\mathscr{M}} &= \mathbf{M}_1\mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}} && \in \{\mathbf{P}_{\mathbf{M}_1\mathbf{X}_2\mathbf{u}\,|\mathscr{M}}\}\,,
\end{aligned}$$

where $\boldsymbol{\theta}_2 = \mathbf{M}_1\mathbf{X}_2\boldsymbol{\beta}_2$ and

$$\mathbf{W}_m = \mathbf{X}_1\mathbf{X}_1' + \boldsymbol{\Sigma} = \mathbf{X}_1\mathbf{X}_1' + \mathbf{X}_2\mathbf{D}\mathbf{X}_2' + \mathbf{V}. \tag{6}$$

The matrices $\dot{\mathbf{M}}_1$ and $\dot{\mathbf{M}}_2$ are defined as

$$\dot{\mathbf{M}}_1 = \mathbf{M}_1(\mathbf{M}_1\mathbf{W}\mathbf{M}_1)^+\mathbf{M}_1\,, \quad \dot{\mathbf{M}}_2 = \mathbf{M}_2(\mathbf{M}_2\mathbf{W}\mathbf{M}_2)^+\mathbf{M}_2\,.$$

Moreover, see, e.g., [19, Ch. 15],

$$\dot{\mathbf{M}}_2 = \mathbf{M}_2(\mathbf{M}_2\mathbf{W}\mathbf{M}_2)^+\mathbf{M}_2 = \mathbf{M}_2(\mathbf{M}_2\mathbf{W}\mathbf{M}_2)^+ = (\mathbf{M}_2\mathbf{W}\mathbf{M}_2)^+.$$

Obviously, denoting $\mathbf{W}_1 = \mathbf{X}_1\mathbf{X}_1' + \mathbf{V}$, we have

$$\mathbf{M}_2\mathbf{W} = \mathbf{M}_2\mathbf{W}_1 = \mathbf{M}_2\mathbf{W}_m\,, \quad \mathbf{M}_1\mathbf{W}_m = \mathbf{M}_1\boldsymbol{\Sigma}\,.$$

It is not necessary to choose $\mathbf{W}$ and $\mathbf{W}_m$ as in (5) and in (6). For example, $\mathbf{W}$ could be replaced with $\mathbf{W}_* = \mathbf{X}\mathbf{U}\mathbf{U}'\mathbf{X}' + \mathbf{V}$ such that $\mathscr{C}(\mathbf{W}_*) = \mathscr{C}(\mathbf{X} : \mathbf{V})$; see, e.g., [19, Sec. 12.3].

The solutions to equations in Lemma 1.1 dealing with $\mathscr{F}$ are unique if and only if $\mathscr{C}(\mathbf{W}) = \mathbb{R}^n$ while those dealing with $\mathscr{M}$ are unique if and only if $\mathscr{C}(\mathbf{W}_m) = \mathbb{R}^n$. The *general* solution for $\mathbf{A}$ in

$$\mathbf{A}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V}\mathbf{M}) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0} : \mathbf{0})$$

can be expressed, e.g., as

$$\mathbf{A}_0 = \mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{F}} + \mathbf{E}\mathbf{Q}_{\mathbf{W}} = \mathbf{M}_2\mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_2 + \mathbf{E}\mathbf{Q}_{\mathbf{W}}\,,$$

where $\mathbf{E} \in \mathbb{R}^{n\times n}$ is free to vary. By the *consistency* of the model $\mathscr{F}$ it is meant that $\mathbf{y}$ lies in $\mathscr{C}(\mathbf{W})$ with probability 1. Thus under the consistent model $\mathscr{F}$ the vector $\mathbf{A}_0\mathbf{y}$ itself is unique once $\mathbf{y}$ has been observed. The consistency in $\mathscr{M}$ means that $\mathbf{y}$ belongs to $\mathscr{C}(\mathbf{W}_m)$. Notice that

$$\mathscr{C}(\mathbf{W}_m) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2\mathbf{D} : \mathbf{V}) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V}) = \mathscr{C}(\mathbf{W}),$$

with equality holding if and only if $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{W}_m) = \mathscr{C}(\mathbf{X}_1 : \mathbf{\Sigma}\mathbf{M}_1)$.

In the consistent linear model $\mathscr{F}$, the estimators $\mathbf{Ay}$ and $\mathbf{By}$ are said to be equal (with probability 1) if

$$\mathbf{Ay} = \mathbf{By} \text{ for all } \mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{X} : \mathbf{VM}) = \mathscr{C}(\mathbf{X}) \oplus \mathscr{C}(\mathbf{VM}), \quad (7)$$

where $\oplus$ refers to the direct sum. In (7) we are dealing with the "statistical" equality of the estimators $\mathbf{Ay}$ and $\mathbf{By}$. In (7) $\mathbf{y}$ refers to a vector in $\mathbb{R}^n$, while in the notation $\mathrm{cov}(\mathbf{Ay})$ we understand $\mathbf{y}$ as a *random* vector. We may consider, for example, the equation

$$\mathbf{G}_{\boldsymbol{\theta}_1 \mid \mathscr{F}}\mathbf{y} = \mathbf{G}_{\boldsymbol{\theta}_1 \mid \mathscr{M}}\mathbf{y} \quad (8)$$

but now we immediately observe some problems in defining the space where $\mathbf{y}$ is varying in (8). We can write, for example,

$$\mathbf{G}_{\boldsymbol{\theta}_1 \mid \mathscr{F}}\mathbf{y} = \mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{F}), \quad \mathbf{G}_{\boldsymbol{\theta}_1 \mid \mathscr{M}}\mathbf{y} = \mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{M}),$$

which are short notations for phrases like "$\mathbf{G}_{\boldsymbol{\theta}_1 \mid \mathscr{M}}\mathbf{y}$ is the BLUE for $\boldsymbol{\theta}_1$ under $\mathscr{F}$" etc. However, writing the equalities like

$$\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{F}) = \mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}),$$

may cause problems when the representations are not unique.

Isotalo et al. [11] found conditions under which an arbitrary representation of the BLUE of $\boldsymbol{\theta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$ under the fixed model $\mathscr{F}$ remains the BLUE for $\boldsymbol{\theta}_1$ under the mixed model $\mathscr{M}$. This kind of property can be denoted shortly as

$$\{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{F})\} \subseteq \{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{M})\},$$

or, equivalently as $\{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{F}}\} \subseteq \{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{M}}\}$, where the sets $\{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{F}}\}$ and $\{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{M}}\}$ are defined as in Lemma 1.1:

$$\mathbf{A} \in \{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{F}}\} \iff \mathbf{A}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0} : \mathbf{0}),$$
$$\mathbf{B} \in \{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{M}}\} \iff \mathbf{B}(\mathbf{X}_1 : \mathbf{\Sigma}\mathbf{M}_1) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0}).$$

In this paper we generalize the results of Isotalo et al. [11] by considering the following relations:

$$\mathrm{BLUE}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{F}) \ \text{ vs } \ \mathrm{BLUP}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M}),$$
$$\mathrm{BLUE}(\mathbf{M}_2\mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F}) \ \text{ vs } \ \mathrm{BLUP}(\mathbf{M}_2\mathbf{X}_2\mathbf{u} \mid \mathscr{M}),$$
$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta} \mid \mathscr{F}) \ \text{ vs } \ \mathrm{BLUP}(\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M}).$$

The case of two linear fixed models $\mathscr{B}_i = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_i\}$, $i = 1, 2$, with different covariance matrices is extensively studied by Mitra and Moore [18]. Haslett et al. [7] provide a review of conditions under which BLUEs/BLUPs in one linear mixed model are also BLUE/BLUPs in another (with possibly different design matrices and covariance structures).

We end this section with a useful lemma.

Lemma 1.2. *Using the earlier notation, the following statements hold:*

(a) $\mathbf{M} = \mathbf{I}_n - \mathbf{P}_{(\mathbf{X}_1:\mathbf{X}_2)} = \mathbf{I}_n - (\mathbf{P}_{\mathbf{X}_2} + \mathbf{P}_{\mathbf{M}_2\mathbf{X}_1}) = \mathbf{M}_2\mathbf{Q}_{\mathbf{M}_2\mathbf{X}_1} = \mathbf{Q}_{\mathbf{M}_2\mathbf{X}_1}\mathbf{M}_2$,
(b) $\mathrm{r}(\mathbf{M}_2\mathbf{X}_1) = \mathrm{r}(\mathbf{X}_1) - \dim \mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2)$,
(c) $\mathrm{r}(\mathbf{AB}) = \mathrm{r}(\mathbf{A}) - \dim \mathscr{C}(\mathbf{A}') \cap \mathscr{C}(\mathbf{B})^{\perp}$,
(d) $\mathscr{C}(\mathbf{W}^{+}\mathbf{X})^{\perp} = \mathscr{C}(\mathbf{WM} : \mathbf{Q}_{\mathbf{W}}) = \mathscr{C}(\mathbf{VM} : \mathbf{Q}_{\mathbf{W}})$,
(e) $\mathscr{C}(\mathbf{X}_2 : \mathbf{\Sigma M}) = \mathscr{C}[\mathbf{M}_2(\mathbf{M}_2\mathbf{W}\mathbf{M}_2)^{+}\mathbf{M}_2\mathbf{X}_1 : \mathbf{Q}_{\mathbf{W}}]^{\perp}$,
(f) $\mathscr{C}[\mathbf{A}(\mathbf{A}'\mathbf{B}^{\perp})^{\perp}] = \mathscr{C}(\mathbf{A}) \cap \mathscr{C}(\mathbf{B})$.

For part (b) and (c), see, e.g., [17, Cor. 6.2]. For (d), see, e.g., [16, Lemma 4] and [20, Sec. 2]. For (e), see [11, Lemma, p. 72], and for (f), see [21, Compl. 7, p. 118].

2. Equality between the BLUEs

Isotalo et al. [11, Sec. 2] proved the following result:

Theorem 2.1. *The following statements hold.*

(a) *An arbitrary* BLUE *for* $\boldsymbol{\theta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$ *under* $\mathscr{F}$ *provides also the* BLUE *for* $\boldsymbol{\theta}_1$ *under the mixed model* $\mathscr{M}$*, i.e.,*
$$\{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{F})\} \subseteq \{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{M})\}, \tag{9}$$
i.e., $\{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{F}}\} \subseteq \{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{M}}\}$*, holds if and only if*
$$\mathscr{C}(\mathbf{\Sigma M}_1) \subseteq \mathscr{C}(\mathbf{X}_2 : \mathbf{VM}). \tag{10}$$

(b) *The reverse relation* $\{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{M})\} \subseteq \{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{F})\}$*, i.e.,* $\{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{M}}\} \subseteq \{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{F}}\}$*, holds if and only if*
$$\mathscr{C}(\mathbf{X}_2 : \mathbf{VM}) \subseteq \mathscr{C}(\mathbf{R} : \mathbf{\Sigma M}_1), \ \text{i.e.,}\ \mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{R} : \mathbf{\Sigma M}_1), \tag{11}$$
where the matrix $\mathbf{R}$ *has property* $\mathscr{C}(\mathbf{R}) = \mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2)$.

Actually, the matrix $\mathbf{R}$ in (11) was erroneously missing in [11]. Notice that the equivalence of the two inclusions in (11) follows from $\mathscr{C}(\mathbf{VM}) = \mathscr{C}(\mathbf{\Sigma M}) \subseteq \mathscr{C}(\mathbf{\Sigma M}_1)$, which is based on
$$\mathscr{C}(\mathbf{M}) = \mathscr{C}(\mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2}) \subseteq \mathscr{C}(\mathbf{M}_1).$$
The inclusion (10) is obviously equivalent to $\mathscr{C}(\mathbf{R} : \mathbf{\Sigma M}_1) \subseteq \mathscr{C}(\mathbf{X}_2 : \mathbf{VM})$ and thereby $\{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{M}}\} = \{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{F}}\}$ holds if and only if
$$\mathscr{C}(\mathbf{R} : \mathbf{\Sigma M}_1) = \mathscr{C}(\mathbf{X}_2 : \mathbf{VM}).$$
Moreover, it is interesting to observe that (10) is equivalent to
$$\mathscr{C}(\mathbf{VM}_1) \subseteq \mathscr{C}(\mathbf{X}_2 : \mathbf{VM}).$$

Namely, writing $\mathbf{P}_{(\mathbf{X}_2:\mathbf{VM})} = \mathbf{P}_{\mathbf{X}_2} + \mathbf{P}_{\mathbf{M}_2\mathbf{VM}}$, it is easy to confirm that

$$\mathbf{P}_{(\mathbf{X}_2:\mathbf{VM})}\mathbf{VM}_1 = \mathbf{VM}_1 \iff \mathbf{P}_{(\mathbf{X}_2:\mathbf{VM})}\mathbf{\Sigma M}_1 = \mathbf{\Sigma M}_1 \,.$$

If $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is estimable under $\mathscr{F}$, i.e., $\mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2) = \{\mathbf{0}\}$, we immdiately observe that (11) simplifies into $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{\Sigma M}_1)$. Moreover, we can obtain the following corollary.

Corollary 2.1. *Let $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is estimable under $\mathscr{F}$. Then the following statements are equivalent:*

(a) $\{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M})\} \subseteq \{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{F})\}$,
(b) $\{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M})\} = \{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{F})\}$,
(c) $\mathscr{C}(\mathbf{X}_2 : \mathbf{VM}) \subseteq \mathscr{C}(\mathbf{\Sigma M}_1)$,
(d) $\mathscr{C}(\mathbf{X}_2 : \mathbf{VM}) = \mathscr{C}(\mathbf{\Sigma M}_1)$,
(e) $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{\Sigma M}_1)$.

Proof. The equivalence of (a), (c) and (e) follows from Theorem 2.1. Assuming the disjointness $\mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2) = \{\mathbf{0}\}$, we observe, using (c) of Lemma 1.2, that

$$\begin{aligned} \mathrm{r}(\mathbf{X}_2 : \mathbf{\Sigma M}) &= \mathrm{r}(\mathbf{X}_2) + \mathrm{r}(\mathbf{\Sigma M}) = \mathrm{r}(\mathbf{X}_2) + \mathrm{r}(\mathbf{\Sigma M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2}) \\ &= \mathrm{r}(\mathbf{X}_2) + \mathrm{r}(\mathbf{\Sigma M}_1) - \dim \mathscr{C}(\mathbf{M}_1\mathbf{\Sigma}) \cap \mathscr{C}(\mathbf{M}_1\mathbf{X}_2) \\ &\geq \mathrm{r}(\mathbf{X}_2) + \mathrm{r}(\mathbf{\Sigma M}_1) - \mathrm{r}(\mathbf{M}_1\mathbf{X}_2) = \mathrm{r}(\mathbf{\Sigma M}_1)\,. \end{aligned} \tag{12}$$

Thereby, if (c) holds, then (12) implies that necessarily (d) holds, which further is equivalent to (b). □

Remark 2.1. Isotalo et al. [11, p. 72] considered also the condition under which there exists at least one representation of the BLUE of $\boldsymbol{\theta}_1$ under $\mathscr{F}$ which is also BLUE of $\boldsymbol{\theta}_1$ under $\mathscr{M}$. This means that there exists a matrix $\mathbf{A}$ such that $\mathbf{A} \in \{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{F}}\} \cap \{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{M}}\}$, i.e., $\mathbf{A}$ satisfies the equation

$$\mathbf{A}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{\Sigma M}_1 : \mathbf{\Sigma M}) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0} : \mathbf{0} : \mathbf{0})\,. \tag{13}$$

It is clear that $\mathbf{A\Sigma M}_1 = \mathbf{0}$ implies $\mathbf{A\Sigma M} = \mathbf{0}$ and so (13) is equivalent to

$$\mathbf{A}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{\Sigma M}_1) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0} : \mathbf{0})\,. \tag{14}$$

Now (14) has a solution for $\mathbf{A}$ if and only if

$$\mathscr{N}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{\Sigma M}_1) \subseteq \mathscr{N}(\mathbf{M}_2\mathbf{X}_1 : \mathbf{0} : \mathbf{0})\,,$$

where $\mathscr{N}(\cdot)$ refers to the nullspace. The corresponding conditions for further relations appearing in this article can be introduced (we will omit them). □

It is interesting to consider the "statistical" equality

$$\mathbf{G}_{\boldsymbol{\theta}_1 \mid \mathscr{F}}\mathbf{y} = \mathbf{G}_{\boldsymbol{\theta}_1 \mid \mathscr{M}}\mathbf{y}$$

in deeper details. In particular we can consider two cases:

$$\mathbf{y} \in \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})\,, \quad \mathbf{y} \in \mathscr{C}(\mathbf{W}_m) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2\mathbf{D} : \mathbf{V})\,.$$

Recall that in the fixed model $\mathscr{F}$ the "permissible observation space" for the response variable $\mathbf{y}$ is $\mathscr{C}(\mathbf{W})$ while in the mixed model $\mathscr{M}$ it is $\mathscr{C}(\mathbf{W}_m)$. Now the following corollary is straightforward to confirm.

Corollary 2.2. *Consider the models $\mathscr{F}$ and $\mathscr{M}$.*

(a) *The following statements are equivalent:*
 (i) $\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}}\mathbf{y} = \mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{F}}\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$,
 (ii) $\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0} : \mathbf{0})$,
 (iii) $\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}} \in \{\mathbf{P}_{\boldsymbol{\theta}_1\,|\mathscr{F}}\}$, *i.e.,* $\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}}\mathbf{y} = \mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{F})$.

(b) *The following statements are equivalent:*
 (i) $(\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}} + \mathbf{EQ}_{\mathbf{W}_m})\mathbf{y} = \mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{F}}\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W})$ *and for all* $\mathbf{E}$,
 (ii) $(\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}} + \mathbf{EQ}_{\mathbf{W}_m})(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0} : \mathbf{0})$ *for all* $\mathbf{E}$,
 (iii) $\{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{M})\} \subseteq \{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{F})\}$.

(c) *The following statements are equivalent:*
 (i) $\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{F}}\mathbf{y} = \mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}}\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W}_m) = \mathscr{C}(\mathbf{X}_1 : \mathbf{\Sigma})$,
 (ii) $(\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{F}} + \mathbf{EQ}_{\mathbf{W}})\mathbf{y} = \mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}}\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W}_m)$ *and for all* $\mathbf{E}$,
 (iii) $\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{F}}(\mathbf{X}_1 : \mathbf{\Sigma M}_1) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0})$,
 (iv) $\{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{F})\} \subseteq \{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{M})\}$.

3. Equality of a particular BLUE and BLUP

In this section we consider the relation

$$\mathrm{BLUE}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{F}) \quad \text{versus} \quad \mathrm{BLUP}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M}).$$

Recall, by (3), that under $\mathscr{M}$ we have

$$\begin{aligned}\mathrm{BLUP}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta} + \mathbf{X}_2\mathbf{u}) &= \mathrm{BLUE}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1) + \mathrm{BLUP}(\mathbf{X}_2\mathbf{u}) \\ &= \mathrm{BLUE}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1) + \mathbf{X}_2\,\mathrm{BLUP}(\mathbf{u}).\end{aligned}$$

By Lemma 1.1, $\mathbf{Ly}$ is the BLUP for $\boldsymbol{\eta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u}$ if and only if

$$\mathbf{L}(\mathbf{X}_1 : \mathbf{\Sigma M}_1) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{X}_2\mathbf{DX}_2'\mathbf{M}_1), \tag{15}$$

where $\mathbf{\Sigma} = \mathbf{X}_2\mathbf{DX}_2' + \mathbf{V}$. The general solution to $\mathbf{L}$ in (15) can be expressed as

$$\begin{aligned}\mathbf{L}_0 &= \mathbf{M}_2\mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_m^-\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{W}_m^+ + \mathbf{X}_2\mathbf{DX}_2'\mathbf{M}_1(\mathbf{M}_1\mathbf{\Sigma M}_1)^+\mathbf{M}_1 + \mathbf{EQ}_{\mathbf{W}_m} \\ &= \mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}} + \mathbf{EQ}_{\mathbf{W}_m},\end{aligned}$$

where $\mathbf{E} \in \mathbb{R}^{n\times n}$ is free to vary and $\mathbf{W}_m = \mathbf{X}_1\mathbf{X}_1' + \mathbf{\Sigma}$. Suppose that $\mathbf{L}_0$ provides also the BLUE for $\boldsymbol{\theta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$ under the fixed model $\mathscr{F}$. Then $\mathbf{L}_0$ has to satisfy, for every $\mathbf{E}$, the fundamental BLUE equation

$$\mathbf{L}_0(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = \mathbf{L}_0(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{\Sigma M}) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0} : \mathbf{0}). \tag{16}$$

Trivially the $\mathbf{X}_1$-part of (16) holds. Moreover, we must have

$$(\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}} + \mathbf{EQ}_{\mathbf{W}_m})(\mathbf{X}_2 : \mathbf{\Sigma M}) = (\mathbf{0} : \mathbf{0}) \quad \text{for all } \mathbf{E},$$

which implies that $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{W}_m) = \mathscr{C}(\mathbf{X}_1 : \mathbf{\Sigma M}_1)$, and thereby

$$\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{W}_m)\,, \quad \mathbf{X}_2 = \mathbf{X}_1\mathbf{A} + \mathbf{\Sigma M}_1\mathbf{B} = \mathbf{X}_1\mathbf{A} + \mathbf{W}_m\mathbf{M}_1\mathbf{B} \tag{17}$$

for some $\mathbf{A}$ and $\mathbf{B}$. We further must have

$$(\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}})(\mathbf{X}_2 : \mathbf{\Sigma M}) = (\mathbf{0} : \mathbf{0})\,. \tag{18}$$

Consider first the $\mathbf{\Sigma M}$-part of (18). In view of (15) we have

$$(\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}})\mathbf{\Sigma M}_1 = \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1\,,$$

which further implies

$$(\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}})\mathbf{\Sigma M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2} = \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2} = \mathbf{0}\,, \tag{19}$$

i.e., $(\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}})\mathbf{\Sigma M} = \mathbf{0}$, and thereby $\mathbf{\Sigma M}$-part of (18) holds.

For the $\mathbf{X}_2$-part in (18) we must have

$$\begin{aligned}(\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}})\mathbf{X}_2 = \mathbf{M}_2\mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_m^{-}\mathbf{X}_1)^{-}\mathbf{X}_1'\mathbf{W}_m^{+}\mathbf{X}_2 \\ + \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1(\mathbf{M}_1\mathbf{\Sigma M}_1)^{+}\mathbf{M}_1\mathbf{X}_2 = \mathbf{0}\,,\end{aligned}$$

which clearly holds if and only if

$$\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{M}}\mathbf{X}_2 = \mathbf{M}_2\mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_m^{-}\mathbf{X}_1)^{-}\mathbf{X}_1'\mathbf{W}_m^{+}\mathbf{X}_2 = \mathbf{0}\,, \tag{20a}$$

$$\mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}}\mathbf{X}_2 = \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1(\mathbf{M}_1\mathbf{\Sigma M}_1)^{+}\mathbf{M}_1\mathbf{X}_2 = \mathbf{0}\,. \tag{20b}$$

Substituting $\mathbf{X}_2 = \mathbf{X}_1\mathbf{A} + \mathbf{W}_m\mathbf{M}_1\mathbf{B}$ into (20a) yields $\mathbf{M}_2\mathbf{X}_1\mathbf{A} = \mathbf{0}$, so that $\mathbf{A} = \mathbf{Q}_{\mathbf{X}_1'\mathbf{M}_2}\mathbf{Z}$ for some $\mathbf{Z}$, and thereby, taking (17) into account,

$$\mathbf{X}_2 = \mathbf{X}_1\mathbf{Q}_{\mathbf{X}_1'\mathbf{M}_2}\mathbf{Z} + \mathbf{\Sigma M}_1\mathbf{B}\,. \tag{21}$$

Moreover, by part (f) of Lemma 1.2, we have

$$\mathscr{C}(\mathbf{X}_1\mathbf{Q}_{\mathbf{X}_1'\mathbf{M}_2}) = \mathscr{C}[\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_2^{\perp})^{\perp}] = \mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2)\,.$$

Consider then (20b). Substituting (21) into (20b) yields

$$\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1(\mathbf{M}_1\mathbf{\Sigma M}_1)^{+}\mathbf{M}_1\mathbf{\Sigma M}_1\mathbf{B} = \mathbf{0}\,,$$

i.e., $\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1\mathbf{B} = \mathbf{0}$, so that $\mathscr{C}(\mathbf{B}) \subseteq \mathscr{C}(\mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2')^{\perp}$, and by (21),

$$\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1\mathbf{Q}_{\mathbf{X}_1'\mathbf{M}_2} : \mathbf{\Sigma M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2'})\,.$$

In light of part (f) of Lemma 1.2 we can further write

$$\mathscr{C}(\mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2'}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2\mathbf{D}\mathbf{X}_2')^{\perp} = \mathscr{C}(\mathbf{X}_1 : \mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2')^{\perp}.$$

Thus, noting that $\mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2' = \mathbf{M}_1(\mathbf{\Sigma} - \mathbf{V})$, we have obtained the following theorem.

Theorem 3.1. *An arbitrary* BLUP *for* $\boldsymbol{\eta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta} + \mathbf{X}_2\mathbf{u}$ *under* $\mathscr{M}$ *provides also the* BLUE *for* $\boldsymbol{\theta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$ *under the fixed model* $\mathscr{F}$*, i.e.,*

$$\{\mathrm{BLUP}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M})\} \subseteq \{\mathrm{BLUE}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{F})\}\,, \tag{22}$$

i.e., $\{\mathbf{P}_{\boldsymbol{\eta}_1\,|\mathscr{M}}\} \subseteq \{\mathbf{P}_{\boldsymbol{\theta}_1\,|\mathscr{F}}\}$ *, if and only if*

$$\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{R} : \boldsymbol{\Sigma}\mathbf{M}_1\mathbf{S})\,, \tag{23}$$

where the matrices $\mathbf{R}$ *and* $\mathbf{S}$ *have properties* $\mathscr{C}(\mathbf{R}) = \mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2)$ *and*

$$\mathscr{C}(\mathbf{S}) = \mathscr{C}[\mathbf{X}_1 : \mathbf{M}_1(\boldsymbol{\Sigma} - \mathbf{V})]^{\perp} = \mathscr{C}(\mathbf{X}_1 : \mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2')^{\perp}. \tag{24}$$

The reverse inclusion to (22) is considered in Theorem 3.2.

Theorem 3.2. *An arbitrary* BLUE *for* $\boldsymbol{\theta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$ *under* $\mathscr{F}$ *provides also the* BLUP *for* $\boldsymbol{\eta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u}$ *under the mixed model* $\mathscr{M}$*, i.e.,*

$$\{\mathrm{BLUE}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{F})\} \subseteq \{\mathrm{BLUP}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M})\}\,,$$

i.e., $\{\mathbf{P}_{\boldsymbol{\theta}_1\,|\mathscr{F}}\} \subseteq \{\mathbf{P}_{\boldsymbol{\eta}_1,|\mathscr{M}}\}$*, if and only if the following two conditions hold:*

(a) $\mathscr{C}(\boldsymbol{\Sigma}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{X}_2 : \boldsymbol{\Sigma}\mathbf{M})$*, i.e.,* $\{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{F})\} \subseteq \{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{M})\}$,
(b) $\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{V}\mathbf{M}_1$ *, i.e.,* $\mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2' = \mathbf{0}$.

Proof. Take an arbitrary member in the class $\{\mathbf{P}_{\boldsymbol{\theta}_1\,|\mathscr{F}}\}$,

$$\mathbf{B}_0 = \mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{F}} + \mathbf{E}\mathbf{Q}_{\mathbf{W}} = \mathbf{M}_2\mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^{-}\mathbf{X}_1'\dot{\mathbf{M}}_2 + \mathbf{E}\mathbf{Q}_{\mathbf{W}}\,,$$

and $\mathbf{E}$ is free to vary and $\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$. Then $\mathbf{B}_0$ provides the BLUP for $\boldsymbol{\eta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u}$ under the mixed model $\mathscr{M}$ if and only if

$$(\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{F}} + \mathbf{E}\mathbf{Q}_{\mathbf{W}})(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1)$$

holds for every $\mathbf{E}$. The $\mathbf{X}_1$-part is clear. The $\boldsymbol{\Sigma}\mathbf{M}_1$-part is

$$(\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{F}} + \mathbf{E}\mathbf{Q}_{\mathbf{W}})\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1\,,$$

i.e.,

$$\mathbf{G}_{\boldsymbol{\theta}_1\,|\mathscr{F}}\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{M}_2\mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^{-}\mathbf{X}_1'\dot{\mathbf{M}}_2\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1\,. \tag{25}$$

It is clear that (25) holds if and only if

$$\mathbf{M}_2\mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^{-}\mathbf{X}_1'\dot{\mathbf{M}}_2\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{0}\,, \tag{26a}$$

$$\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1 = \mathbf{0}\,, \tag{26b}$$

where (26a) is equivalent to

$$\mathbf{X}_1'\dot{\mathbf{M}}_2\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{0}\,, \quad \text{i.e.,} \quad \mathscr{C}(\boldsymbol{\Sigma}\mathbf{M}_1) \subseteq \mathscr{C}(\dot{\mathbf{M}}_2\mathbf{X}_1)^{\perp}. \tag{27}$$

In view of $\mathscr{C}(\boldsymbol{\Sigma}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{W})$, (27) can be written equivalently as

$$\mathscr{C}(\boldsymbol{\Sigma}\mathbf{M}_1) \subseteq \mathscr{C}(\dot{\mathbf{M}}_2\mathbf{X}_1)^{\perp} \cap \mathscr{C}(\mathbf{W})\,. \tag{28}$$

On the other hand, in light of part (e) of Lemma 1.2 we know that

$$\mathscr{C}(\dot{\mathbf{M}}_2\mathbf{X}_1 : \mathbf{Q}_{\mathbf{W}})^{\perp} = \mathscr{C}(\dot{\mathbf{M}}_2\mathbf{X}_1)^{\perp} \cap \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X}_2 : \mathbf{V}\mathbf{M}). \tag{29}$$

Combining (28) and (29) gives

$$\mathscr{C}(\mathbf{\Sigma M}_1) \subseteq \mathscr{C}(\mathbf{X}_2 : \mathbf{VM}) = \mathscr{C}(\mathbf{X}_2 : \mathbf{\Sigma M}) .$$

Moreover, (26b) is equivalent to $\mathbf{VM}_1 = \mathbf{\Sigma M}_1$, which completes the proof. □

What about the equality of the sets $\{\mathbf{P}_{\boldsymbol{\theta}_1 \,|\mathscr{F}}\}$ and $\{\mathbf{P}_{\boldsymbol{\eta}_1 \,|\mathscr{M}}\}$? Requesting that (b) of Theorem 3.2 holds, i.e., $\mathbf{VM}_1 = \mathbf{\Sigma M}_1$, the condition (23) of Theorem 3.1 becomes $\mathscr{C}(\mathbf{X}_2) \subset \mathscr{C}(\mathbf{R} : \mathbf{\Sigma M}_1)$, i.e.,

$$\mathscr{C}(\mathbf{X}_2 : \mathbf{\Sigma M}) \subseteq \mathscr{C}(\mathbf{R} : \mathbf{\Sigma M}_1) , \text{ where } \mathscr{C}(\mathbf{R}) = \mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2) . \tag{30}$$

On the other hand, condition (a) of Theorem 3.2 is equivalent to

$$\mathscr{C}(\mathbf{R} : \mathbf{\Sigma M}_1) \subseteq \mathscr{C}(\mathbf{X}_2 : \mathbf{\Sigma M}) . \tag{31}$$

Now (30) and (31) imply the following result.

Corollary 3.1. *The following statements are equivalent:*

(a) $\{\mathrm{BLUP}(\boldsymbol{\theta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M})\} = \{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{F})\}$,

(b) $\mathscr{C}(\mathbf{X}_2 : \mathbf{\Sigma M}) = \mathscr{C}(\mathbf{R} : \mathbf{\Sigma M}_1)$ *and* $\mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2' = \mathbf{0}$, *i.e.,* $\mathbf{\Sigma M}_1 = \mathbf{VM}_1$, *where* $\mathscr{C}(\mathbf{R}) = \mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2)$.

Notice that if $\boldsymbol{\mu}_1$ is estimable in $\mathscr{F}$ then $\mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2' = \mathbf{0}$ is equivalent to $\mathbf{X}_2\mathbf{D}\mathbf{X}_2' = \mathbf{0}$. Moreover, from Corollary 3.1 we can conclude the following.

Corollary 3.2. *Suppose that* $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ *is estimable under* $\mathscr{F}$. *Then the following three statements are equivalent:*

(a) $\{\mathrm{BLUP}(\boldsymbol{\mu}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M})\} = \{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{F})\}$,

(b) $\{\mathrm{BLUE}(\boldsymbol{\mu}_1 \,|\, \mathscr{M})\} = \{\mathrm{BLUE}(\boldsymbol{\mu}_1 \,|\, \mathscr{F})\}$ *and* $\mathbf{X}_2\mathbf{D}\mathbf{X}_2' = \mathbf{0}$, *i.e.,* $\mathbf{\Sigma} = \mathbf{V}$,

(c) $\mathscr{C}(\mathbf{\Sigma M}_1) = \mathscr{C}(\mathbf{X}_2 : \mathbf{VM})$ *and* $\mathbf{X}_2\mathbf{D}\mathbf{X}_2' = \mathbf{0}$.

Remark 3.1. The property $\mathrm{cov}(\mathbf{X}_2\mathbf{u}) = \mathbf{X}_2\mathbf{D}\mathbf{X}_2' = \mathbf{0}$ together with $\mathrm{E}(\mathbf{u}) = \mathbf{0}$ means that $\mathbf{X}_2\mathbf{u} = \mathbf{0}$ with probability 1. Moreover, if $\mathbf{X}_2\mathbf{D}\mathbf{X}_2' = \mathbf{0}$, then the mixed model $\mathscr{M}$ becomes the *small* fixed model $\mathscr{F}_1 = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{V}\}$ and then any of the conditions in Corollary 3.2 implies the equality

$$\{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{F}_1)\} = \{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{F})\} ,$$

which further is equivalent to $\mathscr{C}(\mathbf{VM}_1) = \mathscr{C}(\mathbf{X}_2 : \mathbf{VM})$. □

4. A further equality of particular BLUE and BLUP

In this section we consider

$$\mathrm{BLUE}(\mathbf{M}_1\mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F}) \quad \text{versus} \quad \mathrm{BLUP}(\mathbf{M}_1\mathbf{X}_2\mathbf{u} \mid \mathscr{M}) .$$

Theorem 4.1. *An arbitrary* BLUP *for* $\mathbf{M}_1\mathbf{X}_2\mathbf{u}$ *under* $\mathscr{M}$ *provides also the* BLUE *for* $\boldsymbol{\theta}_2 = \mathbf{M}_1\mathbf{X}_2\boldsymbol{\beta}_2$ *under the fixed model* $\mathscr{F}$, *i.e.,*

$$\{\mathrm{BLUP}(\mathbf{M}_1\mathbf{X}_2\mathbf{u} \mid \mathscr{M})\} \subseteq \{\mathrm{BLUE}(\mathbf{M}_1\mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F})\} , \tag{32}$$

i.e., $\{\mathbf{P}_{\mathbf{M}_1\mathbf{X}_2\mathbf{u}\,|\mathscr{M}}\} \subseteq \{\mathbf{P}_{\boldsymbol{\theta}_2\,|\mathscr{F}}\}$, *if and only if*

$$\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{\Sigma M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{V}}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{\Sigma M}_1\mathbf{Q}_{(\mathbf{X}_1:\mathbf{V})})\,.$$

Proof. We recall that $\mathbf{Cy}$ is the BLUP for $\mathbf{M}_1\mathbf{X}_2\mathbf{u}$ under $\mathscr{M}$ if and only if

$$\mathbf{C}(\mathbf{X}_1 : \mathbf{\Sigma M}_1) = (\mathbf{0} : \mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1)\,. \tag{33}$$

The general solution to $\mathbf{C}$ in (33) is

$$\begin{aligned}\mathbf{C}_0 &= \mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1(\mathbf{M}_1\mathbf{\Sigma M}_1)^+\mathbf{M}_1 + \mathbf{E}\mathbf{Q}_{\mathbf{W}_m}\\ &= \mathbf{G}_{\mathbf{M}_1\mathbf{X}_2\mathbf{u}\,|\mathscr{M}} + \mathbf{E}\mathbf{Q}_{\mathbf{W}_m}\,,\end{aligned}$$

where $\mathbf{E}$ is free to vary and $\mathbf{W}_m = \mathbf{X}_1\mathbf{X}_1' + \mathbf{\Sigma}$. Suppose that $\mathbf{C}_0$ provides also the BLUE for $\boldsymbol{\theta}_2 = \mathbf{M}_1\mathbf{X}_2\boldsymbol{\beta}_2$ under the fixed model $\mathscr{F}$. Then $\mathbf{C}_0$ has to satisfy, for every $\mathbf{E}$, the fundamental BLUE equation

$$(\mathbf{G}_{\mathbf{M}_1\mathbf{X}_2\mathbf{u}\,|\mathscr{M}} + \mathbf{E}\mathbf{Q}_{\mathbf{W}_m})(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{0} : \mathbf{M}_1\mathbf{X}_2 : \mathbf{0})\,. \tag{34}$$

By (33) the $\mathbf{X}_1$-part of (34) holds. Moreover, we must have

$$(\mathbf{G}_{\mathbf{M}_1\mathbf{X}_2\mathbf{u}\,|\mathscr{M}} + \mathbf{E}\mathbf{Q}_{\mathbf{W}_m})(\mathbf{X}_2 : \mathbf{VM}) = (\mathbf{M}_1\mathbf{X}_2 : \mathbf{0}) \quad \text{for all } \mathbf{E}\,,$$

from which it follows that $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{W}_m) = \mathscr{C}(\mathbf{X}_1 : \mathbf{\Sigma})$, and hence

$$\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{W}_m) \;\text{ and }\; \mathbf{X}_2 = \mathbf{X}_1\mathbf{A} + \mathbf{\Sigma M}_1\mathbf{B} \tag{35}$$

for some $\mathbf{A}$ and $\mathbf{B}$. We further must have

$$\mathbf{G}_{\mathbf{M}_1\mathbf{X}_2\mathbf{u}\,|\mathscr{M}}(\mathbf{X}_2 : \mathbf{VM}) = (\mathbf{M}_1\mathbf{X}_2 : \mathbf{0})\,. \tag{36}$$

Using $\mathbf{VM} = \mathbf{\Sigma M}$, (36) can be written as

$$\mathbf{G}_{\mathbf{M}_1\mathbf{X}_2\mathbf{u}\,|\mathscr{M}}\mathbf{X}_2 = \mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1(\mathbf{M}_1\mathbf{\Sigma M}_1)^+\mathbf{M}_1\mathbf{X}_2 = \mathbf{M}_1\mathbf{X}_2\,, \tag{37a}$$

$$\mathbf{G}_{\mathbf{M}_1\mathbf{X}_2\mathbf{u}\,|\mathscr{M}}\mathbf{\Sigma M} = \mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1(\mathbf{M}_1\mathbf{\Sigma M}_1)^+\mathbf{M}_1\mathbf{\Sigma M} = \mathbf{0}\,. \tag{37b}$$

Now (37b) can be expressed as

$$\mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1(\mathbf{M}_1\mathbf{\Sigma M}_1)^+\mathbf{M}_1\mathbf{\Sigma M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2} = \mathbf{0}\,, \tag{38}$$

which obviously holds.

Consider then (37a):

$$\mathbf{M}_1(\mathbf{\Sigma} - \mathbf{V})\mathbf{M}_1(\mathbf{M}_1\mathbf{\Sigma M}_1)^+\mathbf{M}_1\mathbf{X}_2 = \mathbf{M}_1\mathbf{X}_2\,,$$

from which, in view of $\mathscr{C}(\mathbf{M}_1\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{M}_1\mathbf{\Sigma})$, it follows that

$$\mathbf{M}_1\mathbf{X}_2 - \mathbf{M}_1\mathbf{V}\mathbf{M}_1(\mathbf{M}_1\mathbf{\Sigma M}_1)^+\mathbf{M}_1\mathbf{X}_2 = \mathbf{M}_1\mathbf{X}_2\,, \tag{39}$$

i.e.,

$$\mathbf{V}\mathbf{M}_1(\mathbf{M}_1\mathbf{\Sigma M}_1)^+\mathbf{M}_1\mathbf{X}_2 = \mathbf{0}\,.$$

Substituting $\mathbf{X}_2 = \mathbf{X}_1\mathbf{A} + \mathbf{\Sigma M}_1\mathbf{B}$ into (35) yields $\mathbf{VM}_1\mathbf{B} = \mathbf{0}$, so that $\mathscr{C}(\mathbf{B}) \subseteq \mathscr{C}(\mathbf{M}_1\mathbf{V})^\perp$ and thereby

$$\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{\Sigma M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{V}}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{\Sigma M}_1\mathbf{Q}_{(\mathbf{X}_1:\mathbf{V})})\,,$$

where by part (f) of Lemma 1.2, $\mathscr{C}(\mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{V}}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{V})^{\perp}$. □

Let us consider the reverse inclusion to (32).

Theorem 4.2. *An arbitrary* BLUE *for* $\boldsymbol{\theta}_2 = \mathbf{M}_1\mathbf{X}_2\boldsymbol{\beta}_2$ *under* $\mathscr{F}$ *provides also the* BLUP *for* $\mathbf{M}_1\mathbf{X}_2\mathbf{u}$ *under the mixed model* $\mathscr{M}$, *i.e.,*

$$\{\mathrm{BLUE}(\mathbf{M}_1\mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F})\} \subseteq \{\mathrm{BLUP}(\mathbf{M}_1\mathbf{X}_2\mathbf{u} \mid \mathscr{M})\}\,,$$

i.e., $\{\mathbf{P}_{\boldsymbol{\theta}_2\,|\mathscr{F}}\} \subseteq \{\mathbf{P}_{\mathbf{M}_1\mathbf{X}_2\mathbf{u}\,|\mathscr{M}}\}$, *if and only if*

$$\mathscr{C}(\mathbf{V}\mathbf{M}_1) = \mathscr{C}(\mathbf{V}\mathbf{M})\,.$$

Proof. Take an arbitrary member in the class $\{\mathbf{P}_{\boldsymbol{\theta}_2\,|\mathscr{F}}\}$,

$$\mathbf{N}_0 = \mathbf{G}_{\boldsymbol{\theta}_2\,|\mathscr{F}} + \mathbf{E}\mathbf{Q}_{\mathbf{W}} = \mathbf{M}_1\mathbf{X}_2(\mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{X}_2)^{-}\mathbf{X}_2'\dot{\mathbf{M}}_1 + \mathbf{E}\mathbf{Q}_{\mathbf{W}}\,,$$

where $\mathbf{E}$ is free to vary and $\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$. Then $\mathbf{N}_0$ provides the BLUP for $\mathbf{M}_1\mathbf{X}_2\mathbf{u}$ under the mixed model $\mathscr{M}$ if and only if

$$(\mathbf{G}_{\boldsymbol{\theta}_2\,|\mathscr{F}} + \mathbf{E}\mathbf{Q}_{\mathbf{W}})(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1) = (\mathbf{0} : \mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1)\,,$$

where the $\mathbf{X}_1$-part obviously holds and so we must have

$$\mathbf{G}_{\boldsymbol{\theta}_2\,|\mathscr{F}}\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{M}_1\mathbf{X}_2(\mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{X}_2)^{-}\mathbf{X}_2'\dot{\mathbf{M}}_1\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{M}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1\,. \tag{40}$$

Premultiplying (40) by $\mathbf{X}_2'\dot{\mathbf{M}}_1$ yields an equivalent equation

$$\mathbf{X}_2'\dot{\mathbf{M}}_1\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1 = \mathbf{X}_2'\dot{\mathbf{M}}_1(\boldsymbol{\Sigma} - \mathbf{V})\mathbf{M}_1\,,$$

i.e., $\mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{V}\mathbf{M}_1 = \mathbf{0}$, which means that

$$\mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\dot{\mathbf{M}}_1\mathbf{X}_2)^{\perp}. \tag{41}$$

We know that $\mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{W})$ and hence we can write (41) as

$$\mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\dot{\mathbf{M}}_1\mathbf{X}_2)^{\perp} \cap \mathscr{C}(\mathbf{W})\,. \tag{42}$$

In view of part (e) of Lemma 1.2 we have the following:

$$\mathscr{C}(\mathbf{X}_1 : \mathbf{V}\mathbf{M}) = \mathscr{C}(\dot{\mathbf{M}}_1\mathbf{X}_2 : \mathbf{Q}_{\mathbf{W}})^{\perp} = \mathscr{C}(\dot{\mathbf{M}}_1\mathbf{X}_2)^{\perp} \cap \mathscr{C}(\mathbf{W})\,. \tag{43}$$

Combining (42) and (43) yields

$$\mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V}\mathbf{M})\,,$$

which is obviously equivalent to $\mathscr{C}(\mathbf{V}\mathbf{M}_1) = \mathscr{C}(\mathbf{V}\mathbf{M})$. □

From Theorems 4.1 and 4.2 we get the following result.

Corollary 4.1. *The following statements are equivalent:*

(a) $\{\mathrm{BLUP}(\mathbf{M}_1\mathbf{X}_2\mathbf{u} \mid \mathscr{M})\} = \{\mathrm{BLUE}(\mathbf{M}_1\mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F})\}\,,$

(b) $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{V}})$ *and* $\mathscr{C}(\mathbf{V}\mathbf{M}_1) = \mathscr{C}(\mathbf{V}\mathbf{M})\,.$

5. One further equality between BLUE and BLUP

In this section we consider

$$\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F}) \quad \text{versus} \quad \mathrm{BLUP}(\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M}).$$

Theorem 5.1. *An arbitrary* BLUP *for* $\boldsymbol{\eta} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u}$ *under* $\mathscr{M}$ *provides also the* BLUE *for* $\boldsymbol{\mu} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2$ *under the fixed model* $\mathscr{F}$*, i.e.,*

$$\{\mathrm{BLUP}(\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M})\} \subseteq \{\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F})\}, \tag{44}$$

i.e., $\{\mathbf{P}_{\boldsymbol{\eta}\,|\mathscr{M}}\} \subseteq \{\mathbf{P}_{\boldsymbol{\mu}\,|\mathscr{F}}\}$, *if and only if*

$$\{\mathrm{BLUP}(\mathbf{M}_1\mathbf{X}_2\mathbf{u} \mid \mathscr{M})\} \subseteq \{\mathrm{BLUE}(\mathbf{M}_1\mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F})\}. \tag{45}$$

Proof. The general solution to

$$\mathbf{T}(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1) = (\mathbf{X}_1 : \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1)$$

can be expressed as

$$\begin{aligned}\mathbf{T}_0 &= \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_m^{-}\mathbf{X}_1)^{-}\mathbf{X}_1'\mathbf{W}_m^{+} + \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1(\mathbf{M}_1\boldsymbol{\Sigma}\mathbf{M}_1)^{+}\mathbf{M}_1 + \mathbf{E}\mathbf{Q}_{\mathbf{W}_m}\\ &= \mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}} + \mathbf{E}\mathbf{Q}_{\mathbf{W}_m},\end{aligned}$$

where $\mathbf{E}$ is free to vary and $\mathbf{W}_m = \boldsymbol{\Sigma} + \mathbf{X}_1\mathbf{X}_1'$. Suppose that $\mathbf{T}_0$ provides also the BLUE for $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ under the fixed model $\mathscr{F}$. Then $\mathbf{T}_0$ has to satisfy, for every $\mathbf{E}$, the fundamental BLUE equation

$$\mathbf{T}_0(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V}\mathbf{M}) = (\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{0}). \tag{46}$$

It is obvious that the $\mathbf{X}_1$-part of (46) holds. Moreover, we must have

$$(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}} + \mathbf{E}\mathbf{Q}_{\mathbf{W}_m})(\mathbf{X}_2 : \mathbf{V}\mathbf{M}) = (\mathbf{X}_2 : \mathbf{0}) \quad \text{for all } \mathbf{E},$$

from which it follows that $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{W}_m)$ and that for some $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{X}_2 = \mathbf{X}_1\mathbf{A} + \boldsymbol{\Sigma}\mathbf{M}_1\mathbf{B}. \tag{47}$$

We further must have

$$(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}})(\mathbf{X}_2 : \mathbf{V}\mathbf{M}) = (\mathbf{X}_2 : \mathbf{0}).$$

It is straightforward to show that $(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}})\mathbf{V}\mathbf{M} = \mathbf{0}$, so that we are left with condition

$$\begin{aligned}(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}} + \mathbf{G}_{\mathbf{X}_2\mathbf{u}\,|\mathscr{M}})\mathbf{X}_2 &= \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_m^{-}\mathbf{X}_1)^{-}\mathbf{X}_1'\mathbf{W}_m^{+}\mathbf{X}_2\\ &\quad + \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1(\mathbf{M}_1\boldsymbol{\Sigma}\mathbf{M}_1)^{-}\mathbf{M}_1\mathbf{X}_2 = \mathbf{X}_2.\end{aligned} \tag{48}$$

Substituting $\mathbf{X}_2 = \mathbf{X}_1\mathbf{A} + \boldsymbol{\Sigma}\mathbf{M}_1\mathbf{B} = \mathbf{X}_1\mathbf{A} + \mathbf{W}_m\mathbf{M}_1\mathbf{B}$ into (48) gives

$$\mathbf{X}_1\mathbf{A} + \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1\mathbf{B} = \mathbf{X}_1\mathbf{A} + \boldsymbol{\Sigma}\mathbf{M}_1\mathbf{B},$$

so that we have $\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1\mathbf{B} = \boldsymbol{\Sigma}\mathbf{M}_1\mathbf{B}$, i.e., $\mathbf{V}\mathbf{M}_1\mathbf{B} = \mathbf{0}$ and thereby

$$\mathscr{C}(\mathbf{B}) \subseteq \mathscr{C}(\mathbf{M}_1\mathbf{V})^{\perp}. \tag{49}$$

Combining (47) and (49) gives $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{V}})$, and thus by Theorem 4.1 the proof is completed. □

Consider now the reverse inclusion of (44).

Theorem 5.2. *An arbitrary* BLUE *for* $\boldsymbol{\mu} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2$ *under* $\mathscr{F}$ *provides also the* BLUP *for* $\boldsymbol{\eta} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u}$ *under the mixed model* $\mathscr{M}$*, i.e.,*

$$\{\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F})\} \subseteq \{\mathrm{BLUP}(\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M})\},$$

i.e., $\{\mathbf{P}_{\boldsymbol{\mu}\,|\mathscr{F}}\} \subseteq \{\mathbf{P}_{\boldsymbol{\eta}\,|\mathscr{M}}\}$, *if and only if*

$$\{\mathrm{BLUE}(\mathbf{M}_1\mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F})\} \subseteq \{\mathrm{BLUP}(\mathbf{M}_1\mathbf{X}_2\mathbf{u} \mid \mathscr{M})\}.$$

Proof. Take an arbitrary member in the class $\{\mathbf{P}_{\boldsymbol{\mu}\,|\mathscr{F}}\}$,

$$\mathbf{G}_0 = \mathbf{G} + \mathbf{E}\mathbf{Q}_{\mathbf{W}} = \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+ + \mathbf{E}\mathbf{Q}_{\mathbf{W}},$$

where $\mathbf{E}$ is free to vary and $\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$. Then $\mathbf{G}_0$ provides the BLUP for $\boldsymbol{\eta} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u}$ under the mixed model $\mathscr{M}$ if and only if

$$(\mathbf{G} + \mathbf{E}\mathbf{Q}_{\mathbf{W}})(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1) = (\mathbf{X}_1 : \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1). \tag{50}$$

The $\mathbf{X}_1$-part in (50) is clear. The $\boldsymbol{\Sigma}\mathbf{M}_1$-part gives

$$\mathbf{G}\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1. \tag{51}$$

Premultiplying (51) by $\mathbf{X}'\mathbf{W}^+$ gives an equivalent form

$$\mathbf{X}'\mathbf{W}^+\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{X}'\mathbf{W}^+\mathbf{X}_2\mathbf{D}\mathbf{X}_2'\mathbf{M}_1. \tag{52}$$

Substituting $\mathbf{X}_2\mathbf{D}\mathbf{X}_2' = \boldsymbol{\Sigma} - \mathbf{V}$ into (52) leads to

$$\mathbf{X}'\mathbf{W}^+\boldsymbol{\Sigma}\mathbf{M}_1 = \mathbf{X}'\mathbf{W}^+(\boldsymbol{\Sigma} - \mathbf{V})\mathbf{M}_1,$$

i.e., $\mathbf{X}'\mathbf{W}^+\mathbf{V}\mathbf{M}_1 = \mathbf{0}$, i.e.,

$$\mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{W}^+\mathbf{X})^\perp. \tag{53}$$

Now by part (d) of Lemma 1.2 we know that

$$\mathscr{C}(\mathbf{W}^+\mathbf{X})^\perp = \mathscr{C}(\mathbf{W}\mathbf{M} : \mathbf{Q}_{\mathbf{W}}) = \mathscr{C}(\mathbf{V}\mathbf{M} : \mathbf{Q}_{\mathbf{W}}),$$

and hence (53) becomes

$$\mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{V}\mathbf{M} : \mathbf{Q}_{\mathbf{W}}). \tag{54}$$

Premultiplying (54) by $\mathbf{P}_{\mathbf{W}}$ we obtain $\mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{V}\mathbf{M})$, so that we must have $\mathscr{C}(\mathbf{V}\mathbf{M}_1) = \mathscr{C}(\mathbf{V}\mathbf{M})$, and thus by Theorem 4.2 the proof is completed. □

Combining the theorems of Sections 4 and 5 we get the following interesting result.

Corollary 5.1. *The following statements are equivalent:*

(a) $\{\mathrm{BLUP}(\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M})\} = \{\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F})\}$,
(b) $\{\mathrm{BLUP}(\mathbf{M}_1\mathbf{X}_2\mathbf{u} \mid \mathscr{M})\} = \{\mathrm{BLUE}(\mathbf{M}_1\mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F})\}$,
(c) $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{V}})$ *and* $\mathscr{C}(\mathbf{V}\mathbf{M}_1) = \mathscr{C}(\mathbf{V}\mathbf{M})$.

6. Equality of the covariance matrices

In this section we assume that $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is estimable under $\mathscr{F}$ and we consider the equality of the covariance matrices of the BLUEs of $\boldsymbol{\mu}_1$ under $\mathscr{F}$ and under $\mathscr{M}$, i.e., we are comparing $\operatorname{cov}(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}}\mathbf{y} \mid \mathscr{M})$ and $\operatorname{cov}(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{F}}\mathbf{y} \mid \mathscr{F})$, where

$$\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{F}} = \mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_2 \in \{\mathbf{P}_{\boldsymbol{\mu}_1\,|\mathscr{F}}\}\,,$$
$$\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}} = \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_m^+\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{W}_m^+ \in \{\mathbf{P}_{\boldsymbol{\mu}_1\,|\mathscr{M}}\}\,.$$

It is noteworthy that the covariance matrices of the BLUEs are unique even though the representations of the BLUEs may not be unique.

It can be shown, see, e.g., [13], that

$$\begin{aligned}\operatorname{cov}(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}}\mathbf{y} \mid \mathscr{M}) &= \mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}}\boldsymbol{\Sigma}\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}}' \\ &= \mathbf{X}_1[(\mathbf{X}_1'\mathbf{W}_m^+\mathbf{X}_1)^+ - \mathbf{I}_{p_1}]\mathbf{X}_1' \\ &= \mathbf{X}_1[(\mathbf{X}_1'\mathbf{W}_m^{+1/2}\mathbf{W}_m^{+1/2}\mathbf{X}_1)^+ - \mathbf{I}_{p_1}]\mathbf{X}_1'\,,\end{aligned}$$

where $\mathbf{W}_m^{+1/2}$ refers to the Moore–Penrose inverse of the nonnegative definite square root of $\mathbf{W}_m$, and

$$\begin{aligned}\operatorname{cov}(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{F}}\mathbf{y} \mid \mathscr{F}) &= \mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{F}}\mathbf{V}\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{F}}' \\ &= \mathbf{X}_1[(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^+ - \mathbf{I}_{p_1}]\mathbf{X}_1' \\ &= \mathbf{X}_1\big\{[\mathbf{X}_1'\mathbf{M}_2(\mathbf{M}_2\mathbf{W}_m\mathbf{M}_2)^+\mathbf{M}_2\mathbf{X}_1]^+ - \mathbf{I}_{p_1}\big\}\mathbf{X}_1' \\ &= \mathbf{X}_1[(\mathbf{X}_1'\mathbf{W}_m^{+1/2}\mathbf{P}_{\mathbf{W}_m^{1/2}\mathbf{M}_2}\mathbf{W}_m^{+1/2}\mathbf{X}_1)^+ - \mathbf{I}_{p_1}]\mathbf{X}_1'\,.\end{aligned}$$

The equality $\operatorname{cov}(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}}\mathbf{y} \mid \mathscr{M}) = \operatorname{cov}(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{F}}\mathbf{y} \mid \mathscr{F})$ holds if and only if

$$\begin{aligned}&\mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_m^{+1/2}\mathbf{W}_m^{+1/2}\mathbf{X}_1)^+\mathbf{X}_1' \\ &\qquad = \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_m^{+1/2}\mathbf{P}_{\mathbf{W}_m^{1/2}\mathbf{M}_2}\mathbf{W}_m^{+1/2}\mathbf{X}_1)^+\mathbf{X}_1'\,.\end{aligned} \tag{55}$$

Pre- and postmultiplying (55) by $\mathbf{X}_1^+$ and $(\mathbf{X}_1')^+$, respectively, and using the fact that $\mathbf{P}_{\mathbf{X}_1'} = \mathbf{X}_1^+\mathbf{X}_1$, gives an equivalent form to (55):

$$(\mathbf{X}_1'\mathbf{W}_m^{+1/2}\mathbf{W}_m^{+1/2}\mathbf{X}_1)^+ = (\mathbf{X}_1'\mathbf{W}_m^{+1/2}\mathbf{P}_{\mathbf{W}_m^{1/2}\mathbf{M}_2}\mathbf{W}_m^{+1/2}\mathbf{X}_1)^+,$$

i.e.,

$$\mathbf{X}_1'\mathbf{W}_m^{+1/2}\mathbf{W}_m^{+1/2}\mathbf{X}_1 = \mathbf{X}_1'\mathbf{W}_m^{+1/2}\mathbf{P}_{\mathbf{W}_m^{1/2}\mathbf{M}_2}\mathbf{W}_m^{+1/2}\mathbf{X}_1\,.$$

Now we have the Löwner ordering

$$\mathbf{X}_1'\mathbf{W}_m^{+1/2}(\mathbf{I}_n - \mathbf{P}_{\mathbf{W}_m^{1/2}\mathbf{M}_2})\mathbf{W}_m^{+1/2}\mathbf{X}_1 \geq_{\mathrm{L}} \mathbf{0}\,,$$

where the equality holds if and only if

$$\mathscr{C}(\mathbf{W}_m^{+1/2}\mathbf{X}_1) \subseteq \mathscr{C}(\mathbf{W}_m^{1/2}\mathbf{M}_2)\,. \tag{56}$$

Premultiplying (56) by $\mathbf{W}_m^{1/2}$ gives an equivalent inclusion

$$\mathscr{C}(\mathbf{X}_1) \subseteq \mathscr{C}(\mathbf{W}_m\mathbf{M}_2) = \mathscr{C}(\mathbf{W}_1\mathbf{M}_2), \quad \text{where } \mathbf{W}_1 = \mathbf{X}_1\mathbf{X}_1' + \mathbf{V}. \tag{57}$$

As Isotalo et al. [11, p. 73] point out, the assumption $\mathscr{C}(\mathbf{W}_m) = \mathbb{R}^n$ implies that the BLUE of $\boldsymbol{\mu}_1$ has a unique representation under $\mathscr{F}$ and $\mathscr{M}$. Moreover, following their proof (assuming the estimability of $\boldsymbol{\mu}_1$ under $\mathscr{F}$ it can be shown that the presentations are equal if and only if (57) holds. Thus we can conclude the following result.

Theorem 6.1. *The following statements are equivalent.*

(a) $\operatorname{cov}(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{M}}\mathbf{y} \mid \mathscr{M}) = \operatorname{cov}(\mathbf{G}_{\boldsymbol{\mu}_1\,|\mathscr{F}}\mathbf{y} \mid \mathscr{F})$.
(b) $\mathscr{C}(\mathbf{X}_1) \subseteq \mathscr{C}(\mathbf{W}_m\mathbf{M}_2)$.
(c) *If* $\mathscr{C}(\mathbf{W}_m) = \mathbb{R}^n$*, then the representations of the* BLUE*s of* $\boldsymbol{\mu}_1$ *under the models* $\mathscr{F}$ *and* $\mathscr{M}$ *are equal.*

7. Conclusions

In this article we consider the partitioned fixed linear model $\mathscr{F}\colon \mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}$ and the corresponding mixed model $\mathscr{M}\colon \mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is a random error vector and $\mathbf{u}$ is a random effect vector. Isotalo et al. [11] found conditions under which an arbitrary representation of the best linear unbiased estimator, BLUE, of $\boldsymbol{\theta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$ in the fixed model $\mathscr{F}$ remains BLUE in the mixed model $\mathscr{M}$; here $\mathbf{M}_2$ refers to the orthogonal projector $\mathbf{I}_n - \mathbf{P}_{\mathbf{X}_2}$. The reason to concentrate on estimating $\boldsymbol{\theta}_1 = \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$ is that this approach means that the properties obtained are valid for all parametric functions of the type $\mathbf{K}_1\boldsymbol{\beta}_1$ that are estimable under the partitioned model $\mathscr{F}$ (and thereby under $\mathscr{M}$). In this paper we extend the results concerning further equalities arising from the models $\mathscr{F}$ and $\mathscr{M}$.

The property that BLUE of $\boldsymbol{\theta}_1$ under $\mathscr{F}$ remains BLUE under $\mathscr{M}$ can be denoted shortly as

$$\{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{F})\} \subseteq \{\mathrm{BLUE}(\boldsymbol{\theta}_1 \mid \mathscr{M})\}, \tag{58}$$

or, equivalently as $\{\mathbf{P}_{\boldsymbol{\theta}_1\,|\mathscr{F}}\} \subseteq \{\mathbf{P}_{\boldsymbol{\theta}_1\,|\mathscr{M}}\}$, where, in notation introduced in Section 1,

$$\mathbf{A} \in \{\mathbf{P}_{\boldsymbol{\theta}_1\,|\mathscr{F}}\} \iff \mathbf{A}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0} : \mathbf{0}),$$
$$\mathbf{B} \in \{\mathbf{P}_{\boldsymbol{\theta}_1\,|\mathscr{M}}\} \iff \mathbf{B}(\mathbf{X}_1 : \boldsymbol{\Sigma}\mathbf{M}_1) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0}).$$

In this paper we generalize the results of [11] by considering the following relations:

$$\mathrm{BLUE}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{F}) \ \text{ vs } \ \mathrm{BLUP}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M}),$$
$$\mathrm{BLUE}(\mathbf{M}_2\mathbf{X}_2\boldsymbol{\beta}_2 \mid \mathscr{F}) \ \text{ vs } \ \mathrm{BLUP}(\mathbf{M}_2\mathbf{X}_2\mathbf{u} \mid \mathscr{M}),$$
$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta} \mid \mathscr{F}) \ \text{ vs } \ \mathrm{BLUP}(\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\mathbf{u} \mid \mathscr{M}).$$

As Kala et al. [14, Remark 2] point out, the notation of the type as in (58) is merely symbolic and it is not meant to refer to a set containing only one element which is a single fixed vector resulting from a transformation of an observed vector $\mathbf{y}$, or is a single random vector variable being a specific linear transformation of the random vector $\mathbf{y}$. We are, of course, actually interested in the matrices belonging to classes like $\{\mathbf{P}_{\boldsymbol{\theta}_1 \mid \mathscr{F}}\}$ etc.

There are several related papers concerning the invariance of the BLUEs and/or BLUPs under two models. Mitra and Moore [18] gave an extensive study on the circumstances in which the BLUEs of estimable parametric functions of the fixed parameters in linear model $\{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_1\}$ remain BLUEs under $\{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_2\}$; models differing in covariance matrices. Corresponding considerations related to two mixed models have been made, e.g., by Haslett and Puntanen [5, 6]. In [7], they provide a review of conditions under which BLUEs/BLUPs in one linear mixed model are also BLUE/BLUPs in another. The article [8] explores interesting links between the mixed and fixed linear models. It appears that the concept of the linear model with new future observations is a powerful tool for these considerations. For further references we may mention [15], [22], [25], and [4].

We believe that our results, which are mainly linear-algebraic by nature, can provide some insight into the relations between the fixed and mixed model like $\mathscr{F}$ and $\mathscr{M}$. Some interesting related discussion appears, e.g., in [9, 10].

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