Equalities between the BLUEs and BLUPs under the partitioned linear fixed model and the corresponding mixed model

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Abstract. In this article we consider the partitioned fixed linear model $F: y = \mathbf{X}_1 \beta_1 + \mathbf{X}_2 \beta_2 + \epsilon$ and the corresponding mixed model $M: y = \mathbf{X}_1 \beta_1 + \mathbf{X}_2 u + \epsilon$, where $\epsilon$ is a random error vector and $u$ is a random effect vector. In 2006, Isotalo, Möls, and Puntanen found conditions under which an arbitrary representation of the best linear unbiased estimator (BLUE) of an estimable parametric function of $\beta_1$ in the fixed model $F$ remains BLUE in the mixed model $M$. In this paper we extend the results concerning further equalities arising from models $F$ and $M$.

1. Introduction

Let the partitioned linear fixed effects model be

$F = \{y, \mathbf{X}_1 \beta_1 + \mathbf{X}_2 \beta_2, \mathbf{V}\} = \{y, \mathbf{X} \beta, \mathbf{V}\},$

i.e., the $n$-dimensional observable random vector $y$ is of the form

$y = \mathbf{X}_1 \beta_1 + \mathbf{X}_2 \beta_2 + \epsilon, \quad \text{cov}(\epsilon) = \mathbf{V}, \quad \text{E}(\epsilon) = 0,$

where $\mathbf{X}_1 \in \mathbb{R}^{n \times p_1}$ and $\mathbf{X}_2 \in \mathbb{R}^{n \times p_2}$ are known matrices, $p_1 + p_2 = p$, $\beta_i \in \mathbb{R}^{p_i}$, $i = 1, 2$, are vectors of unknown fixed effects. The covariance matrix $\mathbf{V}$ of the random error vector $\epsilon$ is assumed to be known.

Consider the linear mixed model $M$ which is obtained from $F$ by replacing the fixed vector $\beta_2$ with the random effect vector $u$:

$M: y = \mathbf{X}_1 \beta_1 + \mathbf{X}_2 u + \epsilon, \quad \text{cov}(\epsilon) = \mathbf{V}, \quad \text{E}(\epsilon) = 0,$
where \( \mathbf{X}_1 \) and \( \mathbf{X}_2 \) are as in \( \mathcal{F} \), \( \beta_1 \) is a vector of unknown fixed effects, \( \mathbf{u} \) is an unobservable vector of random effects with \( \text{E}(\mathbf{u}) = \mathbf{0} \), \( \text{cov}(\mathbf{u}) = \mathbf{D} \), \( \text{cov}(\mathbf{\varepsilon}, \mathbf{u}) = \mathbf{0} \); \( \mathbf{V} \) and \( \mathbf{D} \) are assumed to be known. In this situation we have

\[
\text{cov}(\mathbf{\varepsilon}) = \begin{pmatrix} \mathbf{V} & 0 \\ 0 & \mathbf{D} \end{pmatrix}, \quad \text{cov}(\mathbf{y}) = \begin{pmatrix} \Sigma & \mathbf{X}_2 \mathbf{D} \\ \mathbf{D} \mathbf{X}_2' & \mathbf{D} \end{pmatrix}.
\]

Notice that under \( \mathcal{F} \) we have \( \text{cov}(\mathbf{y}) = \mathbf{V} \) but under \( \mathcal{M} \), \( \text{cov}(\mathbf{y}) = \Sigma \).

As for notation, \( r(\mathbf{A}) \), \( \mathbf{A}^- \), \( \mathbf{A}^+ \), \( \mathcal{C}(\mathbf{A}) \), and \( \mathcal{C}(\mathbf{A})^\perp \), denote, respectively, the rank, a generalized inverse, the (unique) Moore–Penrose inverse, the column space, and the orthogonal complement of \( \mathcal{C}(\mathbf{A}) \). By \( \mathbf{A}^\perp \) we denote any matrix satisfying \( \mathcal{C}(\mathbf{A}^\perp) = \mathcal{C}(\mathbf{A})^\perp \). Furthermore, we will write \( \mathbf{P}_\mathbf{A} = \mathbf{A A}^+ = \mathbf{A}^\prime \mathbf{A}^{-1} \mathbf{A}^\prime \) to denote the orthogonal projector onto \( \mathcal{C}(\mathbf{A}) \). The orthogonal projector onto \( \mathcal{C}(\mathbf{A})^\perp \) is denoted as \( \mathbf{Q}_\mathbf{A} = \mathbf{I}_n - \mathbf{P}_\mathbf{A} \), where \( \mathbf{I}_n \) refers to the \( a \times a \) identity matrix and \( a \) is the number of rows of \( \mathbf{A} \). We use the short notations

\[
\mathbf{M} = \mathbf{I}_n - \mathbf{P}_\mathbf{X} \in \{ \mathbf{X}^\perp \}, \quad \mathbf{M}_i = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}_i} \in \{ \mathbf{X}_i^\perp \}, \quad i = 1, 2.
\]

Let \( \mathbf{K} \in \mathbb{R}^{k \times p} \). Then a linear statistic \( \mathbf{A}_i \mathbf{y} \) is said to be a linear unbiased estimator (LUE) for \( \mathbf{K} \beta \) in \( \mathcal{F} \) if its expectation is equal to \( \mathbf{K} \beta \), which happens if and only if \( \mathbf{K}' = \mathbf{X}' \mathbf{A}' \); then \( \mathbf{K} \beta \) is said to be estimable. The LUE \( \mathbf{A}_i \mathbf{y} \) is the best linear unbiased estimator, BLUE, of estimable \( \mathbf{K} \beta \) if \( \mathbf{A}_i \mathbf{y} \) has the smallest covariance matrix in the Löwner sense among all LUEs of \( \mathbf{K} \beta \):

\[
\text{cov}(\mathbf{A}_i \mathbf{y}) \preceq \text{cov}(\mathbf{A}_# \mathbf{y}) \quad \text{for all } \mathbf{A}_# \in \mathbb{R}^{k \times n} : \mathbf{A}_# \mathbf{X} = \mathbf{K}.
\]

Correspondingly, the linear predictor \( \mathbf{B}_i \mathbf{y} \) is said to be unbiased (LUP) for a \( q \)-dimensional random vector \( \mathbf{g} = \mathbf{K}_1 \beta_1 + \mathbf{J} \mathbf{u} \) under \( \mathcal{M} \) if the expected prediction error is zero, i.e., \( \text{E}(\mathbf{g} - \mathbf{B}_i \mathbf{y}) = \mathbf{0} \) for all \( \beta_1 \); here \( \mathbf{K}_1 \in \mathbb{R}^{q \times p_1} \) and \( \mathbf{J} \in \mathbb{R}^{q \times p_2} \). Now a LUP \( \mathbf{B}_i \mathbf{y} \) is the best linear unbiased predictor, BLUP for \( \mathbf{g} \) if it minimizes the covariance matrix of the prediction error among all LUPs, i.e., we have the Löwner ordering

\[
\text{cov}(\mathbf{g} - \mathbf{B}_i \mathbf{y}) \preceq \text{cov}(\mathbf{g} - \mathbf{B}_# \mathbf{y}) \quad \text{for all } \mathbf{B}_# \in \mathbb{R}^{q \times n} : \mathbf{B}_# \mathbf{X}_1 = \mathbf{K}_1.
\]

Suppose we are interested in comparing the BLUE of \( \mathbf{K}_1 \beta_1 \) under \( \mathcal{F} \) and \( \mathcal{M} \). To do this we have to assume that \( \mathbf{K}_1 \beta_1 \) is estimable in both models.

By Groß and Puntanen [21 Lemma 1], \( \mathbf{K}_1 \beta_1 \) is estimable under \( \mathcal{F} \) if and only if \( \mathcal{C}(\mathbf{K}_1') \subset \mathcal{C}(\mathbf{X}_1 \mathbf{M}_2) \), i.e., \( \mathbf{K}_1 = \mathbf{L M}_2 \mathbf{X}_1 \) for some matrix \( \mathbf{L} \). Thus if we wish to consider the estimation of all estimable parametric functions of \( \beta_1 \) under \( \mathcal{F} \), then it is equivalent to consider \( \mathbf{M}_2 \mathbf{X}_1 \beta_1 \). In other words, the reason to concentrate on estimating \( \theta_1 = \mathbf{M}_2 \mathbf{X}_1 \beta_1 \) is that the properties obtained are valid for all parametric functions of the type \( \mathbf{K}_1 \beta_1 \) that are estimable under the partitioned model \( \mathcal{F} \).
Clearly if \( K_1 \beta_1 \) is estimable under \( F \) then it is estimable under \( M \).

It is well known that \( \mu_1 = X_1 \beta_1 \) is estimable in \( F \) if and only if
\[
\mathcal{C}(X_1) \cap \mathcal{C}(X_2) = \{0\}.
\] (1)

This follows from the requirement \( \mathcal{C}(X'_1) \subseteq \mathcal{C}(X'_1 M_2) \), i.e., \( \mathcal{C}(X'_1) = \mathcal{C}(X'_1 M_2) \), which holds if and only if (1) holds.

For Lemma 1.1, characterizing the BLUE, see, e.g., Rao [20, p. 282], and the BLUP, see, e.g., Christensen [1, p. 294], and [12, p. 1015]. For further references, see Haslett et al. [3, 4]. For the general reviews of the BLUP-properties, see, e.g., Tian [23, 24].

**Lemma 1.1.** Consider the models \( F \) and \( M \), and denote \( \Sigma = X_2 DX'_2 + V \). Then the following statements hold.

(a) \( A_1 y \) is the BLUE for estimable \( K \beta \) under \( F \) if and only if
\[
A_1 (X : VM) = (K : 0), \quad \text{i.e.,} \quad A_1 \in \{ P_{K \beta | F} \}.
\]

(b) \( A_2 y \) is the BLUE for estimable \( K_1 \beta_1 \) under \( M \) if and only if
\[
A_2 (X_1 : M_1) = (K_1 : 0), \quad \text{i.e.,} \quad A_2 \in \{ P_{K_1 \beta_1 | M} \}.
\]

(c) \( A_3 y \) is the BLUP for \( J u \) under \( M \) if and only if
\[
A_3 (X_1 : M_1) = (0 : JDJ'M_1), \quad \text{i.e.,} \quad A_3 \in \{ P_{J u | M} \}.
\]

(d) \( A_4 y \) is the BLUP for \( g = K_1 \beta_1 + J u \) under \( M \) if and only if
\[
A_4 (X_1 : M_1) = (K_1 : JDJ'M_1), \quad \text{i.e.,} \quad A_4 \in \{ P_{g | M} \}.
\]

**Remark 1.1.** Notice the difference between the notations like
\[
P_A = AA^+, \quad \{ P_{K_1 \beta_1 | M} \}.
\]

Above \( P_A \) is the (unique) orthogonal projector onto \( \mathcal{C}(A) \), while \( \{ P_{K_1 \beta_1 | M} \} \) is a set of matrices \( A_2 \) satisfying \( A_2 (X_1 : M_1) = (K_1 : 0) \).

If \( A_2 \in \{ P_{K_1 \beta_1 | M} \} \) and \( A_3 \in \{ P_{J u | M} \} \), i.e.,
\[
\begin{pmatrix} A_2 \\ A_3 \end{pmatrix} (X_1 : M_1) = \begin{pmatrix} K_1 & 0 \\ 0 & JDJ'M_1 \end{pmatrix},
\] (2)

then premultiplying (2) by \( (I_q : I_p) \) we immediately see that
\[
A_2 + A_3 \in \{ P_{K_1 \beta_1 + J u | M} \},
\]
i.e., under \( M \) we have
\[
\text{BLUP}(K_1 \beta_1 + J u) = \text{BLUE}(K_1 \beta_1) + \text{BLUP}(J u).
\] (3)

It is well known, see, e.g., Rao [20], that
\[
G = X(X'W^X)^{-1}X'W^+.
\] (4)
where
\[ W = X_1X_1' + X_2X_2' + V = XX' + V \]  \hspace{1cm} (5)
is one solution to the equation \( A(X : VM) = (X : 0) \); recall that \( \mu = X\beta \) is always estimable in \( \mathcal{F} \). The matrix \( G \) is unique for the choice of generalized inverses marked as “−−” but to obtain uniqueness for \( G \) (which somewhat simplifies our considerations) we have to choose the Moore–Penrose inverse \( W^+ \) in the end of the expression \( (4) \).

Below are some solutions to equations appearing in Lemma 1.1 (for references, see, e.g., [19, Ch. 10]):

\[ G_{\mu_1|\mathcal{F}} = X_1(X_1'M_2X_1)^{-1}X_1'M_2 \in \{ P_{\mu_1|\mathcal{F}} \}, \]
\[ G_{\theta_1|\mathcal{F}} = M_2G_{\mu_1|\mathcal{F}} \in \{ P_{\theta_1|\mathcal{F}} \}, \]
\[ G_{\theta_2|\mathcal{F}} = M_1X_2(X_2'M_1X_2)^{-1}X_2'M_1 \in \{ P_{\theta_2|\mathcal{F}} \}, \]
\[ G_{\mu_1|\mathcal{M}} = X_1(X_1'W_mX_1)^{-1}X_1'W^+_m \in \{ P_{\mu_1|\mathcal{M}} \}, \]
\[ G_{\theta_1|\mathcal{M}} = M_2G_{\mu_1|\mathcal{M}} \in \{ P_{\theta_1|\mathcal{M}} \}, \]
\[ G_{X_2u|\mathcal{M}} = X_2DX_2'M_1(M_1\Sigma M_1)^+M_1 \in \{ P_{X_2u|\mathcal{M}} \}, \]
\[ G_{M_1X_2u|\mathcal{M}} = M_1G_{X_2u|\mathcal{M}} \in \{ P_{M_1X_2u|\mathcal{M}} \}, \]
where \( \theta_2 = M_1X_2\beta_2 \) and
\[ W_m = X_1X_1' + \Sigma = X_1X_1' + X_2DX_2' + V. \]  \hspace{1cm} (6)
The matrices \( \tilde{M}_1 \) and \( \tilde{M}_2 \) are defined as
\[ \tilde{M}_1 = M_1(M_1WM_1)^+M_1, \quad \tilde{M}_2 = M_2(M_2WM_2)^+M_2. \]
Moreover, see, e.g., [19, Ch. 15],
\[ \tilde{M}_2 = M_2(M_2WM_2)^+M_2 = M_2(M_2WM_2)^+ = (M_2WM_2)^+. \]
Obviously, denoting \( W_1 = X_1X_1' + V \), we have
\[ M_2W = M_2W_1 = M_2W_m, \quad M_1W_m = M_1\Sigma. \]

It is not necessary to choose \( W \) and \( W_m \) as in \( (5) \) and in \( (6) \). For example, \( W \) could be replaced with \( W_* = XUU'X' + V \) such that \( \mathcal{C}(W_*) = \mathcal{C}(X : V) \); see, e.g., [19, Sec. 12.3].

The solutions to equations in Lemma 1.1 dealing with \( \mathcal{F} \) are unique if and only if \( \mathcal{C}(W) = \mathbb{R}^n \) while those dealing with \( \mathcal{M} \) are unique if and only if \( \mathcal{C}(W_m) = \mathbb{R}^n \). The general solution for \( A \) in
\[ A(X_1 : X_2 : VM) = (M_2X_1 : 0 : 0) \]
can be expressed, e.g., as
\[ A_0 = G_{\theta_1|\mathcal{F}} + EQ_W = M_2X_1(X_1'M_2X_1)^{-1}X_1'M_2 + EQ_W, \]
where \( E \in \mathbb{R}^{n \times n} \) is free to vary. By the consistency of the model \( \mathcal{F} \) it is meant that \( y \) lies in \( \mathcal{C}(W) \) with probability 1. Thus under the consistent
In the consistent linear model $\mathcal{F}$, the estimators $Ay$ and $By$ are said to be equal (with probability 1) if

$$Ay = By \text{ for all } y \in \mathcal{C}(X : V) = \mathcal{C}(X : VM) = \mathcal{C}(X) \oplus \mathcal{C}(VM),$$

where $\oplus$ refers to the direct sum. In (7) we are dealing with the “statistical” equality of the estimators $Ay$ and $By$. In (7) $y$ refers to a vector in $\mathbb{R}^n$, while in the notation $\text{cov}(Ay)$ we understand $y$ as a random vector. We may consider, for example, the equation

$$G_{\theta_1 | \mathcal{F}} = \text{BLUE}(\theta_1 | \mathcal{F}), \quad G_{\theta_1 | \mathcal{M}} = \text{BLUE}(\theta_1 | \mathcal{M}),$$

which are short notations for phrases like “$G_{\theta_1 | \mathcal{M}}$ is the BLUE for $\theta_1$ under $\mathcal{F}$” etc. However, writing the equalities like

$$\text{BLUE}(\mu_1 | \mathcal{F}) = \text{BLUE}(\mu_1 | \mathcal{M}),$$

may cause problems when the representations are not unique.

Isotalo et al. [11] found conditions under which an arbitrary representation of the BLUE of $\theta_1 = M_2X_1\beta_1$ under the fixed model $\mathcal{F}$ remains the BLUE for $\theta_1$ under the mixed model $\mathcal{M}$. This kind of property can be denoted shortly as

$$\{\text{BLUE}(\theta_1 | \mathcal{F})\} \subseteq \{\text{BLUE}(\theta_1 | \mathcal{M})\},$$

or, equivalently as $\{P_{\theta_1 | \mathcal{F}}\} \subseteq \{P_{\theta_1 | \mathcal{M}}\}$, where the sets $\{P_{\theta_1 | \mathcal{F}}\}$ and $\{P_{\theta_1 | \mathcal{M}}\}$ are defined as in Lemma 1.1

$$A \in \{P_{\theta_1 | \mathcal{F}}\} \iff A(X_1 : X_2 : VM) = (M_2X_1 : 0 : 0),$$

$$B \in \{P_{\theta_1 | \mathcal{M}}\} \iff B(X_1 : VM) = (M_2X_1 : 0).$$

In this paper we generalize the results of Isotalo et al. [11] by considering the following relations:

$$\text{BLUE}(M_2X_1\beta_1 | \mathcal{F}) \text{ vs } \text{BLUP}(M_2X_1\beta_1 + X_2u | \mathcal{M}),$$

$$\text{BLUE}(M_2X_2\beta_2 | \mathcal{F}) \text{ vs } \text{BLUP}(M_2X_2u | \mathcal{M}),$$

$$\text{BLUE}(X\beta | \mathcal{F}) \text{ vs } \text{BLUP}(X_1\beta_1 + X_2u | \mathcal{M}).$$

The case of two linear fixed models $\mathcal{B}_i = \{y_i, X_i, V_i\}$, $i = 1, 2$, with different covariance matrices is extensively studied by Mitra and Moore [18]. Haslett et al. [7] provide a review of conditions under which BLUEs/BLUPs
in one linear mixed model are also BLUE/BLUPs in another (with possibly different design matrices and covariance structures).

We end this section with a useful lemma.

**Lemma 1.2.** Using the earlier notation, the following statements hold:

(a) \( M = I_n - P_{(X_1;X_2)} = I_n - (P_{X_2} + P_{M_2X_1}) = M_2Q_{M_2X_1} = Q_{M_2X_1}M_2 \),
(b) \( r(M_2X_1) = r(X_1) - \dim \mathcal{C}(X_1) \cap \mathcal{C}(X_2) \),
(c) \( r(AB) = r(A) - \dim \mathcal{C}(A') \cap \mathcal{C}(B)^\perp \),
(d) \( \mathcal{C}(W^+ X)^\perp = \mathcal{C}(W M : Q_W) = \mathcal{C}(V M : Q_W) \),
(e) \( \mathcal{C}(X_2 : \Sigma M) = \mathcal{C}[M_2(M_2WM_2)^+M_2X_1 : Q_W]^\perp \),
(f) \( \mathcal{C}[A(A'B)^\perp]^\perp = \mathcal{C}(A) \cap \mathcal{C}(B) \).

For part (b) and (c), see, e.g., [17, Cor. 6.2]. For (d), see, e.g., [16, Lemma 4] and [20, Sec. 2]. For (e), see [11, Lemma, p. 72], and for (f), see [21, Compl. 7, p. 118].

2. Equality between the BLUEs

Isotalo et al. [11, Sec. 2] proved the following result:

**Theorem 2.1.** The following statements hold.

(a) An arbitrary BLUE for \( \theta_1 = M_2X_1\beta_1 \) under \( F \) provides also the BLUE for \( \theta_1 \) under the mixed model \( M \), i.e.,

\[
\{ \text{BLUE}(\theta_1 | F) \} \subseteq \{ \text{BLUE}(\theta_1 | M) \},
\]

i.e., \( \{ P_{\theta_1|F} \} \subseteq \{ P_{\theta_1|M} \} \), holds if and only if

\[
\mathcal{C}(\Sigma M_1) \subseteq \mathcal{C}(X_2 : VM).
\]

(b) The reverse relation \( \{ \text{BLUE}(\theta_1 | M) \} \subseteq \{ \text{BLUE}(\theta_1 | F) \} \), i.e.,

\[
\{ P_{\theta_1|M} \} \subseteq \{ P_{\theta_1|F} \},
\]

holds if and only if

\[
\mathcal{C}(X_2 : VM) \subseteq \mathcal{C}(R : \Sigma M_1), \text{ i.e., } \mathcal{C}(X_2) \subseteq \mathcal{C}(R : \Sigma M_1),
\]

where the matrix \( R \) has property \( \mathcal{C}(R) = \mathcal{C}(X_1) \cap \mathcal{C}(X_2) \).

Actually, the matrix \( R \) in (11) was erroneously missing in [11]. Notice that the equivalence of the two inclusions in (11) follows from \( \mathcal{C}(VM) = \mathcal{C}(\Sigma M) \subseteq \mathcal{C}(\Sigma M_1) \), which is based on

\[
\mathcal{C}(M) = \mathcal{C}(M_1QM_1X_2) \subseteq \mathcal{C}(M_1).
\]

The inclusion \( \mathcal{C}(R : \Sigma M_1) \subseteq \mathcal{C}(X_2 : VM) \) and thereby \( \{ P_{\theta_1|M} \} = \{ P_{\theta_1|F} \} \) holds if and only if

\[
\mathcal{C}(R : \Sigma M_1) \subseteq \mathcal{C}(X_2 : VM).
\]

Moreover, it is interesting to observe that \( \mathcal{C}(VM_1) \) is equivalent to

\[
\mathcal{C}(X_2 : VM).
\]
Namely, writing \( P_{(X_2:VM)} = P_{X_2} + P_{M_2:VM} \), it is easy to confirm that
\[
P_{(X_2:VM)}V_{M_1} = V_{M_1} \iff P_{(X_2:VM)}\Sigma_{M_1} = \Sigma_{M_1}.
\]

If \( \mu_1 = X_1\beta_1 \) is estimable under \( \mathcal{F} \), i.e., \( \mathcal{C}(X_1) \cap \mathcal{C}(X_2) = \{0\} \), we immediately observe that (11) simplifies into \( \mathcal{C}(X_2) \subseteq \mathcal{C}(\Sigma_{M_1}) \). Moreover, we can obtain the following corollary.

**Corollary 2.1.** Let \( \mu_1 = X_1\beta_1 \) is estimable under \( \mathcal{F} \). Then the following statements are equivalent:

- (a) \( \mathcal{C}(X_1) \subseteq \mathcal{C}(\mu_1 | \mathcal{M}) \)
- (b) \( \mathcal{C}(\mu_1 | \mathcal{M}) = \{\text{BLUE}(\mu_1 | \mathcal{F})\} \)
- (c) \( \mathcal{C}(X_2 : VM) \subseteq \mathcal{C}(\Sigma_{M_1}) \)
- (d) \( \mathcal{C}(X_2 : VM) = \mathcal{C}(\Sigma_{M_1}) \)
- (e) \( \mathcal{C}(X_2) \subseteq \mathcal{C}(\Sigma_{M_1}) \)

**Proof.** The equivalence of (a), (c) and (e) follows from Theorem 2.1. Assuming the disjointness \( \mathcal{C}(X_1) \cap \mathcal{C}(X_2) = \{0\} \), we observe, using (c) of Lemma 1.2 that
\[
r(X_2 : \Sigma M) = r(X_2) + r(\Sigma M) = r(X_2) + r(\Sigma M_1 Q_{M_1 X_2})
\]
\[
= r(X_2) + r(\Sigma M_1) - \dim \mathcal{C}(M_1 \Sigma) \cap \mathcal{C}(M_1 X_2)
\]
\[
\geq r(X_2) + r(\Sigma M_1) - r(M_1 X_2) = r(\Sigma M_1).
\]

Thereby, if (c) holds, then (12) implies that necessarily (d) holds, which further is equivalent to (b).

**Remark 2.1.** Isotalo et al. [11, p. 72] considered also the condition under which there exists at least one representation of the BLUE of \( \theta_1 \) under \( \mathcal{F} \) which is also BLUE of \( \theta_1 \) under \( \mathcal{M} \). This means that there exists a matrix \( \mathcal{A} \) such that \( \mathcal{A} \in \{P_{\theta_1} | \mathcal{F}\} \cap \{P_{\theta_1} | \mathcal{M}\} \), i.e., \( \mathcal{A} \) satisfies the equation
\[
\mathcal{A}(X_1 : X_2 : \Sigma M_1 : \Sigma M) = (M_2 X_1 : 0 : 0 : 0).
\]

It is clear that \( \mathcal{A} \Sigma M_1 = 0 \) implies \( \mathcal{A} \Sigma M = 0 \) and so (13) is equivalent to
\[
\mathcal{A}(X_1 : X_2 : \Sigma M_1) = (M_2 X_1 : 0 : 0).
\]

Now (14) has a solution for \( \mathcal{A} \) if and only if
\[
\mathcal{N}(X_1 : X_2 : \Sigma M_1) \subseteq \mathcal{N}(M_2 X_1 : 0 : 0),
\]
where \( \mathcal{N}(\cdot) \) refers to the nullspace. The corresponding conditions for further relations appearing in this article can be introduced (we will omit them).

It is interesting to consider the “statistical” equality
\[
\mathcal{G}_{\theta_1 | \mathcal{F}} y = \mathcal{G}_{\theta_1 | \mathcal{M}} y
\]
in deeper details. In particular we can consider two cases:
\[
y \in \mathcal{C}(W) = \mathcal{C}(X_1 : X_2 : V), \quad y \in \mathcal{C}(W_m) = \mathcal{C}(X_1 : X_2 D : V).
\]
Recall that in the fixed model $\mathcal{F}$ the “permissible observation space” for the response variable $y$ is $\mathcal{C}(\mathbf{W})$ while in the mixed model $\mathcal{M}$ it is $\mathcal{C}(\mathbf{W}_m)$. Now the following corollary is straightforward to confirm.

**Corollary 2.2.** Consider the models $\mathcal{F}$ and $\mathcal{M}$.

(a) The following statements are equivalent:
   (i) $G_{\theta_1 | \mathcal{F}} y = G_{\theta_1 | \mathcal{F}} y$ for all $y \in \mathcal{C}(\mathbf{W}) = \mathcal{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$,
   (ii) $G_{\theta_1 | \mathcal{F}}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{M}_2 \mathbf{X}_1 : 0 : 0)$,
   (iii) $G_{\theta_1 | \mathcal{F}} \in \{P_{\theta_1 | \mathcal{F}}\}$, i.e., $G_{\theta_1 | \mathcal{F}} y = \text{BLUE}(\theta_1 | \mathcal{F})$.

(b) The following statements are equivalent:
   (i) $(G_{\theta_1 | \mathcal{F}} + \text{EQ}_{\mathbf{W}_m}) y = G_{\theta_1 | \mathcal{F}} y$ for all $y \in \mathcal{C}(\mathbf{W})$ and for all $\mathbf{E}$,
   (ii) $(G_{\theta_1 | \mathcal{F}} + \text{EQ}_{\mathbf{W}_m})(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{M}_2 \mathbf{X}_1 : 0 : 0)$ for all $\mathbf{E}$,
   (iii) $\{\text{BLUE}(\theta_1 | \mathcal{F})\} \subseteq \{\text{BLUE}(\theta_1 | \mathcal{F})\}$.

(c) The following statements are equivalent:
   (i) $G_{\theta_1 | \mathcal{F}} y = G_{\theta_1 | \mathcal{F}} y$ for all $y \in \mathcal{C}(\mathbf{W}_m) = \mathcal{C}(\mathbf{X}_1 : \Sigma)$,
   (ii) $(G_{\theta_1 | \mathcal{F}} + \text{EQ}_{\mathbf{W}_m}) y = G_{\theta_1 | \mathcal{F}} y$ for all $y \in \mathcal{C}(\mathbf{W}_m)$ and for all $\mathbf{E}$,
   (iii) $G_{\theta_1 | \mathcal{F}}(\mathbf{X}_1 : \Sigma \mathbf{M}_1) = (\mathbf{M}_2 \mathbf{X}_1 : 0)$,
   (iv) $\{\text{BLUE}(\theta_1 | \mathcal{F})\} \subseteq \{\text{BLUE}(\theta_1 | \mathcal{F})\}$.

**3. Equality of a particular BLUE and BLUP**

In this section we consider the relation

\[ \text{BLUE}(\mathbf{M}_2 \mathbf{X}_1 \beta_1 | \mathcal{F}) \text{ versus } \text{BLUP}(\mathbf{M}_2 \mathbf{X}_1 \beta_1 + \mathbf{X}_2 \mathbf{u} | \mathcal{M}). \]

Recall, by (3), that under $\mathcal{M}$ we have

\[ \text{BLUP}(\mathbf{M}_2 \mathbf{X}_1 \beta + \mathbf{X}_2 \mathbf{u}) = \text{BLUE}(\mathbf{M}_2 \mathbf{X}_1 \beta_1) + \text{BLUP}(\mathbf{X}_2 \mathbf{u}) \]

\[ = \text{BLUE}(\mathbf{M}_2 \mathbf{X}_1 \beta_1) + \mathbf{X}_2 \text{BLUP}(\mathbf{u}). \]

By Lemma [1.1] $\mathbf{L} y$ is the BLUP for $\mathbf{\eta}_1 = \mathbf{M}_2 \mathbf{X}_1 \beta_1 + \mathbf{X}_2 \mathbf{u}$ if and only if

\[ \mathbf{L}(\mathbf{X}_1 : \Sigma \mathbf{M}_1) = (\mathbf{M}_2 \mathbf{X}_1 : \mathbf{X}_2 \mathbf{D} \mathbf{X}_1 \mathbf{M}_1), \quad (15) \]

where $\Sigma = \mathbf{X}_2 \mathbf{D} \mathbf{X}_1 + \mathbf{V}$. The general solution to $\mathbf{L}$ in (15) can be expressed as

\[ \mathbf{L}_0 = \mathbf{M}_2 \mathbf{X}_1 (\mathbf{X}_1 \mathbf{W}_m \mathbf{X}_1)^{-1} \mathbf{X}_1 \mathbf{W}_m^+ + \mathbf{X}_2 \mathbf{D} \mathbf{X}_1 \mathbf{M}_1 (\mathbf{M}_1 \Sigma \mathbf{M}_1)^+ \mathbf{M}_1 + \text{EQ}_{\mathbf{W}_m}, \]

where $\mathbf{E} \in \mathbb{R}^{n \times n}$ is free to vary and $\mathbf{W}_m = \mathbf{X}_1 \mathbf{X}_1^+ + \Sigma$. Suppose that $\mathbf{L}_0$ provides also the BLUE for $\theta_1 = \mathbf{M}_2 \mathbf{X}_1 \beta_1$ under the fixed model $\mathcal{F}$. Then $\mathbf{L}_0$ has to satisfy, for every $\mathbf{E}$, the fundamental BLUE equation

\[ \mathbf{L}_0(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = \mathbf{L}_0(\mathbf{X}_1 : \mathbf{X}_2 : \Sigma \mathbf{M}) = (\mathbf{M}_2 \mathbf{X}_1 : 0 : 0). \quad (16) \]

Trivially the $\mathbf{X}_1$-part of (16) holds. Moreover, we must have

\[ (G_{\theta_1 | \mathcal{F}} + G_{\mathbf{X}_2 \mathbf{u} | \mathcal{F}} + \text{EQ}_{\mathbf{W}_m})(\mathbf{X}_2 : \Sigma \mathbf{M}) = (0 : 0) \quad \text{for all } \mathbf{E}, \]
which implies that \( \mathcal{C}(X_2) \subseteq \mathcal{C}(W_m) = \mathcal{C}(X_1 : \Sigma M_1) \), and thereby
\[
\mathcal{C}(W) = \mathcal{C}(W_m), \quad X_2 = X_1A + \Sigma M_1B = X_1A + W_mB_1B
\] (17)
for some \( A \) and \( B \). We further must have
\[
(G_{\theta_1|_{\mathcal{M}}} + G_{X_2u|_{\mathcal{M}}})(X_2 : \Sigma M) = (0 : 0). \tag{18}
\]
Consider first the \( \Sigma M \)-part of (18). In view of (15) we have
\[
(G_{\theta_1|_{\mathcal{M}}} + G_{X_2u|_{\mathcal{M}}})\Sigma M_1 = X_2DX_2'M_1,
\]
which further implies
\[
(G_{\theta_1|_{\mathcal{M}}} + G_{X_2u|_{\mathcal{M}}})\Sigma M_1Q_{M_1}X_2 = X_2DX_2'M_1Q_{M_1}X_2 = 0, \tag{19}
\]
i.e., \( (G_{\theta_1|_{\mathcal{M}}} + G_{X_2u|_{\mathcal{M}}})\Sigma M = 0 \), and thereby \( \Sigma M \)-part of (18) holds. For the \( X_2 \)-part in (18) we must have
\[
(G_{\theta_1|_{\mathcal{M}}} + G_{X_2u|_{\mathcal{M}}})X_2 = M_2X_1(X'_1W_m^{-1}X_1)\Sigma M_1X_2 = 0,
\]
which clearly holds if and only if
\[
G_{\theta_1|_{\mathcal{M}}}X_2 = M_2X_1(X'_1W_m^{-1}X_1)\Sigma M_1X_2 = 0, \tag{20a}
\]
\[
G_{X_2u|_{\mathcal{M}}}X_2 = X_2DX_2'M_1(M_1\Sigma M_1)^{\perp}M_1X_2 = 0. \tag{20b}
\]
Substituting \( X_2 = X_1A + W_mB_1B \) into (20a) yields \( M_2X_1A = 0 \), so that \( A = Q_{X'_1M_2}Z \) for some \( Z \), and thereby, taking (17) into account,
\[
X_2 = X_1Q_{X'_1M_2}Z + \Sigma M_1B. \tag{21}
\]
Moreover, by part (f) of Lemma 1.2 we have
\[
\mathcal{C}(X_1Q_{X'_1M_2}) = \mathcal{C}(X_1X'_1X_2^{-1}) = \mathcal{C}(X_1) \cap \mathcal{C}(X_2).
\]
Consider then (20b). Substituting (21) into (20b) yields
\[
X_2DX_2'M_1(M_1\Sigma M_1)^{\perp}M_1B = 0, \quad \text{i.e., } X_2DX_2'M_1B = 0, \quad \text{so that } \mathcal{C}(B) \subseteq \mathcal{C}(M_1X_2DX_2'), \quad \text{and by (21),}
\]
\[
\mathcal{C}(X_2) \subseteq \mathcal{C}(X_1Q_{X'_1M_2} : \Sigma M_1Q_{M_1X_2DX_2'}).
\]
In light of part (f) of Lemma 1.2 we can further write
\[
\mathcal{C}(M_1Q_{M_1X_2DX_2'}) = \mathcal{C}(X_1 : X_2DX_2')^{\perp} = \mathcal{C}(X_1 : M_1X_2DX_2')^{\perp}.
\]
Thus, noting that \( M_1X_2DX_2' = M_1(\Sigma - V) \), we have obtained the following theorem.
Theorem 3.1. An arbitrary BLUP for \( \eta_1 = M_2 X_1 \beta + X_2 u \) under \( \mathcal{M} \) provides also the BLUE for \( \theta_1 = M_2 X_1 \beta_1 \) under the fixed model \( \mathcal{F} \), i.e.,

\[
\{ \text{BLUE}(M_2 X_1 \beta_1 | \mathcal{F}) \} \subseteq \{ \text{BLUE}(M_2 X_1 \beta | \mathcal{M}) \},
\]

i.e., \( \{ P_{\eta_1 | \mathcal{M}} \} \subseteq \{ P_{\theta_1 | \mathcal{F}} \} \), if and only if

\[
\mathcal{C}(X_2) \subseteq \mathcal{C}(R : \Sigma M_1 S),
\]

where the matrices \( R \) and \( S \) have properties \( \mathcal{C}(R) = \mathcal{C}(X_1) \cap \mathcal{C}(X_2) \) and

\[
\mathcal{C}(S) = \mathcal{C}[X_1 : M_1(\Sigma - V)]^\perp = \mathcal{C}(X_1 : M_1 X_2 D X_2')^\perp.
\]

The reverse inclusion to (22) is considered in Theorem 3.2.

**Theorem 3.2.** An arbitrary BLUE for \( \theta_1 = M_2 X_1 \beta_1 \) under \( \mathcal{F} \) provides also the BLUP for \( \eta_1 = M_2 X_1 \beta_1 + X_2 u \) under the mixed model \( \mathcal{M} \), i.e.,

\[
\{ \text{BLUE}(M_2 X_1 \beta_1 | \mathcal{M}) \} \subseteq \{ \text{BLUE}(M_2 X_1 \beta_1 + X_2 u | \mathcal{M}) \},
\]

i.e., \( \{ P_{\theta_1 | \mathcal{F}} \} \subseteq \{ P_{\eta_1 | \mathcal{M}} \} \), if and only if the following two conditions hold:

(a) \( \mathcal{C}(\Sigma M_1) \subseteq \mathcal{C}(X_2 : \Sigma M) \), i.e., \( \{ \text{BLUE}(\theta_1 | \mathcal{F}) \} \subseteq \{ \text{BLUE}(\theta_1 | \mathcal{M}) \} \),

(b) \( \Sigma M_1 = VM_1 \), i.e., \( M_1 X_2 D X_2' = 0 \).

**Proof.** Take an arbitrary member in the class \( \{ P_{\theta_1 | \mathcal{F}} \} \),

\[
B_0 = G_{\theta_1 | \mathcal{F}} + EQ_W = M_2 X_1 (X_1' M_2 X_1)^{-1} X_1' M_2 + EQ_W,
\]

and \( E \) is free to vary and \( \mathcal{C}(W) = \mathcal{C}(X_1 : X_2 : V) \). Then \( B_0 \) provides the BLUP for \( \eta_1 = M_2 X_1 \beta_1 + X_2 u \) under the mixed model \( \mathcal{M} \) if and only if

\[
(G_{\theta_1 | \mathcal{F}} + EQ_W)(X_1 : \Sigma M_1) = (M_2 X_1 : X_2 D X_2' M_1)
\]

holds for every \( E \). The \( X_1 \)-part is clear. The \( \Sigma M_1 \)-part is

\[
(G_{\theta_1 | \mathcal{F}} + EQ_W)\Sigma M_1 = X_2 D X_2' M_1,
\]

i.e.,

\[
G_{\theta_1 | \mathcal{F}} \Sigma M_1 = M_2 X_1 (X_1' M_2 X_1)^{-1} X_1' M_2 \Sigma M_1 = X_2 D X_2' M_1.
\]

It is clear that (25) holds if and only if

\[
M_2 X_1 (X_1' M_2 X_1)^{-1} X_1' M_2 \Sigma M_1 = 0,
\]

\[
X_2 D X_2' M_1 = 0,
\]

where (26a) is equivalent to

\[
X_1' M_2 \Sigma M_1 = 0,
\]

i.e., \( \mathcal{C}(\Sigma M_1) \subseteq \mathcal{C}(M_2 X_1)^\perp \).

In view of \( \mathcal{C}(\Sigma M_1) \subseteq \mathcal{C}(W), \) (27) can be written equivalently as

\[
\mathcal{C}(\Sigma M_1) \subseteq \mathcal{C}(M_2 X_1)^\perp \cap \mathcal{C}(W).
\]

On the other hand, in light of part (e) of Lemma 1.2 we know that

\[
\mathcal{C}(M_2 X_1 : Q_W)^\perp = \mathcal{C}(M_2 X_1)^\perp \cap \mathcal{C}(W) = \mathcal{C}(X_2 : VM).
\]
Theorem 3.1 becomes
\[ C(\Sigma M_1) \subseteq C(X_2 : VM) = C(X_2 : \Sigma M). \]

Moreover, (26b) is equivalent to \( VM_1 = \Sigma M_1 \), which completes the proof. □

What about the equality of the sets \( \{P_{\theta_1 | \mathcal{F}}\} \) and \( \{P_{\theta_1 | \mathcal{M}}\} \)? Requesting that (b) of Theorem 3.2 holds, i.e., \( VM_1 = \Sigma M_1 \), the condition (23) of Theorem 3.1 becomes \( C(X_2) \subseteq C(R : \Sigma M_1) \), i.e.,
\[ C(X_2 : \Sigma M) \subseteq C(R : \Sigma M_1), \] where \( C(R) = C(X_1) \cap C(X_2). \)

On the other hand, condition (a) of Theorem 3.2 is equivalent to
\[ C(R : \Sigma M_1) \subseteq C(X_2 : \Sigma M). \]

Now (30) and (31) imply the following result.

**Corollary 3.1.** The following statements are equivalent:

(a) \( \{\text{BLUE}(\theta_1 + X_2u | \mathcal{M})\} = \{\text{BLUE}(\theta_1 | \mathcal{F})\} \),

(b) \( C(X_2 : \Sigma M) = C(R : \Sigma M_1) \) and \( M_1X_2DX_2' = 0 \), i.e., \( \Sigma M_1 = VM_1 \), where \( C(R) = C(X_1) \cap C(X_2) \).

Notice that if \( \mu_1 \) is estimable in \( \mathcal{F} \) then \( M_1X_2DX_2' = 0 \) is equivalent to \( X_2DX_2' = 0 \). Moreover, from Corollary 3.1 we can conclude the following.

**Corollary 3.2.** Suppose that \( \mu_1 = X_1\beta_1 \) is estimable under \( \mathcal{F} \). Then the following three statements are equivalent:

(a) \( \{\text{BLUE}(\mu_1 + X_2u | \mathcal{M})\} = \{\text{BLUE}(\mu_1 | \mathcal{F})\} \),

(b) \( \{\text{BLUE}(\mu_1 | \mathcal{M})\} = \{\text{BLUE}(\mu_1 | \mathcal{F})\} \) and \( X_2DX_2' = 0 \), i.e., \( \Sigma = V \),

(c) \( C(\Sigma M_1) = C(X_2 : VM) \) and \( X_2DX_2' = 0 \).

**Remark 3.1.** The property \( \text{cov}(X_2u) = X_2DX_2' = 0 \) together with \( E(u) = 0 \) means that \( X_2u = 0 \) with probability 1. Moreover, if \( X_2DX_2' = 0 \), then the mixed model \( \mathcal{M} \) becomes the small fixed model \( \mathcal{F}_1 = \{y, X_1\beta_1, V\} \) and then any of the conditions in Corollary 3.2 implies the equality
\[ \{\text{BLUE}(\mu_1 | \mathcal{F}_1)\} = \{\text{BLUE}(\mu_1 | \mathcal{F})\}, \]
which further is equivalent to \( C(VM_1) = C(X_2 : VM) \). □

**4. A further equality of particular BLUE and BLUP**

In this section we consider
\[ \text{BLUE}(M_1X_2\beta_2 | \mathcal{F}) \quad \text{versus} \quad \text{BLUE}(M_1X_2u | \mathcal{M}). \]

**Theorem 4.1.** An arbitrary BLUP for \( M_1X_2u \) under \( \mathcal{M} \) provides also the BLUP for \( \theta_2 = M_1X_2\beta_2 \) under the fixed model \( \mathcal{F} \), i.e.,
\[ \{\text{BLUE}(M_1X_2u | \mathcal{M})\} \subseteq \{\text{BLUE}(M_1X_2\beta_2 | \mathcal{F})\}, \quad (32) \]
for some VM from which it follows that $X$.

By (33) the $E\theta$ also the BLUE for $\theta$.

The general solution to $C$ is

$$C_0 = M_1X_2DX_2'M_1(M_1\Sigma M_1)^+M_1 + EQW_m,$$

where $E$ is free to vary and $W_m = X_1X_1' + \Sigma$. Suppose that $C_0$ provides also the BLUE for $\theta_2 = M_1X_2\beta_2$ under the fixed model $M$. Then $C_0$ has to satisfy, for every $E$, the fundamental BLUE equation

$$(G_{M_1X_2u\mid\theta} + EQW_m)(X_1 : X_2 : VM) = (0 : M_1X_2 : 0).$$

By (33) the $X_1$-part of (34) holds. Moreover, we must have

$$(G_{M_1X_2u\mid\theta} + EQW_m)(X_2 : VM) = (M_1X_2 : 0)$$

for some $A$ and $B$. We further must have

$$G_{M_1X_2u\mid\theta}(X_2 : VM) = (M_1X_2 : 0).$$

Using $VM = \Sigma M_1$, (36) can be written as

$$G_{M_1X_2u\mid\theta}X_2 = M_1X_2DX_2'M_1(M_1\Sigma M_1)^+M_1X_2 = M_1X_2,$$

$$(37a)$$

$$G_{M_1X_2u\mid\theta}\Sigma M = M_1X_2DX_2'M_1(M_1\Sigma M_1)^+M_1\Sigma M = 0.$$  

$$(37b)$$

Now (37b) can be expressed as

$$M_1X_2DX_2'M_1(M_1\Sigma M_1)^+M_1\Sigma M_1Q_{M_1X_2} = 0,$$

which obviously holds.

Consider then (37a):

$$M_1(\Sigma - V)M_1(M_1\Sigma M_1)^+M_1X_2 = M_1X_2,$$

from which, in view of $\mathscr{C}(M_1X_2) \subseteq \mathscr{C}(M_1\Sigma)$, it follows that

$$M_1X_2 - M_1VM_1(M_1\Sigma M_1)^+M_1X_2 = M_1X_2,$$

i.e.,

$$VM_1(M_1\Sigma M_1)^+M_1X_2 = 0.$$  

Substituting $X_2 = X_1A + \Sigma M_1B$ into (33) yields $VM_1B = 0$, so that $\mathscr{C}(B) \subseteq \mathscr{C}(M_1V)^+$ and thereby

$$\mathscr{C}(X_2) \subseteq \mathscr{C}(X_1 : \Sigma M_1Q_{M_1X_1}) = \mathscr{C}(X_1 : \Sigma M_1Q_{(X_1; V)}).$$
where by part (f) of Lemma 1.2, $\mathcal{C}(M_1Q_{M_1}V) = \mathcal{C}(X_1 : V)^\perp$. \hfill \Box

Let us consider the reverse inclusion to (32).

**Theorem 4.2.** An arbitrary BLUP for $\theta_2 = M_1X_2B_2$ under $\mathcal{F}$ provides also the BLUP for $M_1X_2u$ under the mixed model $\mathcal{M}$, i.e.,

$$\{\text{BLUE}(M_1X_2\beta_2 \mid \mathcal{F})\} \subseteq \{\text{BLUP}(M_1X_2u \mid \mathcal{M})\},$$

i.e., $\{P_{\theta_2 \mid \mathcal{F}}\} \subseteq \{P_{M_1X_2u \mid \mathcal{M}}\}$, if and only if $\mathcal{C}(VM_1) = \mathcal{C}(VM)$.

**Proof.** Take an arbitrary member in the class $\{P_{\theta_2 \mid \mathcal{F}}\}$,

$$N_0 = G_{\theta_2 \mid \mathcal{F}} + EQW = M_1X_2(X_2'\hat{M}_1X_2)^{-1}X_2'\hat{M} - X_2'\hat{M}_1 + EQW,$$

where $E$ is free to vary and $\mathcal{C}(W) = \mathcal{C}(X_1 : X_2 : V)$. Then $N_0$ provides the BLUP for $M_1X_2u$ under the mixed model $\mathcal{M}$ if and only if

$$(G_{\theta_2 \mid \mathcal{F}} + EQW)(X_1 : \Sigma M_1) = (0 : M_1X_2DX_2'M_1),$$

where the $X_1$-part obviously holds and so we must have

$$G_{\theta_2 \mid \mathcal{F}}\Sigma M_1 = M_1X_2(X_2'\hat{M}_1X_2)^{-1}X_2'\hat{M}_1\Sigma M_1 = M_1X_2DX_2'M_1. \quad (40)$$

Premultiplying (40) by $X_2'\hat{M}_1$ yields an equivalent equation

$$X_2'\hat{M}_1\Sigma M_1 = X_2'\hat{M}_1X_2DX_2'M_1 = X_2'\hat{M}_1(\Sigma - V)M_1,$$

i.e., $X_2'\hat{M}_1VM_1 = 0$, which means that

$$\mathcal{C}(VM_1) \subseteq \mathcal{C}(\hat{M}_1X_2)^\perp. \quad (41)$$

We know that $\mathcal{C}(VM_1) \subseteq \mathcal{C}(W)$ and hence we can write (41) as

$$\mathcal{C}(VM_1) \subseteq \mathcal{C}(\hat{M}_1X_2)^\perp \cap \mathcal{C}(W). \quad (42)$$

In view of part (e) of Lemma 1.2 we have the following:

$$\mathcal{C}(X_1 : VM) = \mathcal{C}(\hat{M}_1X_2 : QW)^\perp = \mathcal{C}(\hat{M}_1X_2)^\perp \cap \mathcal{C}(W). \quad (43)$$

Combining (42) and (43) yields

$$\mathcal{C}(VM_1) \subseteq \mathcal{C}(X_1 : VM),$$

which is obviously equivalent to $\mathcal{C}(VM_1) = \mathcal{C}(VM)$. \hfill \Box

From Theorems 4.1 and 4.2 we get the following result.

**Corollary 4.1.** The following statements are equivalent:

(a) $\{\text{BLUP}(M_1X_2u \mid \mathcal{M})\} = \{\text{BLUE}(M_1X_2\beta_2 \mid \mathcal{F})\},$

(b) $\mathcal{C}(X_2) \subseteq \mathcal{C}(X_1 : \Sigma M_1Q_{M_1}V)$ and $\mathcal{C}(VM_1) = \mathcal{C}(VM).$
5. One further equality between BLUE and BLUP

In this section we consider

\[ \text{BLUE}(X_1\beta_1 + X_2\beta_2 \mid \mathcal{F}) \quad \text{versus} \quad \text{BLUP}(X_1\beta_1 + X_2u \mid \mathcal{M}) . \]

**Theorem 5.1.** An arbitrary BLUP for \( \eta = X_1\beta_1 + X_2u \) under \( \mathcal{M} \) provides also the BLUE for \( \mu = X_1\beta_1 + X_2\beta_2 \) under the fixed model \( \mathcal{F} \), i.e.,

\[ \{ \text{BLUP}(X_1\beta_1 + X_2u \mid \mathcal{M}) \} \subseteq \{ \text{BLUE}(X_1\beta_1 + X_2\beta_2 \mid \mathcal{F}) \} , \quad (44) \]

i.e., \( \{ \mathcal{P}_{\eta \mid \mathcal{M}} \} \subseteq \{ \mathcal{P}_{\mu \mid \mathcal{F}} \} \), if and only if

\[ \{ \text{BLUP}(M_1X_2u \mid \mathcal{M}) \} \subseteq \{ \text{BLUE}(M_1X_2\beta_2 \mid \mathcal{F}) \} . \quad (45) \]

**Proof.** The general solution to

\[ T(X_1 : \Sigma M_1) = (X_1 : X_2DX_2M_1) \]

can be expressed as

\[ T_0 = X_1(X'_1W_mX_1)^{-1}X'_1W_m + X_2DX_2M_1(M_1\Sigma M_1)^{-1}M_1 + EQW_m \]

where \( E \) is free to vary and \( W_m = \Sigma + X_1X'_1 \). Suppose that \( T_0 \) provides also the BLUE for \( \mu = X\beta \) under the fixed model \( \mathcal{F} \). Then \( T_0 \) has to satisfy, for every \( E \), the fundamental BLUE equation

\[ T_0(X_1 : X_2 : VM) = (X_1 : X_2 : 0) . \quad (46) \]

It is obvious that the \( X_1 \)-part of (46) holds. Moreover, we must have

\[ (G_{\mu_1 \mid \mathcal{M}} + G_{X_2u \mid \mathcal{M}} + EQW_m)(X_2 : VM) = (X_2 : 0) \] for all \( E \),

from which it follows that \( \mathcal{C}(X_2) \subseteq \mathcal{C}(W_m) \) and that for some \( A \) and \( B \),

\[ X_2 = X_1A + \Sigma M_1B . \quad (47) \]

We further must have

\[ (G_{\mu_1 \mid \mathcal{M}} + G_{X_2u \mid \mathcal{M}})(X_2 : VM) = (X_2 : 0) . \]

It is straightforward to show that \( (G_{\mu_1 \mid \mathcal{M}} + G_{X_2u \mid \mathcal{M}})VM = 0 \), so that we are left with condition

\[ (G_{\mu_1 \mid \mathcal{M}} + G_{X_2u \mid \mathcal{M}})X_2 = X_1(X'_1W_mX_1)^{-1}X'_1W_mX_2 \]

\[ + X_2DX_2M_1(M_1\Sigma M_1)^{-1}M_1X_2 = X_2 . \quad (48) \]

Substituting \( X_2 = X_1A + \Sigma M_1B = X_1A + W_mM_1B \) into (48) gives

\[ X_1A + X_2DX_2M_1B = X_1A + \Sigma M_1B , \]

so that we have \( X_2DX_2M_1B = \Sigma M_1B \), i.e., \( VM_1B = 0 \) and thereby

\[ \mathcal{C}(B) \subseteq \mathcal{C}(M_1V)^\perp . \quad (49) \]

Combining (47) and (49) gives \( \mathcal{C}(X_2) \subseteq \mathcal{C}(X_1 : \Sigma M_1Q_{M_1V}) \), and thus by Theorem 4.1 the proof is completed. □
Consider now the reverse inclusion of (44).

**Theorem 5.2.** An arbitrary BLUE for $\mu = X_1\beta_1 + X_2\beta_2$ under $\mathcal{F}$ provides also the BLUP for $\eta = X_1\beta_1 + X_2u$ under the mixed model $\mathcal{M}$, i.e.,

$$\{\text{BLUE}(X_1\beta_1 + X_2\beta_2 \mid \mathcal{F})\} \subseteq \{\text{BLUP}(X_1\beta_1 + X_2u \mid \mathcal{M})\},$$

i.e., $\{P_{\mu \mid \mathcal{F}}\} \subseteq \{P_{\eta \mid \mathcal{M}}\}$, if and only if

$$\{\text{BLUE}(M_1X_2\beta_2 \mid \mathcal{F})\} \subseteq \{\text{BLUP}(M_1X_2u \mid \mathcal{M})\}.$$

**Proof.** Take an arbitrary member in the class $\{P_{\mu \mid \mathcal{F}}\}$,

$$G_0 = G + EQ_W = X(X'W^-X)^{-1}X'W^+ + EQ_W,$$

where $E$ is free to vary and $\mathcal{C}(W) = \mathcal{C}(X_1 : X_2 : V)$. Then $G_0$ provides the BLUP for $\eta = X_1\beta_1 + X_2u$ under the mixed model $\mathcal{M}$ if and only if

$$(G + EQ_W)(X_1 : \Sigma M_1) = (X_1 : X_2DX_2'M_1). \quad (50)$$

The $X_1$-part in (50) is clear. The $\Sigma M_1$-part gives

$$G\Sigma M_1 = X(X'W^-X)^{-1}X'W^+\Sigma M_1 = X_2DX_2'M_1. \quad (51)$$

Premultiplying (51) by $X'W^+$ gives an equivalent form

$$X'W^+\Sigma M_1 = X'W^+X_2DX_2'M_1. \quad (52)$$

Substituting $X_2DX_2' = \Sigma - V$ into (52) leads to

$$X'W^+\Sigma M_1 = X'W^+(\Sigma - V)M_1,$$

i.e., $X'W^+VM_1 = 0$, i.e.,

$$\mathcal{C}(VM_1) \subseteq \mathcal{C}(W^+X)\perp. \quad (53)$$

Now by part (d) of Lemma 1.2 we know that

$$\mathcal{C}(W^+X)\perp = \mathcal{C}(WM : Q_W) = \mathcal{C}(VM : Q_W),$$

and hence (53) becomes

$$\mathcal{C}(VM_1) \subseteq \mathcal{C}(VM : Q_W). \quad (54)$$

Premultiplying (54) by $P_W$ we obtain $\mathcal{C}(VM_1) \subseteq \mathcal{C}(VM)$, so that we must have $\mathcal{C}(VM_1) = \mathcal{C}(VM)$, and thus by Theorem 1.2 the proof is completed. 

Combining the theorems of Sections 4 and 5 we get the following interesting result.

**Corollary 5.1.** The following statements are equivalent:

(a) $\{\text{BLUP}(X_1\beta_1 + X_2u \mid \mathcal{M})\} = \{\text{BLUE}(X_1\beta_1 + X_2\beta_2 \mid \mathcal{F})\},$

(b) $\{\text{BLUP}(M_1X_2u \mid \mathcal{M})\} = \{\text{BLUE}(M_1X_2\beta_2 \mid \mathcal{F})\},$

(c) $\mathcal{C}(X_2) \subseteq \mathcal{C}(X_1 : \Sigma M_1 Q_{M_1}V)$ and $\mathcal{C}(VM_1) = \mathcal{C}(VM).$
6. Equality of the covariance matrices

In this section we assume that \( \mu_1 = X_1 \beta_1 \) is estimable under \( \mathcal{F} \) and we consider the equality of the covariance matrices of the BLUEs of \( \mu_1 \) under \( \mathcal{F} \) and under \( \mathcal{M} \), i.e., we are comparing \( \text{cov}(G_{\mu_1} \mid \mathcal{F} y \mid \mathcal{F}) \) and \( \text{cov}(G_{\mu_1} \mid \mathcal{M} y \mid \mathcal{F}) \), where

\[
G_{\mu_1} \mid \mathcal{F} = X_1 (X_1' \hat{M}_2 X_1)^{-1} X_1' \hat{M}_2 \in \{P_{\mu_1} \mid \mathcal{F}\},
\]

\[
G_{\mu_1} \mid \mathcal{M} = X_1 (X_1' W_m^+ X_1)^{-1} X_1' W_m^+ \in \{P_{\mu_1} \mid \mathcal{M}\}.
\]

It is noteworthy that the covariance matrices of the BLUEs are unique even though the representations of the BLUEs may not be unique.

It can be shown, see, e.g., [13], that

\[
\text{cov}(\mu_1 \mid \mathcal{F} y \mid \mathcal{M}) = G_{\mu_1} \mid \mathcal{M} \Sigma G_{\mu_1} \mid \mathcal{F}.
\]

where \( W_m^{+1/2} \) refers to the Moore–Penrose inverse of the nonnegative definite square root of \( W_m \), and

\[
\text{cov}(G_{\mu_1} \mid \mathcal{M} y \mid \mathcal{F}) = G_{\mu_1} \mid \mathcal{F} V G_{\mu_1} \mid \mathcal{F} = X_1 (X_1' W_m^{+1/2} X_1)^{+} X_1' = X_1 (X_1' W_m^{+1/2} X_1)^{+} X_1.
\]

The equality \( \text{cov}(G_{\mu_1} \mid \mathcal{M} y \mid \mathcal{F}) = \text{cov}(G_{\mu_1} \mid \mathcal{F} y \mid \mathcal{F}) \) holds if and only if

\[
X_1 (X_1' W_m^{+1/2} X_1)^{+} X_1' = X_1 (X_1' W_m^{+1/2} P_{W_m^{+1/2} M_2} W_m^{+1/2} X_1)^{+} X_1'.
\]

Pre- and postmultiplying (55) by \( X_1^{+} \) and \( (X_1')^{+} \), respectively, and using the fact that \( P_{X_1} = X_1^{+} X_1 \), gives an equivalent form to (55):

\[
(X_1' W_m^{+1/2} X_1)^{+} = (X_1' W_m^{+1/2} P_{W_m^{+1/2} M_2} W_m^{+1/2} X_1)^{+},
\]

i.e.,

\[
X_1' W_m^{+1/2} W_m^{+1/2} X_1 = X_1' W_m^{+1/2} P_{W_m^{+1/2} M_2} W_m^{+1/2} X_1.
\]

Now we have the Löwner ordering

\[
X_1' W_m^{+1/2} (I_n - P_{W_m^{+1/2} M_2}) W_m^{+1/2} X_1 \geq_{L} 0,
\]

where the equality holds if and only if

\[
\mathcal{C}(W_m^{+1/2} X_1) \subseteq \mathcal{C}(W_m^{+1/2} M_2).
\]

(56)
Premultiplying (56) by \( W_{1/2} \) gives an equivalent inclusion

\[
\mathcal{C}(X_1) \subseteq \mathcal{C}(W_m M_2) = \mathcal{C}(W_1 M_2), \quad \text{where } W_1 = X_1 X_1' + V. \tag{57}
\]

As Isotalo et al. [11, p. 73] point out, the assumption \( \mathcal{C}(W_m) = \mathbb{R}^n \) implies that the BLUE of \( \mu_1 \) has a unique representation under \( \mathcal{F} \) and \( \mathcal{M} \). Moreover, following their proof (assuming the estimability of \( \mu_1 \) under \( \mathcal{F} \)), it can be shown that the presentations are equal if and only if (57) holds. Thus we can conclude the following result.

**Theorem 6.1.** The following statements are equivalent.

(a) \( \text{cov}(G_{\mu_1 | \mathcal{M}} y | \mathcal{M}) = \text{cov}(G_{\mu_1 | \mathcal{F}} y | \mathcal{F}) \).

(b) \( \mathcal{C}(X_1) \subseteq \mathcal{C}(W_m M_2) \).

(c) If \( \mathcal{C}(W_m) = \mathbb{R}^n \), then the representations of the BLUEs of \( \mu_1 \) under the models \( \mathcal{F} \) and \( \mathcal{M} \) are equal.

7. Conclusions

In this article we consider the partitioned fixed linear model \( \mathcal{F} : y = X_1 \beta_1 + X_2 \beta_2 + \varepsilon \) and the corresponding mixed model \( \mathcal{M} : y = X_1 \beta_1 + X_2 u + \varepsilon \), where \( \varepsilon \) is a random error vector and \( u \) is a random effect vector. Isotalo et al. [11] found conditions under which an arbitrary representation of the best linear unbiased estimator, BLUE, of \( \theta_1 = M_2 X_1 \beta_1 \) in the fixed model \( \mathcal{F} \) remains BLUE in the mixed model \( \mathcal{M} \); here \( M_2 \) refers to the orthogonal projector \( I_n - P_{X_2} \). The reason to concentrate on estimating \( \theta_1 = M_2 X_1 \beta_1 \) is that this approach means that the properties obtained are valid for all parametric functions of the type \( K \beta_1 \) that are estimable under the partitioned model \( \mathcal{F} \) (and thereby under \( \mathcal{M} \)). In this paper we extend the results concerning further equalities arising from the models \( \mathcal{F} \) and \( \mathcal{M} \).

The property that BLUE of \( \theta_1 \) under \( \mathcal{F} \) remains BLUE under \( \mathcal{M} \) can be denoted shortly as

\[
\{\text{BLUE}(\theta_1 | \mathcal{F})\} \subseteq \{\text{BLUE}(\theta_1 | \mathcal{M})\}, \tag{58}
\]

or, equivalently as \( \{P_{\theta_1 | \mathcal{F}}\} \subseteq \{P_{\theta_1 | \mathcal{M}}\} \), where, in notation introduced in Section [11]

\[
A \in \{P_{\theta_1 | \mathcal{F}}\} \iff A(X_1 : X_2 : VM) = (M_2 X_1 : 0 : 0),
\]

\[
B \in \{P_{\theta_1 | \mathcal{M}}\} \iff B(X_1 : \Sigma M_1) = (M_2 X_1 : 0).
\]

In this paper we generalize the results of [11] by considering the following relations:

\[
\text{BLUE}(M_2 X_1 \beta_1 | \mathcal{F}) \text{ vs } \text{BLUP}(M_2 X_1 \beta_1 + X_2 u | \mathcal{M}),
\]

\[
\text{BLUE}(M_2 X_2 \beta_2 | \mathcal{F}) \text{ vs } \text{BLUP}(M_2 X_2 u | \mathcal{M}),
\]

\[
\text{BLUE}(X \beta | \mathcal{F}) \text{ vs } \text{BLUP}(X_1 \beta_1 + X_2 u | \mathcal{M}).
\]
As Kala et al. [14, Remark 2] point out, the notation of the type as in (58) is merely symbolic and it is not meant to refer to a set containing only one element which is a single fixed vector resulting from a transformation of an observed vector $y$, or is a single random vector variable being a specific linear transformation of the random vector $y$. We are, of course, actually interested in the matrices belonging to classes like $\{P_{\theta_1} \mid \mathcal{F}\}$ etc.

There are several related papers concerning the invariance of the BLUEs and/or BLUPs under two models. Mitra and Moore [18] gave an extensive study on the circumstances in which the BLUEs of estimable parametric functions of the fixed parameters in linear model $\{y, X\beta, V_1\}$ remain BLUEs under $\{y, X\beta, V_2\}$: models differing in covariance matrices. Corresponding considerations related to two mixed models have been made, e.g., by Haslett and Puntanen [5, 6]. In [7], they provide a review of conditions under which BLUEs/BLUPs in one linear mixed model are also BLUE/BLUPs in another. The article [8] explores interesting links between the mixed and fixed linear models. It appears that the concept of the linear model with new future observations is a powerful tool for these considerations. For further references we may mention [15], [22], [25], and [4].

We believe that our results, which are mainly linear-algebraic by nature, can provide some insight into the relations between the fixed and mixed model like $\mathcal{F}$ and $\mathcal{M}$. Some interesting related discussion appears, e.g., in [9] [10].

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References