Extrapolation to mixed norm spaces and applications

Kwok-Pun Ho

ABSTRACT. This paper establishes extrapolation theory to mixed norm spaces. By applying this extrapolation theory, we obtain the mapping properties of the Rubio de Francia Littlewood–Paley functions and the geometrical maximal functions on mixed norm spaces. As special cases of these results, we have the mapping properties on the mixed norm Lebesgue spaces with variable exponents and the mixed norm Lorentz spaces.

1. Introduction

The main theme of this paper is the extension of the Rubio de Francia extrapolation theory to mixed norm spaces.

The study of mixed norm spaces was initialized in [1]. The mixed norm Lebesgue spaces have been extended to mixed norm Lorentz spaces in [8] and to mixed norm rearrangement-invariant spaces in [3]. The mixed norm spaces have applications on Sobolev's inequality, Littlewood's inequality [9] and martingale Hardy spaces [4, 27]. The Rubio de Francia extrapolation theory was introduced in [23, 24, 25]. It is a powerful tool in analysis, especially in the study of nonlinear operators. It has been extended to rearrangement-invariant Banach functions spaces in [6], the Herz spaces with variable exponents in [14] and the Morrey–Banach spaces in [15].

In this paper, we further extend the extrapolation theory to mixed norm spaces. In [11, Theorem 3.2], we present an approach to the extension of extrapolation theory to mixed norm spaces, but this approach has a technical mistake. The main result of this paper corrects this mistake and establishes the extrapolation theory to mixed norm spaces with an assumption on the

boundedness of the strong maximal operator. The main result in this paper does not only correct the mistake in [11, Theorem 3.2], our main result also gives a refinement on the extrapolation theory to mixed norm spaces. To apply the extrapolation theory to mixed norm spaces, we do not need to use the density argument or the approximation argument to obtain the boundedness of operators on the entire mixed norm space. We use the ideas from [15] to obtain this refinement.

We apply our extrapolation theory to mixed norm spaces to the Rubio de Francia Littlewood–Paley functions and the geometrical maximal functions. In particular, we study two special cases of the mixed norm spaces, namely, the mixed norm Lebesgue spaces with variable exponents and the mixed norm Lorentz spaces.

This paper is organized as follows. The definition of a mixed norm spaces and some of the basic properties of mixed norm spaces are presented in Section 2. The extrapolation theory to mixed norm spaces and its applications on Rubio de Francia Littlewood–Paley functions and the geometrical maximal functions are established in Section 3.

2. Definitions

Let $B(x_0, r) = \{x \in \mathbb{R} : |x - x_0| < r\}$ denote the open ball with center $x_0 \in \mathbb{R}$ and radius $r > 0$. Let $\mathbb{B} = \{B(x_0, r) : z \in \mathbb{R}, r > 0\}$.

For any $r, s > 0$ and $z = (x, y) \in \mathbb{R} \times \mathbb{R}$, define $R(z, r, s) = B(x, r) \times B(y, s)$. Write $\mathcal{R} = \{R(z, r, s) : z \in \mathbb{R} \times \mathbb{R}, r, s > 0\}$.

Let $\mathcal{M}$ and L^1_{loc} denote the space of Lebesgue measurable functions and the space of locally integrable functions on $\mathbb{R} \times \mathbb{R}$, respectively.

For any $f \in L^1_{loc}$, the strong maximal operator $M_S f$ is given by

$$M_S f(z) = \sup_{R \ni z} \frac{1}{|R|} \int_R |f(u)| du$$

where the supremum is taken over all $R \in \mathcal{R}$ containing z.

We recall the Muckenhoupt weight function for the Hardy–Littlewood maximal operator.

Definition 1. For $1 < p < \infty$, a locally integrable function $\omega : \mathbb{R} \to [0, \infty)$ is said to be an A_p weight if

$$[\omega]_{A_p} = \sup_{B \in \mathbb{B}} \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} < \infty$$

where $p' = \frac{p}{p-1}$. A locally integrable function $\omega : \mathbb{R} \to [0, \infty)$ is said to be an A_1 weight if for any $B \in \mathbb{B}$

$$\frac{1}{|B|} \int_B \omega(y) dy \leq C\omega(x), \quad a.e.\ x \in B$$

for some constants $C > 0$. The infimum of all such C is denoted by $[\omega]_{A_1}$. Write $A_\infty = \cup_{p \geq 1} A_p$.

We recall the Muckenhoupt weight function for a strong maximal operator from [10, Chapter IV, Section 6].

Definition 2. For $1 < p < \infty$, a locally integrable function $\omega : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is said to be an A_p^* weight if

$$[\omega]_{A_p^*} = \sup_{R \in \mathcal{R}} \left(\frac{1}{|R|} \int_R \omega(x) dx \right) \left(\frac{1}{|R|} \int_R \omega(x)^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} < \infty$$

where $p' = \frac{p}{p-1}$. A locally integrable function $\omega : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is said to be an A_1^* weight if for any $R \in \mathcal{R}$

$$\frac{1}{|R|} \int_R \omega(y) dy \leq C \omega(x), \quad a.e.\ x \in R$$

for some constants $C > 0$. The infimum of all such C is denoted by $[\omega]_{A_1^*}$.

We recall the definition of a Banach function space from [2, Chapter 1, Definitions 1.1 and 1.3].

Definition 3. A Banach space $X \subset \mathcal{M}$ is said to be a Banach function space if it satisfies

(1) $\|f\|_X = 0 \iff f = 0$ a.e.,
(2) $|g| \leq |f|$ a.e. $\Rightarrow \|g\|_X \leq \|f\|_X$,
(3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \|f_n\|_X \uparrow \|f\|_X$,
(4) $\chi_E \in \mathcal{M}$ and $|E| < \infty \Rightarrow \chi_E \in X$,
(5) $\chi_E \in \mathcal{M}$ and $|E| < \infty \Rightarrow \int_E |f(x)| dx < C_E \|f\|_X, \forall f \in X$ for some $C_E > 0$.

We recall the definition of an associate space from [2, Chapter 1, Definitions 2.1 and 2.3].

Definition 4. Let X be a Banach function space. The associate space of X, X', is the collection of all Lebesgue measurable functions f such that

$$\|f\|_{X'} = \sup \left\{ \int_{\mathbb{R}} |f(t)g(t)| dt : g \in X, \|g\|_X \leq 1 \right\} < \infty.$$

According to [2, Chapter 1, Theorems 1.7 and 2.2], when X is a Banach function space, X' is also a Banach function space. The above definition yields the Hölder inequality

$$\int_{\mathbb{R}} |f(x)g(x)| dx \leq \|f\|_X \|g\|_{X'}. \tag{1}$$

In addition, the Lorentz–Luxemburg theorem [2, Chapter 1, Theorem 2.7] guarantees that

$$X = (X')'. \tag{2}$$

Definition 5. Let X_1, X_2 be Banach function spaces. The mixed norm space (X_1, X_2) consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{(X_1,X_2)} = \|\|f\|_{X_1}\|_{X_2} < \infty.$$

As X_1 and X_2 satisfy item (3) of Definition 3, for any $f_n(x,y) \uparrow f(x,y)$, we have $\|f_n(\cdot, y)\|_{X_1} \uparrow \|f(\cdot, y)\|_{X_1}$ and, hence, $\|f_n\|_{(X_1,X_2)} \uparrow \|f\|_{(X_1,X_2)}$.

In view of the Luxemburg–Gribanov theorem [19, 29] and the Luxemburg representation theorem (2), $\|f(\cdot, y)\|_{X_1}$ is Lebesgue measurable, therefore, $\|\|f\|_{X_1}\|_{X_2}$ is well defined.

According to (1), we have

$$\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)g(x,y)|dxdy \leq \int_{\mathbb{R}} \|f(\cdot,y)\|_{X_1}\|g(\cdot,y)\|_{X_1'}dy$$
$$\leq \|\|f\|_{X_1}\|_{X_2}\|\|g\|_{X_1'}\|_{X_2'}.$$

Thus, we obtain the Hölder inequality for mixed norm space:

$$\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)g(x,y)|dxdy \leq \|f\|_{(X_1,X_2)}\|g\|_{(X_1',X_2')}. \tag{3}$$

Let X_1, X_2 be Banach function spaces. The space $(X_1, X_2)'$ consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{(X_1,X_2)'} = \sup_{\substack{g\in(X_1,X_2) \\ \|g\|_{(X_1,X_2)}\leq 1}} \int_{\mathbb{R}\times\mathbb{R}} |f(x,y)g(x,y)|dxdy < \infty.$$

Even though (X_1, X_2) is not necessarily a Banach function space on $\mathbb{R} \times \mathbb{R}$, we also call $(X_1, X_2)'$ the associate space of (X_1, X_2).

We have the following identification of the associate space of (X_1, X_2).

Proposition 1. *Let X_1, X_2 be Banach function spaces. We have*

$$(X_1, X_2)' = (X_1', X_2').$$

Proof. The Hölder inequality (3) gives $(X_1', X_2') \hookrightarrow (X_1, X_2)'$.

We prove the reverse embedding. Let $f \in (X_1, X_2)'$. Let

$$\chi_N = \chi_{B(0,N)\times B(0,N)}\chi_{\{(x,y)\in\mathbb{R}\times\mathbb{R}:|f(x,y)|\leq N\}}$$

and $f_N = |f|\chi_N$. Obviously, $f_N \in (X_1', X_2') \cap (X_1, X_2)'$.

In view of Definition 4, we have

$$\|f_N\|_{(X_1',X_2')} = \sup_{\substack{h\in X_2 \\ \|h\|_{X_2}\leq 1}} \int_{\mathbb{R}} \sup_{\substack{k(\cdot,y)\in X_1 \\ \|k(\cdot,y)\|_{X_1}\leq 1}} \int_{\mathbb{R}} f_N(x,y)k(x,y)dxh(y)dy.$$

Therefore, for any $\epsilon > 0$, there exist $H \in X_2$ with $\|H\|_{X_2} \leq 1$ and $K(\cdot, y) \in X_1$ with $\|K(\cdot, y)\|_{X_1} \leq 1$ such that

$$\|f_N\|_{(X_1',X_2')} \leq (1+\epsilon)\int_{\mathbb{R}\times\mathbb{R}} |f_N(x,y)K(x,y)|dx|H(y)|dy.$$

We find that

$$\|KH\|_{(X_1,X_2)} = \|\|KH\|_{X_1}\|_{X_2} = \|\|K\|_{X_1}H\|_{X_2} \le \|H\|_{X_2} \le 1.$$

Consequently, the Hölder inequality for (X_1, X_2) gives

$$\|f_N\|_{(X_1',X_2')} \le (1+\epsilon)\|f_N\|_{(X_1,X_2)'}\|KH\|_{(X_1,X_2)} \le (1+\epsilon)\|f_N\|_{(X_1,X_2)'}.$$

As $\epsilon > 0$ is arbitrary, we have $\|f_N\|_{(X_1',X_2')} \le \|f_N\|_{(X_1,X_2)'}$. Since $f_N \uparrow |f|$, we have $\|f\|_{(X_1',X_2')} \le \|f\|_{(X_1,X_2)'}$ which gives $(X_1, X_2)' \hookrightarrow (X_1', X_2')$. $\square$

The above result generalizes [3, Theorem 3.12] to Banach function spaces.

Furthermore, the Lorentz–Luxemburg theorem (2) and Proposition 1 assures that

$$(X_1, X_2)'' = (X_1'', X_2'') = (X_1, X_2). \tag{4}$$

Definition 6. Let X_1, X_2 be Banach function spaces. We write $(X_1, X_2) \in \mathbb{M}_S$ if the strong maximal operator M_S is bounded on (X_1, X_2). We write $(X_1, X_2) \in \mathbb{M}$ if the Hardy–Littlewood maximal operator M is bounded on (X_1, X_2).

We recall the definition of the r-convexification for Banach lattices. The reader is referred to [18, Definition 1.a.1] for the definition of a Banach lattice. For any $0 < r < \infty$ and a Banach lattice X, the r-convexification of X, X^r, is defined as

$$X^r = \{f : |f|^r \in X\}.$$

The vector space X^r is equipped with the quasi-norm $\|f\|_{X^r} = \|\,|f|^r\|_X^{1/r}$. The reader is referred to [18, Volume II, pp. 53–54] for the case $1 \le r < \infty$ and [20, Section 2.2] for the general case for further details of the p-convexification. The reader is alerted that in [20], the r-convexification of X is called as the $\frac{1}{r}$-th power of X.

It is easy to see that, for any $r > 0$, we have

$$(X_1, X_2)^r = (X_1^r, X_2^r).$$

3. Main results

Let $p_0 > 0$ and let $\mathcal{F}$ denote a family of ordered pairs of non-negative, Lebesgue measurable functions (f, g). We say that the inequality

$$\int_{\mathbb{R}\times\mathbb{R}} f(x,y)^{p_0}\omega(x,y)dxdy \le C \int_{\mathbb{R}\times\mathbb{R}} f(x,y)^{p_0}\omega(x,y)dxdy$$

holds for any $(f, g) \in \mathcal{F}$ and $\omega_0 \in A_1^*$ if it is valid for any pair in $\mathcal{F}$ such that the left-hand side is finite and the constant C depends only on p_0 and $[\omega_0]_{A_1^*}$.

Theorem 1. *Let X_1, X_2 be Banach function spaces. Given a family $\mathcal{F}$, suppose that, for some $0 < p_0 < \infty$ and for every $\omega_0 \in A_1^*$, we have*

$$\int_{\mathbb{R} \times \mathbb{R}} f(x,y)^{p_0} \omega_0(x,y) dx dy \leq C \int_{\mathbb{R} \times \mathbb{R}} f(x,y)^{p_0} \omega_0(x,y) dx dy$$

for any $(f,g) \in \mathcal{F}$ where C depends only on p_0 and $[\omega_0]_{A_1^*}$.

Suppose that there exists $p_0 \leq q_0 < \infty$ such that X_1^{1/q_0} and X_2^{1/q_0} are Banach function spaces. If

$$((X_1^{1/q_0})', (X_2^{1/q_0})') \in \mathbb{M}_S, \tag{5}$$

then

$$\|f\|_{(X_1, X_2)} \leq C \|g\|_{(X_1, X_2)}. \tag{6}$$

Proof. When $g \notin (X_1, X_2)$, we have $\|g\|_{(X_1, X_2)} = \infty$ and (6) holds. Therefore, we assume that $(f,g) \in \mathcal{F}$ with $g \in (X_1, X_2)$. In view of [11, Theorem 3.1], for any $\omega \in A_1^*$, we have

$$\int_{\mathbb{R} \times \mathbb{R}} f(x,y)^{q_0} \omega(x,y) dx dy \leq C \int_{\mathbb{R} \times \mathbb{R}} f(x,y)^{q_0} \omega(x,y) dx dy. \tag{7}$$

Write $Y_i = X_i^{1/q_0}$, $i = 1, 2$. For any $h \in L_{loc}^1$, define

$$\mathcal{R}_S h = \sum_{k=0}^{\infty} \frac{M_S^k h}{2^k \|M_S\|_{(Y_1', Y_2')}^k}$$

where M_S^k is the k-iteration of M_S, $M_S^0 h = |h|$ and $\|M_S\|_{(Y_1', Y_2')}$ denotes the operator norm of M_S on (Y_1', Y_2').

Since $(Y_1', Y_2') \in \mathbb{M}_S$, the definitions of A_1^* and $\mathcal{R}_S$ yield

$$h \leq \mathcal{R}_S h \tag{8}$$

$$\|\mathcal{R}_S h\|_{(Y_1', Y_2')} \leq 2\|h\|_{(Y_1', Y_2')} \tag{9}$$

$$[\mathcal{R}_S h]_{A_1^*} \leq 2\|M_S\|_{(Y_1', Y_2')}. \tag{10}$$

As $f_N \in (X_1, X_2)$, according to Proposition 1, we have a nonnegative $G \in (Y_1, Y_2)'$ with $\|G\|_{(Y_1, Y_2)'} \leq 1$ such that

$$\|f_N\|_{(X_1, X_2)}^{q_0} = \||f_N|^{q_0}\|_{(Y_1, Y_2)} \leq 2 \int_{\mathbb{R} \times \mathbb{R}} (f_N(\cdot, y))^{q_0} G(x,y) dx dy$$

$$\leq 4 \int_{\mathbb{R} \times \mathbb{R}} (f(x,y))^{q_0} \mathcal{R}_S G(x,y) dx dy$$

$$\leq 4 \int_{\mathbb{R} \times \mathbb{R}} (g(x,y))^{q_0} \mathcal{R}_S G(x,y) dx dy$$

where we use (7) and (10) in the last inequality.

The Hölder inequality and (9) give

$$\|f_N\|_{(X_1,X_2)}^{q_0} \leq 4\|g^{q_0}\|_{(Y_1,Y_2)}\|\mathcal{R}_S G\|_{(Y_1,Y_2)'}$$
$$\leq C\|g\|_{(X_1,X_2)}^{q_0}\|G\|_{(Y_1,Y_2)'} \leq C\|g\|_{(X_1,X_2)}^{q_0}.$$

As $f_N \uparrow f$, we have established (6). $\square$

The above result corrects a technical mistake in [11, Theorem 3.2]. In [11, Theorem 3.2], it is stated that there exist h_1, h_2 such that

$$\|f\|_{(X_1,X_2)} \leq 4 \int_{\mathbb{R} \times \mathbb{R}} f(x,y)^{q_0} h_1(x) h_2(y) dx dy.$$

The above inequality is not valid in general. Consequently, the assumption in [11, Theorem 3.2] is insufficient to guarantee the validity of the results in [11, Theorem 3.2]. Theorem 1 corrects this mistake and gives the extrapolation theory for the mixed norm spaces for (X_1, X_2) satisfying (5).

There are a number of mixed norm spaces satisfying (5). For instance, the mixed norm Lebesgue spaces with variable exponents $(L^{p(\cdot)}, L^q)$ and the mixed norm Lorentz spaces $(L_{p_1,q_1}, L_{p_2,q_2})$. We will obtain these results at the end of this section.

In order to apply the above result to (X_1, X_2), we need to show that a subset of this space, for example, the space of bounded functions with compact support, is dense in (X_1, X_2). We can further refine the above result so that this density argument is not required. We use the idea from [15] to establish the following result.

Theorem 2. *Let* $p_0 \in (0, \infty)$, X_1, X_2 *be Banach function spaces. If* X_1^{1/p_0} *and* X_2^{1/p_0} *are Banach function spaces,* $((X_1^{1/p_0})', (X_2^{1/p_0})') \in \mathbb{M}_S$ *and for every*

$$\omega \in \{\mathcal{R}_S h : h \in (X_1^{1/p_0}, X_2^{1/p_0})'\},$$

the operator $T : L^{p_0}(\omega) \to L^{p_0}(\omega)$ *is bounded, then* $T : (X_1, X_2) \to (X_1, X_2)$ *is bounded.*

Proof. Let $f \in (X_1, X_2)$. for any $h \in (X_1^{1/p_0}, X_2^{1/p_0})'$, (9) yields

$$\int_{\mathbb{R} \times \mathbb{R}} |f(x,y)|^{p_0} \mathcal{R}_S h(x,y) dx dy$$
$$\leq \||f|^{p_0}\|_{(X_1^{1/p_0}, X_2^{1/p_0})} \|\mathcal{R}_S h\|_{(X_1^{1/p_0}, X_2^{1/p_0})'} \leq \|f\|_{(X_1,X_2)} \|h\|_{(X_1^{1/p_0}, X_2^{1/p_0})'}.$$

That is,

$$(X_1, X_2) \hookrightarrow \bigcap_{h \in (X_1^{1/p_0}, X_2^{1/p_0})'} L^{p_0}(\mathcal{R}_S h). \tag{11}$$

For any $h \in (X_1^{1/p_0}, X_2^{1/p_0})'$, (8) gives

$$\int_{\mathbb{R} \times \mathbb{R}} |Tf(x,y)|^{p_0} |h(x,y)| dx dy \leq \int_{\mathbb{R} \times \mathbb{R}} |Tf(x,y)|^{p_0} \mathcal{R}_S h(x,y) dx dy.$$

The boundedness of $T : L^{p_0}(\mathcal{R}_S h) \to L^{p_0}(\mathcal{R}_S h)$ and the embedding (11) yield

$$\int_{\mathbb{R} \times \mathbb{R}} |Tf(x,y)|^{p_0} |h(x,y)| dx dy \leq C \int_{\mathbb{R} \times \mathbb{R}} |f(x,y)|^{p_0} \mathcal{R}_S h(x,y) dx dy.$$

Consequently, the Hölder inequality and (9) assert that

$$\int_{\mathbb{R} \times \mathbb{R}} |Tf(x,y)|^{p_0} |h(x,y)| dx dy$$
$$\leq C \||f|^{p_0}\|_{(X_1^{1/p_0}, X_2^{1/p_0})} \|\mathcal{R}_S h\|_{(X_1^{1/p_0}, X_2^{1/p_0})'} \leq C \|f\|_{(X_1, X_2)}^{p_0} \|h\|_{(X_1^{1/p_0}, X_2^{1/p_0})'}.$$

By taking the supremum over $h \in (X_1^{1/p_0}, X_2^{1/p_0})'$ with $\|h\|_{(X_1^{1/p_0}, X_2^{1/p_0})'} \leq 1$, (4) yields

$$\begin{aligned}
\|Tf\|_{(X_1, X_2)}^{p_0} &= \||Tf|^{p_0}\|_{(X_1^{1/p_0}, X_2^{1/p_0})} \\
&= \||Tf|^{p_0}\|_{(X_1^{1/p_0}, X_2^{1/p_0})''} \\
&= \sup_{\|h\|_{(X_1^{1/p_0}, X_2^{1/p_0})'} \leq 1} \int_{\mathbb{R} \times \mathbb{R}} |Tf(x,y)|^{p_0} |h(x,y)| dx dy \\
&\leq C \|f\|_{(X_1, X_2)}^{p_0},
\end{aligned}$$

which establishes the boundedness of $T : (X_1, X_2) \to (X_1, X_2)$. $\square$

The embedding (11) is crucial for the above theorem. With this embedding, we can get rid of the density argument. We can directly use the boundedness of $T : L^{p_0}(\omega) \to L^{p_0}(\omega)$ to obtain our desired result. Notice that in the above theorem, we do not require T to satisfy any linearity assumption. Therefore, the above theorem applies to nonlinear operators.

The above extrapolation theory requires an assumption for the boundedness of the strong maximal operator. We have another extrapolation theory for mixed norm spaces which requires the boundedness of the Hardy–Littlewood maximal operator.

Theorem 3. *Let* $p_0 \in (0, \infty)$, X_1, X_2 *be Banach function spaces. If* X_1^{1/p_0} *and* X_2^{1/p_0} *are Banach function spaces,* $((X_1^{1/p_0})', (X_2^{1/p_0})') \in \mathbb{M}$ *and for every*

$$\omega \in \{\mathcal{R} h : h \in (X_1^{1/p_0}, X_2^{1/p_0})'\},$$

where

$$\mathcal{R} h = \sum_{k=0}^{\infty} \frac{M^k h}{2^k \|M\|_{(Y_1', Y_2')}^k}$$

and $Y_i = X_i^{1/p_0}$, $i = 1, 2$, the operator $T : L^{p_0}(\omega) \to L^{p_0}(\omega)$ is bounded, then $T : (X_1, X_2) \to (X_1, X_2)$ is bounded.

As the proof of the preceding theorem is similar to the proof of Theorem 2, for brevity, we omit the details. The reader may consult [15, Theorem 3.3] for the proof of the analogue of the preceding results on Morrey–Banach spaces.

Similarly to (10), we see that $\{\mathcal{R}h : h \in (X_1^{1/p_0}, X_2^{1/p_0})'\} \subset A_1$. Since $A_1^* \subset A_1$, the above theorem requires a relaxed condition on the function spaces, namely $((X_1^{1/p_0})', (X_2^{1/p_0})') \in \mathbb{M}$, while it imposes a stronger condition on the weight functions.

We use Theorem 2 to study the mapping properties of the Rubio de Francia Littlewood–Paley function and the geometric maximal function on mixed norm spaces.

We first consider the Rubio de Francia Littlewood–Paley function. Let $\mathcal{S}'$ be the space of Schwartz distributions on $\mathbb{R} \times \mathbb{R}$. For any $f \in \mathcal{S}'$, the Fourier transform of f is denoted by $\hat{f}$.

Let $W = \{R_j\}_{j \in \mathbb{N}}$ be a set of disjoint rectangles in $\mathbb{R} \times \mathbb{R}$ with sides parallel to the coordinate axes. The Littlewood–Paley function associated with W is given by

$$\triangle_W f(x) = \Big(\sum_{R_j \in W} |S_{R_j} f(x)|^2 \Big)^{1/2}$$

where $(S_{R_j} f)\hat{} = \chi_{R_j} \hat{f}$. The operator $\triangle_W$ is the extension of the Littlewood–Paley function for arbitrary intervals, introduced by Rubio de Francia in [22], to the product domain $\mathbb{R} \times \mathbb{R}$. The following result for $\triangle_W$ is given in [19, Section 4].

Theorem 4. *Let $p \in (2, \infty)$. If $\omega \in A_{p/2}^*$, then*

$$\int_{\mathbb{R} \times \mathbb{R}} |\triangle_W f(x, y)|^p \omega(x, y) dx dy \le C \int_{\mathbb{R} \times \mathbb{R}} |f(x, y)^p \omega(x, y) dx dy.$$

We have the following mapping properties of $\triangle_W$ on mixed norm spaces.

Theorem 5. *Let $p_0 \in (2, \infty)$, X_1, X_2 be Banach function spaces. If X_1^{1/p_0} and X_2^{1/p_0} are Banach function spaces and $(X_1^{1/p_0}, X_2^{1/p_0})' \in \mathbb{M}_S$, then there exists a constant $C > 0$ such that for any $f \in (X_1, X_2)$,*

$$\|\triangle_W f\|_{(X_1, X_2)} \le C \|f\|_{(X_1, X_2)}.$$

Proof. In view of (10), we have

$$\{\mathcal{R}_S h : h \in (X_1^{1/p_0}, X_2^{1/p_0})'\} \subset A_1^* \subset A_{p_0/2}^*.$$

Theorem 4 assures that for every $\omega \in \{\mathcal{R}_S h : h \in (X_1^{1/p_0}, X_2^{1/p_0})'\}$, the operator $\triangle_W : L^{p_0}(\omega) \to L^{p_0}(\omega)$ is bounded. Therefore, we are allowed to

apply Theorem 2 to obtain the boundedness of $\triangle_W : (X_1, X_2) \to (X_1, X_2)$.
$\square$

Our extrapolation theory, Theorem 2, applies to those operators having the weighted norm inequality for weights belonging to A_1^*. The Rubio de Francia Littlewood–Paley function $\triangle_W$ is an example of this class of operators. As $A_1^* \subset A_1$, Theorem 2 applies to those operators having the weighted norm inequality for A_1. The next examples, the geometric maximal functions, are nonlinear operators which possess the weighted norm inequalities with weights belonging to A_1.

For any Lebesgue measurable function f on $\mathbb{R} \times \mathbb{R}$, the geometric maximal function $M_0 f$ is defined as

$$M_0 f(x) = \sup_I \exp \left(\frac{1}{|I|} \int_I \log |f(y)| dy \right)$$

where the supremum is taken over all cubes I containing x with its sides parallel to the coordinates axes. For any locally integrable function f, M_0^* is defined by

$$M_0^* f(x) = \lim_{r \to 0} (M(|f|^r))^{1/r}(x)$$

where M is the Hardy–Littlewood maximal function.

The most remarkable feature for the geometrical maximal operators is that they are not linear, not sublinear nor quasi-linear while Theorem 2 does not only apply to linear operators or sublinear operators. Theorem 2 also applies to the geometrical maximal operators.

We have the following weighted norm inequalities for M_0 and M_0^* from [26], [5, Theorem 1.7], respectively.

Theorem 6. *Let $p \in [1, \infty)$ and $\omega \in A_\infty$. We have a constant $C > 0$ such that for any $f \in L^p(\omega)$*

$$\int_{\mathbb{R}^2} (M_0 f(x))^p \omega(x) dx \le C \int_{\mathbb{R}^2} |f(x)|^p \omega(x) dx.$$

Theorem 7. *Let $p \in [1, \infty)$ and $\omega \in A_\infty$. We have a constant $C > 0$ such that for any $f \in L^p(\omega)$*

$$\int_{\mathbb{R}^2} (M_0^* f(x))^p \omega(x) dx \le C \int_{\mathbb{R}^2} |f(x)|^p \omega(x) dx, \quad \forall f \in L^p(\omega).$$

Theorem 8. *Let X_1, X_2 be Banach function spaces. If $(X_1, X_2)' \in \mathbb{M}_S$, then*

$$\|M_0 f\|_{(X_1, X_2)} \le C \|f\|_{(X_1, X_2)},$$
$$\|M_0^* f\|_{(X_1, X_2)} \le C \|f\|_{(X_1, X_2)}.$$

Since $A_1^* \subset A_1 \subset A_\infty$, the above results follow from Theorems 2, 6 and 7.

3.1. Lebesgue spaces with variable exponents. This section establish the boundedness of the strong maximal operator on the mixed norm Lebesgue spaces with variable exponent $(L^{p(\cdot)}, L^q)$. It is a consequence of the UMD property satisfied by the variable exponents obtained recently in [17].

For brevity, the reader is referred to [16, Definition 4.2.1] for the definition of the UMD property. For more information on the UMD property, see [16, Chapter 4].

We now give the definition of Lebesgue spaces with variable exponents. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ be a Lebesgue measurable function. Define

$$p_- = \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^n\} \text{ and } p_+ = \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^n\}.$$

Definition 7. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ be a Lebesgue measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \rho(|f(x)|/\lambda) \le 1\right\} < \infty$$

where

$$\rho(f) = \int_{\{x \in \mathbb{R}^n : p(x) \ne \infty\}} |f(x)|^{p(x)} dx + \|f \chi_{\{x \in \mathbb{R}^n : p(x) = \infty\}}\|_{L^\infty}.$$

We call $p(x)$ the exponent function of $L^{p(\cdot)}$.

According to [7, Theorem 3.2.13], $L^{p(\cdot)}$ is a Banach function space. In addition, we have $(L^{p(\cdot)})' = L^{p'(\cdot)}$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ [7, Theorem 3.2.13]. The reader is referred to [7, Section 3] for more information on Lebesgue spaces with variable exponents.

We recall a well-,known example on the exponent function $p(\cdot)$ for which $p(\cdot) \in \mathbb{M}$. We recall the class of log-Hölder continuity functions $C^{\log}$, [7, Definition 2.1]. We write $p(\cdot) \in C^{\log}$ if it satisfies

$$|p(x) - p(y)| \le C \frac{1}{\log(1/|x - y|)}, \quad |x - y| \le \frac{1}{2}, \tag{12}$$

$$|p(x) - p(y)| \le C \frac{1}{\log(e + |x|)}, \quad |y| \ge |x|. \tag{13}$$

It is easy to see that

$$p(\cdot) \in C^{\log} \Leftrightarrow p'(\cdot) \in C^{\log}. \tag{14}$$

In view of [7, Theorem 4.3.8], whenever $p(\cdot) \in C^{\log}$ and $p_- > 1$, then $L^{p(\cdot)} \in \mathbb{M}$.

We now establish the boundedness of the strong maximal operator on the mixed norm Lebesgue spaces with variable exponents $(L^{p(\cdot)}, L^q)$.

Let M_1, M_2 be the Hardy–Littlewood maximal operators on $\mathbb{R}$ with respect to x and on $\mathbb{R}$ with respect to y, respectively, $z = (x,y) \in \mathbb{R} \times \mathbb{R}$. It is well known that

$$M_S f(x,y) \leq M_1(M_2 f)(x,y), \tag{15}$$

see [10, p.452].

Theorem 9. *Let $q \in (1,\infty)$ and $p : \mathbb{R} \to (1,\infty]$ be a Lebesgue measurable function. If $L^{p(\cdot)} \in \mathbb{M}$ with $1 < p_- \leq p_+ < \infty$, then $(L^{p(\cdot)}, L^q) \in \mathbb{M}_S$.*

Proof. According to (15), we have

$$\|\|M_S f\|_{L^{p(\cdot)}}\|_{L^q} \leq \|\|M_1(M_2 f)\|_{L^{p(\cdot)}}\|_{L^q}.$$

As $p_1(\cdot) \in \mathbb{M}$, we obtain

$$\|\|M_1(M_2 f)\|_{L^{p(\cdot)}}\|_{L^q} \leq C \|\|M_2 f\|_{L^{p(\cdot)}}\|_{L^q}$$

$$= C \left(\int_{\mathbb{R}} \|M_2 f(\cdot, y)\|_{L^{p(\cdot)}}^q dy \right)^{1/q}$$

for some $C > 0$.

Since $1 < p_- \leq p_+ < \infty$, [17, Corollary 3.6] guarantees that $L^{p_1(\cdot)}$ has UMD property. In view of [10, Chapter V, Section 7.6] and [21, Theorem 3], we have

$$\|\|M_1(M_2 f)\|_{L^{p(\cdot)}}\|_{L^q} \leq C \left(\int_{\mathbb{R}} \|M_2 f(\cdot, y)\|_{L^{p(\cdot)}}^q dy \right)^{1/q}$$

$$\leq C \left(\int_{\mathbb{R}} \|f(\cdot, y)\|_{L^{p(\cdot)}}^q dy \right)^{1/q}.$$

That is, $(L^{p(\cdot)}, L^q) \in \mathbb{M}_S$. □

We can use the idea of the proof of the preceding theorem to establish the boundedness of M on $(L^{p(\cdot)}(\mathbb{R}^{n_1}), L^q(\mathbb{R}^{n_2}))$, $n_i \in \mathbb{N}$, $i = 1, 2$ because $Mf \leq M_1(M_2 f)$ where M is the Hardy–Littlewood maximal operator on $\mathbb{R}^{n_1+n_2}$ and M_i are the Hardy–Littlewood maximal operators on $\mathbb{R}^{n_i}$, $i = 1, 2$, respectively. Thus, we can apply Theorem 3 to $(L^{p(\cdot)}(\mathbb{R}^{n_1}), L^q(\mathbb{R}^{n_2}))$ provided that $L^{p(\cdot)}(\mathbb{R}^{n_1}) \in \mathbb{M}$ and $q \in (1,\infty)$.

As [11, Theorem 4.3] relies on [11, Theorem 3.2], the boundedness of the strong maximal operator on $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ are still open while the preceding theorem assures that the strong maximal operator are bounded on the mixed norm Lebesgue spaces with variable exponent $(L^{p(\cdot)}, L^q)$. In addition, for those results in [12, 13] that rely on [11, Theorem 3.2], in view of Theorem 1, they are valid for the mixed norm Lebesgue spaces with variable exponent $(L^{p(\cdot)}, L^q)$. The results for $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ are still open.

The above discussion inspires an open question. Whenever $p_1(\cdot), p_2(\cdot) \in C^{\log}$, whether we have a constant $C > 0$ such that for any $f \in (L^{p_1(\cdot)}, L^{p_2(\cdot)})$

$$\|\|M_2 f(\cdot, y)\|_{L^{p_1(\cdot)}}\|_{L^{p_2(\cdot)}} \leq C\|\|f\|\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})}. \qquad (16)$$

In view of the extrapolation theory for Lebesgue spaces with variable exponent [7, Theorem 7.2.1], we find that if for any $\omega \in A_1$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} \|M_2 f(\cdot, y)\|_{L^{p_1(\cdot)}} \omega(y) dy \leq C \int_{\mathbb{R}} \|f(\cdot, y)\|_{L^{p_1(\cdot)}} \omega(y) dy,$$

then (16) holds.

In addition, a further generalization of the above question is that if X is a Banach function space having UMD property, for any $\omega \in A_1$, whether we have

$$\int_{\mathbb{R}} \|M_2 f(\cdot, y)\|_X \omega(y) dy \leq C \int_{\mathbb{R}} \|f(\cdot, y)\|_X \omega(y) dy.$$

We have the following results for the Rubio de Francia Littlewood–Paley function and the geometric maximal function on mixed norm Lebesgue spaces with variable exponents.

Theorem 10. *Let $p(\cdot)$ be a Lebesgue measurable function and $q \in (1, \infty)$.*

(1) If $q \in (2, \infty)$ and $p(\cdot) \in C^{\log}$ with $2 < p_- \leq p_+ < \infty$, then there exists a constant $C > 0$ such that

$$\|\triangle_W f\|_{(L^{p(\cdot)}, L^q)} \leq C\|f\|_{(L^{p(\cdot)}, L^q)}.$$

(2) If $p(\cdot) \in C^{\log}$ with $1 \leq p_- \leq p_+ < \infty$, then there exists a constant $C > 0$ such that

$$\|M_0 f\|_{(L^{p(\cdot)}, L^q)} \leq C\|f\|_{(L^{p(\cdot)}, L^q)},$$
$$\|M_0^* f\|_{(L^{p(\cdot)}, L^q)} \leq C\|f\|_{(L^{p(\cdot)}, L^q)}.$$

Proof. Take $p_0 \in (2, \min(p_-, q))$. Since $p(\cdot) \in C^{\log}$, we find that $(p(\cdot)/p_0)' \in C^{\log}$ and $(p(\cdot)/p_0)'_- > 1$. Proposition 1 and Theorem 9 assure that $(L^{(p(\cdot)/p_0)}, L^{q/p_0})' = (L^{(p(\cdot)/p_0)'}, (L^{q/p_0})') \in \mathbb{M}_S$. Therefore, Theorem 5 gives the boundedness of $\triangle$ on $(L^{p(\cdot)}, L^q)$.

Similarly, since $p'(\cdot) \in C^{\log}$ with $(p'(\cdot))_- > 1$, we find that $(L^{p(\cdot)}, L^q)' = (L^{p'(\cdot)}, L^{q'}) \in \mathbb{M}_S$. Thus, Theorem 8 yields the boundedness of M_0 and M_0^* on $(L^{p(\cdot)}, L^q)$. $\qquad \square$

3.2. Lorentz spaces. In this section, we apply Theorem 2 to the mixed norm spaces generated by Lorentz spaces. We obtain our result by showing that (16) is valid when $L^{p_1(\cdot)}$ and $L^{p_2(\cdot)}$ are replaced by the Lorentz spaces.

We briefly recall the definition of Lorentz spaces from [2, Chapter 4, Definition 4.1]. For any Lebesgue measurable function f, define $\mu_f(\lambda) = |\{x \in \mathbb{R} : |f(x)| > \lambda\}|$, $\lambda \geq 0$ and $f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}$, $t \geq 0$.

Definition 8. Let $p, q \in (1, \infty)$. The Lorentz space $L_{p,q}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty (t^{\frac{1}{p} - \frac{1}{q}} f^*(t))^q dt \right)^{1/q} < \infty.$$

The Lorentz space is a Banach function space. For the studies of the mixed norm Lorentz spaces, the reader is referred to [8].

We now show that the strong maximal operator is bounded on the mixed norm Lorentz space $(L_{p_1,q_1}, L_{p_2,q_2})$.

Theorem 11. *Let* $p_i, q_i \in (1, \infty)$, $i = 1, 2$. *We have* $(L_{p_1,q_1}, L_{p_2,q_2}) \in \mathbb{M}_S$.

Proof. Let $r \in (1, \infty)$. As $p_1, q_1 \in (1, \infty)$, the Hardy–Littlewood maximal operator M_1 is bounded on L_{p_1,q_1}. Therefore, we have

$$\|\|M_1(M_2 f)\|_{L_{p_1,q_1}}\|_{L^r} \le C \|\|M_2 f\|_{L_{p_1,q_1}}\|_{L^r}$$

$$= C \left(\int_{\mathbb{R}} \|M_2 f(\cdot, y)\|_{L_{p_1,q_1}}^r dy \right)^{1/r}$$

for some $C > 0$.

Since L_{p_1,q_1} has UMD property, we have

$$\left(\int_{\mathbb{R}} \|M_2 f(\cdot, y)\|_{L_{p_1,q_1}}^r dy \right)^{1/r} \le C \left(\int_{\mathbb{R}} \|f(\cdot, y)\|_{L_{p_1,q_1}}^r dy \right)^{1/r}. \tag{17}$$

Let $q \in (0, \infty)$, $\theta \in (0, 1)$ and $(\cdot, \cdot)_{\theta, q}$ be the real interpolation functor [28, Section 1.3.2]. In view of the interpolation of vector-valued Lorentz spaces [28, Section 1.18.6, Theorem 2], we have

$$((L_{p_1,q_1}, L^{r_1}), (L_{p_1,q_1}, L^{r_2}))_{\theta, q_2} = (L_{p_1,q_1}, L_{p_2,q_2}) \tag{18}$$

where $\frac{1}{p_2} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}$. Notice that in [28], the mixed norm space (L_{p_1,q_1}, L^{r_1}) is denoted by $L^{r_1}(L_{p_1,p_2})$.

According to (17), $M_2 : (L_{p_1,q_1}, L^r) \to (L_{p_1,q_1}, L^r)$ is a bounded sublinear operator, (18) assures that $M_2 : (L_{p_1,q_1}, L_{p_2,q_2}) \to (L_{p_1,q_1}, L_{p_2,q_2})$ is bounded. Consequently, as M_1 is bounded on L_{p_1,q_1}, we find that

$$\|\|M_S f\|_{L_{p_1,q_1}}\|_{L_{p_2,q_2}} \le \|\|M_1(M_2 f)\|_{L_{p_1,q_1}}\|_{L_{p_2,q_2}}$$

$$\le C \|\|M_2 f\|_{L_{p_1,q_1}}\|_{L_{p_2,q_2}}$$

$$\le C \|\|f\|_{L_{p_1,q_1}}\|_{L_{p_2,q_2}}$$

for some $C > 0$. $\qquad \square$

The above result shows that the mixed norm Lorentz spaces belong to $\mathbb{M}_S$. Therefore, we are allowed to apply Theorem 2 to obtain the following mapping properties on mixed norm Lorentz spaces.

Theorem 12. *Let $p_i, q_i \in (1, \infty)$, $i = 1, 2$.*

(1) If $p_i, q_i \in (2, \infty)$, $i = 1, 2$, then there exists a constant $C > 0$ such that

$$\|\triangle_W f\|_{(L_{p_1,q_1}, L_{p_2,q_2})} \le C\|f\|_{(L_{p_1,q_1}, L_{p_2,q_2})}.$$

(2) There exists a constant $C > 0$ such that

$$\|M_0 f\|_{(L_{p_1,q_1}, L_{p_2,q_2})} \le C\|f\|_{(L_{p_1,q_1}, L_{p_2,q_2})},$$
$$\|M_0^* f\|_{(L_{p_1,q_1}, L_{p_2,q_2})} \le C\|f\|_{(L_{p_1,q_1}, L_{p_2,q_2})}.$$

The above results follow from the duality $L_{p,q} = L_{p',q'}$ [2, Chapter 4, Corollary 4.8], Theorems 5, 8 and 11. For brevity, we leave the details to the readers.

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Department of Mathematics and Information Technology, The Education University of Hong Kong, 10 Lo Ping Road, Tai Po, Hong Kong, China
E-mail address: vkpho@eduhk.hk