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Weak module amenability for the second dual of a Banach algebra

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Abstract. In this paper we study the weak module amenability of Banach algebras which are Banach modules over another Banach algebra with compatible actions. We show that for every module derivation $D : A \longmapsto (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$ if $D^{**}(\mathcal{A}^{**}) \subseteq WAP(\frac{\mathcal{A}}{J_{\mathcal{A}}})$, then weak module amenability of $\mathcal{A}^{**}$ implies that of $\mathcal{A}$. Also we prove that under certain conditions for the module derivation D, if $\mathcal{A}^{**}$ is weak module amenable then $\mathcal{A}$ is also weak module amenable.

1. Introduction

The concept of amenability for Banach algebras was introduced by B. E. Johnson [13]. He showed that the group algebra $L^1(G)$ is amenable if and only if G as a group is amenable. Subsequently, various generalizations of this notion − such as Banach modules − were studied by a number of authors (see [1, 5, 6, 10, 16]). Amini [1] used this fact and developed the concept of module amenability for a Banach algebra $\mathcal{A}$ to the case where there is an extra $\mathfrak{A}$-module structure on $\mathcal{A}$. He showed that for an inverse semigroup S with the set of idempotents E, $l^1(S)$ is $l^1(E)$-module amenable if and only if S is amenable.

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-module. A *derivation* $D : \mathcal{A} \longrightarrow X$ is a bounded linear operator such that, $D(ab) = a.D(b)+D(a).b$ for all $a, b \in \mathcal{A}$. Also, a derivation D is said to be *inner* if there exists $x \in X$ such that $D(a) = a.x - x.a$ for every $a \in A$.

A Banach algebra $\mathcal{A}$ is said to be *amenable* if every derivation from $\mathcal{A}$ into each dual Banach $\mathcal{A}$-module is inner [13].

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A Banach algebra $\mathcal{A}$ is called *weakly amenable* [2] if every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is inner. F. Gourdeau has shown that the amenability of $\mathcal{A}^{**}$ implies the amenability of $\mathcal{A}$ [12]. However, for weak amenability this result is not proved yet.

Problem: Let $\mathcal{A}^{**}$ be a weakly amenable Banach algebra Can we conclude that $\mathcal{A}$ is also weakly amenable?

The above problem has been solved in certain settings. For example, in each of the following cases the above problem has a positive answer.

1) $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$ [11].

2) $\mathcal{A}$ is a dual Banach algebra [4, 10].

3) Every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ satisfies $D^{**}(\mathcal{A}^{**}) \subseteq WAP(\mathcal{A})$ [9].

4) $\mathcal{A}$ is a right ideal of $\mathcal{A}^{**}$ with $\mathcal{A}^{**}\Box\mathcal{A} = \mathcal{A}^{**}$ [9].

In Section 2, we prove that if $\mathcal{A}$ is a Banach $\mathfrak{A}$-module and $J_{\mathcal{A}}$ is the closed ideal of $\mathcal{A}$ generated by $\{(a.\alpha)b - a(\alpha.b) : a, b \in \mathcal{A}, \alpha \in \mathfrak{A}\}$ and every derivation $D : \mathcal{A} \longrightarrow (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$ satisfies $D^{**}(\mathcal{A}^{**}) \subseteq WAP(\frac{\mathcal{A}}{J_{\mathcal{A}}})$, then weak module amenability of $\mathcal{A}^{**}$ implies weak module amenability of $\mathcal{A}$. The case $\mathfrak{A} = \mathbb{C}$ yields our main theorem. In fact this result can be considered as an extension of [9, Theorem 2.1]. Also, we show that if $\mathcal{A}$ is a Banach $\mathfrak{A}$-module, such that $J_{\mathcal{A}}^{\perp\perp} \subseteq \mathcal{A}$, $\mathcal{A}^{**}.\frac{\mathcal{A}}{J_{\mathcal{A}}^{\perp\perp}} = \frac{\mathcal{A}^{**}}{J_{\mathcal{A}}^{\perp\perp}}$ and $\frac{\mathcal{A}}{J_{\mathcal{A}}^{\perp\perp}}.\mathcal{A}^{**} \subseteq \frac{\mathcal{A}}{J_{\mathcal{A}}^{\perp\perp}}$, then weak module amenability of $\mathcal{A}^{**}$ implies weak module amenability of $\mathcal{A}$. This result can be considered as an extension of [9, Theorem 2.4].

2. Main results

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras and let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module such that

$$(\alpha.a)b = \alpha.(ab), \ (ab).\alpha = a(b.\alpha) \qquad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

If X is a Banach $\mathcal{A}$-module and a Banach $\mathfrak{A}$-module with compatible actions, such that

$$\alpha.(a.x) = (\alpha.a).x, \ (a.x).\alpha = a.(x.\alpha) \quad (a \in \mathcal{A}, x \in X, \alpha \in \mathfrak{A}),$$

and with similar operations for right actions, then X is called an $\mathcal{A}$-$\mathfrak{A}$-*module*. If, moreover,

$$\alpha.x = x.\alpha \qquad (\alpha \in \mathfrak{A}, x \in X),$$

then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-*module*.

If X is a (commutative) Banach $\mathcal{A}$-$\mathfrak{A}$-module so is X^*, with the following actions:

$$\langle \alpha.f, x \rangle = \langle f, x.\alpha \rangle, \ \langle f.\alpha, x \rangle = \langle f, \alpha.x \rangle$$
$$\langle a.f, x \rangle = \langle f, x.a \rangle, \ \langle f.a, x \rangle = \langle f, a.x \rangle \qquad (a \in \mathcal{A}, x \in X, \alpha \in \mathfrak{A}, f \in X^*).$$

Let X and Y be $\mathcal{A}$-$\mathfrak{A}$-modules, and let $\phi : X \to Y$ satisfies the following conditions:

$$\phi(\alpha.x) = \alpha.\phi(x),\ \phi(x.\alpha) = \phi(x).\alpha$$
$$\phi(a.x) = a.\phi(x),\ \phi(x.a) = \phi(x).a \qquad (a \in \mathcal{A}, x \in X, \alpha \in \mathfrak{A}).$$

Then ϕ is called a *module bihomomorphism.*

Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then the projective tensor product $\mathcal{A}\hat{\otimes}X$ is an $\mathcal{A}$-$\mathfrak{A}$-module with the following actions:

$$a.(b \otimes x) = (ab) \otimes x,\ (b \otimes x).a = b \otimes (x.a)$$
$$\alpha.(b \otimes x) = (\alpha.b) \otimes x,\ (b \otimes x).\alpha = b \otimes (x.\alpha) \qquad (a, b \in \mathcal{A}, x \in Y, \alpha \in \mathfrak{A}).$$

Now, define $\pi_X : \mathcal{A}\hat{\otimes}X \to X$ by

$$\pi_X(a \otimes x) = a.x \qquad (a \in \mathcal{A}, x \in X).$$

It is clear that π_X is an $\mathcal{A}$-$\mathfrak{A}$-module bihomomorphism.

Let I_X be the closed $\mathcal{A}$-$\mathfrak{A}$-submodule of the projective tensor product $A\hat{\otimes}X$ generated by

$$\{(a.\alpha) \otimes x - a \otimes (\alpha.x) : a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X\}.$$

Also let J_X be the closed submodule of X generated by $\pi(I_X)$, that is,

$$J_X = \overline{\langle \pi_X(I_X) \rangle} = \overline{\{(a.\alpha).x - a.(\alpha.x) : a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X\}}.$$

In a particular case, when $X = \mathcal{A}$, $J_{\mathcal{A}}$ is the closed submodule in $\mathcal{A}$ generated by $\{(a.\alpha)b - a(\alpha.b)\}$ for $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$.

Definition 1. The closed $\mathcal{A}$-$\mathfrak{A}$-modules ${J_X}^{\perp}$ of X^* and $J^{\perp}_{{J_X}^{\perp}}$ of X^{**} are called the *first and the second module dual* of X, respectively.

In the case when X is a commutative $\mathfrak{A}$-module, then ${J_X}^{\perp} = X^*$ and $J^{\perp}_{{J_X}^{\perp}} = X^{**}$.

Remark 1. Since $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^* \simeq {J_{\mathcal{A}}}^{\perp}$, we have

$$\langle \tilde{f}, a + J_{\mathcal{A}} \rangle = \langle f, a \rangle \qquad (a \in A),$$

when $f \in {J_{\mathcal{A}}}^{\perp}$ is the corresponding element $\tilde{f} \in (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$. Since $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^{**} \simeq \frac{A^{**}}{{J_{\mathcal{A}}}^{\perp\perp}}$, we have

$$\langle \tilde{F}, \tilde{f} \rangle = \langle F, f \rangle \qquad (\tilde{f} \simeq f \in {J_{\mathcal{A}}}^{\perp}),$$

where $F + {J_{\mathcal{A}}}^{\perp\perp} \in \frac{A^{**}}{{J_{\mathcal{A}}}^{\perp\perp}}$ is the corresponding element $\tilde{F} \in (\frac{\mathcal{A}}{J_{\mathcal{A}}})^{**}$.
Note that $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$ is an $\mathcal{A}$-module, where the actions of $\mathcal{A}$ on $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$ are defined by:

$$\langle \tilde{f}.a, b + J_{\mathcal{A}} \rangle = \langle \tilde{f}, ab + J_{\mathcal{A}} \rangle,\ \langle a.\tilde{f}, b + J_{\mathcal{A}} \rangle = \langle \tilde{f}, ba + J_{\mathcal{A}} \rangle \quad (a, b \in \mathcal{A}, \tilde{f} \in (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*).$$

Therefore the second module dual of $\mathcal{A}$ is a closed submodule of $\frac{\mathcal{A}^{**}}{J_{\mathcal{A}}{}^{\perp\perp}}$.

Definition 2. Let $\mathcal{A}$ and $\mathfrak{A}$ be two Banach algebras and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A bounded linear map $D : \mathcal{A} \to X$ is a *module derivation* if D satisfies the following relations:

$$D(ab) = D(a).b + a.D(b),$$
$$D(\alpha.a) = \alpha.D(a),\ D(a.\alpha) = D(a).\alpha \qquad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Lemma 1. [15, Theorem 2.4] *Let X^* be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D : \mathcal{A} \to X^*$ be a module derivation, then $D(\mathcal{A}) \subseteq J_X{}^{\perp}$.*

Proof. For each $a, b \in \mathcal{A}, \alpha \in \mathfrak{A}$ *and* $x \in X$, we have $(a.\alpha).x - a.(\alpha.x) \in J_X$. Hence

$$\langle D(b), (a.\alpha).x - a.(\alpha.x)\rangle = \langle D(b).(a.\alpha) - (D(b).a).\alpha, x\rangle = 0.$$

□

Motivated by Definition 2.1 and 2.3, the concept of module amenability was introduced and studied in [1, 2].

Definition 3. A Banach algebra $\mathcal{A}$ is called *weakly module amenable* (as an $\mathfrak{A}$-module) if $J_{\mathcal{A}}^{\perp}$ is a commutative $\mathfrak{A}$-module and each linear module derivation $D : \mathcal{A} \to J_{\mathcal{A}}{}^{\perp}$ is inner.

Lemma 2. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module, then $J^{\perp}_{J_{\mathcal{A}}{}^{\perp}}$ is a closed ideal of $\mathcal{A}^{**}$.*

Proof. Let $a \in \mathcal{A}$, $\alpha \in \mathfrak{A}$, $f \in J_{\mathcal{A}}^{\perp}$ and $F \in J^{\perp}_{J_{\mathcal{A}}{}^{\perp}}$. If $a^{**} \in \mathcal{A}^{**}$, there exists a bounded net a_i in $\mathcal{A}$ such that w^*-$\lim a_i = a^{**}$. Since $J_{J_{\mathcal{A}}{}^{\perp}}$ is an $\mathcal{A}$-module, $a_i\big((a.\alpha)f - a(\alpha.f)\big) \in J_{J_{\mathcal{A}}{}^{\perp}}$ for each i. It follows that

$$\begin{aligned}\langle F\Box a^{**}, (a.\alpha)f - a(\alpha.f)\rangle &= \langle F, a^{**}((a.\alpha)f - a(\alpha.f))\rangle \\ &= \lim\langle F, a_i((a.\alpha)f - a(\alpha.f))\rangle = 0.\end{aligned}$$

Hence $F\Box a^{**} \in J^{\perp}_{J_{\mathcal{A}}{}^{\perp}}$. Similarly $a^{**}\Box F \in J^{\perp}_{J_{\mathcal{A}}{}^{\perp}}$. □

Remark 2. If $\mathcal{A}$ is an $\mathfrak{A}$-module Banach algebra, then $J^{\perp}_{J_{\mathcal{A}}{}^{\perp}} \cong \frac{\mathcal{A}^{**}}{J_{\mathcal{A}}^{\perp\perp}}$.

Proof. Let $f \in J_{\mathcal{A}}^{\perp}$. Then

$$\begin{aligned}\langle (a.\alpha)f - a(\alpha.f), b\rangle &= \langle f, b(a.\alpha)\rangle - \langle \alpha.f, ba\rangle \\ &= \langle f, (ba).\alpha\rangle - \langle f, (ba).\alpha\rangle = \langle f, (ba).\alpha - (ba).\alpha\rangle = 0\end{aligned}$$

for every $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. Therefore, $(a.\alpha)f - a(\alpha.f) = 0$.

Since $(a.\alpha)f - a(\alpha.f)$ is a basic member of $J_{J_{\mathcal{A}}{}^{\perp}}$, we have $J_{J_{\mathcal{A}}{}^{\perp}} = 0$. Thanks to lemma 2.6, $J^{\perp}_{J_{\mathcal{A}}{}^{\perp}}$ is a closed ideal in $\frac{\mathcal{A}^{**}}{J_{\mathcal{A}}^{\perp\perp}}$, so that $J^{\perp}_{J_{\mathcal{A}}{}^{\perp}} \cong \frac{\mathcal{A}^{**}}{J_{\mathcal{A}}^{\perp\perp}}$. □

For a Banach algebra $\mathcal{A}$, let $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ be a Banach $\mathcal{A}$-module whose left and right module actions are

$$\pi_1 : \mathcal{A} \times \tfrac{\mathcal{A}}{J_{\mathcal{A}}} \longrightarrow \tfrac{\mathcal{A}}{J_{\mathcal{A}}}, \qquad \pi_1(a, b + J_{\mathcal{A}}) = ab + J_{\mathcal{A}}$$

and

$$\pi_2 : \tfrac{\mathcal{A}}{J_{\mathcal{A}}} \times \mathcal{A} \longrightarrow \tfrac{\mathcal{A}}{J_{\mathcal{A}}}, \qquad \pi_2(b + J_{\mathcal{A}}, \mathcal{A}) = ba + J_{\mathcal{A}}$$

for $a, b \in \mathcal{A}$.

We denote $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ with the above operations by $(\pi_1, \frac{\mathcal{A}}{J_{\mathcal{A}}}, \pi_2)$. Then $(\pi_2^{r*r}, (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*, \pi_1^*)$ is a Banach $\mathcal{A}$-module [8], which is called the dual of $(\pi_1, \frac{\mathcal{A}}{J_{\mathcal{A}}}, \pi_2)$. Here $\pi_2^{r*r} : \mathcal{A} \times (\frac{\mathcal{A}}{J_{\mathcal{A}}})^* \longrightarrow (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$ and $\pi_1^* : (\frac{\mathcal{A}}{J_{\mathcal{A}}})^* \times \mathcal{A} \longrightarrow (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$ are given by $\pi_2^{r*r}(a, f) = a.f, \pi_1^*(f, a) = f.a \quad (a \in \mathcal{A}, f \in (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*)$. Since $(\pi_2^{r*r***}, (\frac{\mathcal{A}}{J_{\mathcal{A}}})^{***}, \pi_1^{****})$ is the second dual of $(\pi_2^{r*r}, (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*, \pi_1^*)$ (as a Banach $\mathcal{A}$-module), $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^{***}$ is an $\mathcal{A}^{**}$-module.

Lemma 3. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module, X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D : \mathcal{A} \longrightarrow X$ be a module derivation. Then $D^{**} : \mathcal{A}^{**} \longrightarrow X^{**}$ is a module derivation.*

Proof. By [15, Lemma 2.9]. □

Remark 3. [15, Remark 3] Let $\mathcal{A}$ be a Banach algebra, then

$$(J_{\mathcal{A}}^{\perp})^{**} \cong (\frac{\mathcal{A}}{J_{\mathcal{A}}})^{***} \cong (\frac{\mathcal{A}^{**}}{J_{\mathcal{A}}^{\perp\perp}})^*.$$

For a Banach algebra $\mathcal{A}$, we have

$$\widehat{(\frac{\mathcal{A}}{J_{\mathcal{A}}})^*} \cong \widehat{J_{\mathcal{A}}^{\perp}} \subseteq (\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})^* \cong J_{\mathcal{A}^{**}}^{\perp}.$$

Proof. If $f \in J_{\mathcal{A}}^{\perp}$, then $f\mid_{J_{\mathcal{A}}} = 0$. We will show that $\hat{f} \in J_{\mathcal{A}^{**}}^{\perp}$. Take some $a^{**} \in \mathcal{A}^{**}$, let a_i be a bounded net in $\mathcal{A}$ such that $w^*\text{-}\lim a_i = a^{**}$, and let $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. Then

$$\begin{aligned}
\langle \hat{f}, (a.\alpha)a^{**} - a(\alpha.a^{**})\rangle &= \langle (a.\alpha)a^{**} - a(\alpha.a^{**}), f\rangle \\
&= \lim_i \langle (a.\alpha)\hat{a_i} - a(\alpha.\hat{a_i}), f\rangle \\
&= \lim_i \langle (a.\alpha)\widehat{a_i - a}(\alpha.a_i), f\rangle \\
&= \lim_i \langle f, (a.\alpha)a_i - a(\alpha.a_i)\rangle = 0.
\end{aligned}$$

Since $\hat{f}$ is linear and continuous, $\hat{f} \in J_{\mathcal{A}^{**}}^{\perp}$. □

Definition 4. The collection of all $f \in J_{\mathcal{A}}^{\perp}$ such that $a^{**} + J_{\mathcal{A}}^{\perp\perp} \longmapsto \langle b^{**}\Box a^{**} + J_{\mathcal{A}}^{\perp\perp}, f\rangle$ is w^*-continuous on $J_{J_{\mathcal{A}}^{\perp}}^{\perp}$ for every $b^{**} + J_{\mathcal{A}}^{\perp\perp} \in J_{J_{\mathcal{A}}^{\perp}}^{\perp}$ is denoted by $WAP(\frac{\mathcal{A}}{J_{\mathcal{A}}})$.

Definition 5. Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module, then $\mathcal{A}$ is called *ideal Arens regular* if $m\Box n = m\Diamond n$ for any $m, n \in J^{\perp}_{J_{\mathcal{A}}{}^{\perp}}$.

Remark 4. If $\mathcal{A}$ is a commutative $\mathfrak{A}$-module then $J_{\mathcal{A}}^{\perp\perp} = 0$. Therefore $\mathcal{A}^{**} = J^{\perp}_{J_{\mathcal{A}}{}^{\perp}}$. This shows that our definition is compatible with the original definition of Arens regularity [3].

Now we are ready to state the first main result of this section.

Theorem 1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $\mathcal{A}^{**}$ be weak module amenable. For every module derivation $D : \mathcal{A} \longrightarrow (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$ if $D^{**}(\mathcal{A}^{**}) \subseteq WAP(\frac{\mathcal{A}}{J_{\mathcal{A}}})$, then $\mathcal{A}$ is weak module amenable.*

Proof. Let $\varphi : \frac{\mathcal{A}}{J_{\mathcal{A}}} \longrightarrow \frac{\mathcal{A}^{**}}{J_{\mathcal{A}}^{\perp\perp}}$ be defined by $\varphi(a + J_{\mathcal{A}}) = \hat{a} + J_{\mathcal{A}}^{\perp\perp}$ for each $a \in \mathcal{A}$. We prove that φ is well-defined. Let $a_1 + J_{\mathcal{A}} = a_2 + J_{\mathcal{A}}$ for $a_1 + J_{\mathcal{A}}$, $a_2 + J_{\mathcal{A}} \in \frac{\mathcal{A}}{J_{\mathcal{A}}}$, then we have $a_1 - a_2 \in J_{\mathcal{A}}$. Therefore, for each $f \in J_{\mathcal{A}}^{\perp}$ we have

$$\langle \widehat{a_1 - a_2}, f\rangle = \langle f, a_1 - a_2\rangle = 0,$$

thus $\widehat{a_1 - a_2} \in J_{\mathcal{A}}^{\perp\perp}$. Hence $\hat{a_1} + J_{\mathcal{A}}^{\perp\perp} = \hat{a_2} + J_{\mathcal{A}}^{\perp\perp}$. Now let $D : \mathcal{A} \longrightarrow J_{\mathcal{A}}^{\perp}$ be a module derivation. By Remark 3, we may assume that $\varphi^* \circ D^{**}$ is a map from $\mathcal{A}^{**}$ to $(\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})^*$. Hence, we have to show that $\varphi^* \circ D^{**}$ is a module derivation. By Lemma 3, $D^{**} : \mathcal{A}^{**} \longrightarrow (J_{\mathcal{A}}^{\perp})^{**}$ is a module derivation, i.e.,

$$D^{**}(a^{**}\Box b^{**}) = \pi_1^{****}(D^{**}(a^{**}), b^{**}) + \pi_2^{r*r***}(a^{**}, D^{**}(b^{**}))$$

for every $a^{**}, b^{**} \in \mathcal{A}$. Hence for each $a^{**}, b^{**} \in \mathcal{A}$,

$$\varphi^* \circ D^{**}(a^{**}\Box b^{**}) = \varphi^*(\pi_1^{****}(D^{**}(a^{**}), b^{**})) + \varphi^*(\pi_2^{r*r***}(a^{**}, D^{**}(b^{**})).$$

Let a_i, b_j be bounded nets in $\mathcal{A}$ such that w^*-$\lim_i a_i = a^{**}$ and w^*-$\lim_j b_j = b^{**}$. For each $a + J_{\mathcal{A}} \in \frac{\mathcal{A}}{J_{\mathcal{A}}}$, we have

$$\begin{aligned}
\langle \varphi^*(\pi_2^{r*r***}(a^{**}, D^{**}(b^{**}))), a + J_{\mathcal{A}}\rangle &= \langle \pi_2^{r*r***}(a^{**}, D^{**}(b^{**})), \varphi(a + J_{\mathcal{A}})\rangle \\
&= \lim_i \lim_j \langle \pi_2^{r*r}(\hat{a}_i, D^{**}(\hat{b_j})), \hat{a} + J_{\mathcal{A}}^{\perp\perp}\rangle \\
&= \lim_i \lim_j \langle D^{**}(\hat{b_j}), \pi_2(\hat{a} + J_{\mathcal{A}}^{\perp\perp}, \hat{a}_i)\rangle \\
&= \lim_i \lim_j \langle D^{**}(\hat{b_j}), \widehat{aa_i} + J_{\mathcal{A}}^{\perp\perp}\rangle \\
&= \lim_i \lim_j \langle D^{**}(\hat{b_j}), \varphi(\pi_2(a + J_{\mathcal{A}}, a_i))\rangle \\
&= \lim_i \lim_j \langle \varphi^*(D^{**}(\hat{b_j})), \pi_2^r(a_i, a + J_{\mathcal{A}})\rangle \\
&= \lim_i \lim_j \langle \pi_2^{r*r}(a_i, \varphi^*(D^{**}(\hat{b_j}))), a + J_{\mathcal{A}}\rangle \\
&= \langle \pi_2^{r*r***}(a^{**}, \varphi^*(D^{**}(b^{**})), a + J_{\mathcal{A}}\rangle.
\end{aligned}$$

Therefore,

$$\varphi^*(\pi_2^{r*r***}(a^{**}, D^{**}(b^{**}))) = \pi_2^{r*r***}(a^{**}, \varphi^*(D^{**}(b^{**}))).$$

Also

$$\begin{aligned}
\langle \varphi^*(\pi_1^{****}(D^{**}(a^{**}), b^{**})), a + J_{\mathcal{A}} \rangle &= \langle \pi_1^{****}(D^{**}(a^{**}), b^{**}), \hat{a} + J_{\mathcal{A}}^{\perp\perp} \rangle \\
&= \lim_j \lim_i \langle \pi_1^*(D(\hat{a}_i), \hat{b_j}), \hat{a} + J_{\mathcal{A}}^{\perp\perp} \rangle \\
&= \lim_j \lim_i \langle D(\hat{a}_i), \pi_1(\hat{b_j}, \hat{a} + J_{\mathcal{A}}^{\perp\perp}) \rangle \\
&= \lim_j \lim_i \langle D(\hat{a}_i), \widehat{b_j a} + J_{\mathcal{A}}^{\perp\perp} \rangle \\
&= \lim_j \lim_i \langle D(\hat{a}_i), \varphi(b_j a + J_{\mathcal{A}}) \rangle \\
&= \lim_j \langle D^{**}(a^{**}), \varphi(b_j a + J_{\mathcal{A}}) \rangle \\
&= \langle D^{**}(a^{**}), \varphi(b^{**} a + J_{\mathcal{A}}) \rangle \\
&= \langle D^{**}(a^{**}), \varphi(\pi_1^{***}(b^{**}, a + J_{\mathcal{A}})) \rangle \\
&= \langle \varphi^*(D^{**}(a^{**})), \pi_1^{***}(b^{**}, a + J_{\mathcal{A}}) \rangle \\
&= \langle \pi_1^{****}(\varphi^*(D^{**}(a^{**})), b^{**})), a + J_{\mathcal{A}} \rangle.
\end{aligned}$$

Therefore,

$$\varphi^*(\pi_1^{****}(D^{**}(a^{**}), b^{**})) = \pi_1^{****}(\varphi^*(D^{**}(a^{**}), b^{**})).$$

Hence $\varphi^* \circ D^{**}$ is a derivation. For each $\alpha \in \mathfrak{A}$ and $a^{**} \in \mathcal{A}^{**}$, we have

$$\varphi^* \circ D^{**}(\alpha.a^{**}) = \alpha.\varphi^* \circ D^{**}(a^{**}), \quad \varphi^* \circ D^{**}(a^{**}.\alpha) = \varphi^* \circ D^{**}(a^{**}).\alpha.$$

Thus $\varphi^* \circ D^{**} : \mathcal{A}^{**} \longrightarrow (\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})^*$ is a module derivation.

As $\mathcal{A}^{**}$is weakly module amenable, there exist $F \in (\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})^*$ such that $\varphi^* \circ D^{**} = \delta_F$. Let $f = \varphi^*(F) \in (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$. It follows that $D = \delta_f$; i.e D is inner. Therefore $\mathcal{A}$ is weakly module amenable. □

Corollary 1. *Let $\mathcal{A}$ be Banach $\mathfrak{A}$-module which is ideal Arens regular. Suppose that every module derivation from $\mathcal{A}$ to $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$ is weakly compact. If $\mathcal{A}^{**}$ is weakly module amenable, then so is $\mathcal{A}$.*

Proof. Since $\mathcal{A}$ is ideal Arens regular, we have $WAP(\frac{\mathcal{A}}{J_{\mathcal{A}}}) = (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$. Let $D : \mathcal{A} \longrightarrow (\frac{\mathcal{A}}{J_{\mathcal{A}}})^*$ be a module derivation. Since D is weakly compact, $D^{**}(\mathcal{A}^{**}) \subseteq (\frac{\mathcal{A}}{J_{\mathcal{A}}})^* = WAP(\frac{\mathcal{A}}{J_{\mathcal{A}}})$. By Theorem 1, the weak module amenability of $\mathcal{A}^{**}$ implies that of $\mathcal{A}$. □

Corollary 2. [9, Corollary 2.1] *Let $\mathcal{A}$ be Arens regular and suppose that every derivation from $\mathcal{A}$ to $\mathcal{A}^*$ is weakly compact. If $\mathcal{A}^{**}$ is weakly amenable, then so is $\mathcal{A}$.*

Proof. Take $\mathfrak{A} = \mathbb{C}$ in Corollary 1. □

We are ready to state the second main result of this section.

Theorem 2. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module such that $J_{\mathcal{A}}^{\perp\perp} \subseteq \mathcal{A}$, $\mathcal{A}^{**}.\frac{\mathcal{A}}{J_{\mathcal{A}}^{\perp\perp}} = \frac{\mathcal{A}^{**}}{J_{\mathcal{A}}^{\perp\perp}}$ and $\frac{\mathcal{A}}{J_{\mathcal{A}}^{\perp\perp}}.\mathcal{A}^{**} \subseteq \frac{\mathcal{A}}{J_{\mathcal{A}}^{\perp\perp}}$. If $\mathcal{A}^{**}$ is weakly module amenable, then so is $\mathcal{A}$.*

Proof. Let $\varphi : \frac{\mathcal{A}}{J_{\mathcal{A}}} \longrightarrow \frac{\mathcal{A}^{**}}{J_{\mathcal{A}}^{\perp\perp}}$ be defined by $\varphi(a + J_{\mathcal{A}}) = \hat{a} + J_{\mathcal{A}}^{\perp\perp}$ for each $a \in \mathcal{A}$. Applying the same argument that was used in the proof of Theorem 1, we see that φ is well defined. We prove that $\varphi^* \circ D^{**} : A^{**} \longrightarrow (\frac{\mathcal{A}^{**}}{J_{\mathcal{A}^{**}}})^*$ is a module derivation.

Let $D : \mathcal{A} \longrightarrow J_{\mathcal{A}}^{\perp}$ be a module derivation. By Lemma 3, $D^{**} : \mathcal{A}^{**} \longrightarrow (J_{\mathcal{A}}^{\perp})^{**}$ is a module derivation, i.e.

$$D^{**}(a^{**}\Box b^{**}) = \pi_1^{****}(D^{**}(a^{**}), b^{**}) + \pi_2^{r*r***}(a^{**}, D^{**}(b^{**}))\ (a^{**}, b^{**} \in \mathcal{A}).$$

Thus, for each $a^{**}, b^{**} \in \mathcal{A}$,

$$\varphi^* \circ D^{**}(a^{**}\Box b^{**}) = \varphi^*(\pi_1^{****}(D^{**}(a^{**}), b^{**})) + \varphi^*(\pi_2^{r*r***}(a^{**}, D^{**}(b^{**})) \tag{1}$$

By the proof of Theorem 1, we obtain

$$\varphi^*(\pi_2^{r*r***}(a^{**}, D^{**}(b^{**}))) = \pi_2^{r*r***}(a^{**}, \varphi^*(D^{**}(b^{**}))).$$

For proving the second part of the equation (1), we have

$$\begin{aligned}\langle \varphi^*(\pi_1^{****}(D^{**}(a^{**}), b^{**}), a + J_{\mathcal{A}}\rangle &= \langle \pi_1^{****}(D^{**}(a^{**}), b^{**}), \varphi(a + J_{\mathcal{A}})\rangle \\ &= \langle \pi_1^{****}(D^{**}(a^{**}), b^{**}), \hat{a} + J_{\mathcal{A}}^{\perp\perp}\rangle \\ &= \langle D^{**}(a^{**}), \pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}^{\perp\perp})\rangle.\end{aligned}$$

We claim that

$$\pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}^{\perp\perp}) = \varphi(\pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}^{\perp\perp})). \tag{2}$$

For each $a \in \mathcal{A}$,

$$\begin{aligned}\langle D^{**}(a^{**}), \pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}^{\perp\perp})\rangle &= \langle D^{**}(a^{**}), \varphi(\pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}^{\perp\perp}))\rangle \\ &= \langle \varphi^*(D^{**}(a^{**})), \pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}^{\perp\perp}))\rangle \\ &= \langle \pi_1^{****} \circ \varphi^*(D^{**}(a^{**}), b^{**})), \hat{a} + J_{\mathcal{A}}^{\perp\perp}\rangle.\end{aligned}$$

This shows that

$$\varphi^*(\pi_1^{****}(D^{**}(a^{**}), b^{**})) = \pi_1^{****}(\varphi^*(D^{**}(a^{**}), b^{**})).$$

Proof of the claim (2):

Let $f \in J_{\mathcal{A}}^{\perp}$. Since $\hat{a} + J_{\mathcal{A}}^{\perp\perp} \in \frac{\mathcal{A}^{**}}{J_{\mathcal{A}}^{\perp\perp}}$, there exist members $d^{**} \in \mathcal{A}^{**}$ and $c \in \mathcal{A}$ such that $\hat{a} + J_{\mathcal{A}}^{\perp\perp} = d^{**}\Box c + J_{\mathcal{A}}^{\perp\perp}$ and $c\Box\hat{a} + J_{\mathcal{A}}^{\perp\perp} \in \frac{\mathcal{A}}{J_{\mathcal{A}}^{\perp\perp}}$, hence

$$\langle \pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}^{\perp\perp}), f\rangle = \langle \pi_1^{***}(b^{**}, d^{**}\Box c + J_{\mathcal{A}}^{\perp\perp}), f\rangle$$

$$
\begin{aligned}
&= \lim_i \lim_j \langle \pi_1(\hat{b_i}, \hat{d_j}\Box c + J_{\mathcal{A}}^{\perp\perp}), f\rangle \\
&= \lim_i \lim_j \langle \widehat{b_i d_j c} + J_{\mathcal{A}}^{\perp\perp}, f\rangle \\
&= \lim_i \lim_j \langle (b_i d_j c + J_{\mathcal{A}}), \varphi^*(f)\rangle \\
&= \lim_i \lim_j \langle \varphi^*(f), b_i d_j c + J_{\mathcal{A}}\rangle.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\langle \varphi(\pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}^{\perp\perp}), f\rangle &= \langle \pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}^{\perp\perp}, \varphi^*(f)\rangle \\
&= \langle \pi_1^{***}(b^{**}, d^{**}\Box c + J_{\mathcal{A}}^{\perp\perp}, \varphi^*(f)\rangle \\
&= \lim_i \lim_j \langle \pi_1(\hat{b_i}, \hat{d_j}\Box c + J_{\mathcal{A}}^{\perp\perp}), \varphi^*(f)\rangle \\
&= \lim_i \lim_j \langle \widehat{b_i d_j c} + J_{\mathcal{A}}^{\perp\perp}, \varphi^*(f)\rangle \\
&= \lim_i \lim_j \langle b_i \widehat{d_j c + J_{\mathcal{A}}}, \varphi^*(f)\rangle \\
&= \lim_i \lim_j \langle \varphi^*(f), b_i d_j c + J_{\mathcal{A}}\rangle.
\end{aligned}
$$

Hence

$$
\pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}^{\perp\perp}) = \varphi(\pi_1^{***}(b^{**}, \hat{a} + J_{\mathcal{A}}^{\perp\perp})).
$$

□

Corollary 3. [9, Theorem 2.4] *Let $\mathcal{A}$ be a right ideal in $\mathcal{A}^{**}$ and suppose that $\mathcal{A}^{**}\Box\mathcal{A} = \mathcal{A}^{**}$. If $\mathcal{A}^{**}$ is weakly amenable, then $\mathcal{A}$ is weakly amenable.*

Proof. Take $\mathfrak{A} = \mathbb{C}$ in Theorem 2. □

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