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On the uniqueness of two different classes of meromorphic functions under the sharing of two sets of rational functions

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Abstract. We study the uniqueness problem of two special classes of meromorphic functions sharing two sets of rational functions. One of the considered classes has the property to include the Selberg class L-functions, while the other class is comprising of arbitrary meromorphic functions having finitely many poles. We obtain a number of results which extend and improve a number of earlier results such as Li [Proc. Amer. Math. Soc. **138** (2010), 2071–2077], Lin and Lin [Filomat **30** (2016), 3795–3806] and others. We have also been able to replace the strict CM (IM) sharing of the sets in our theorems to almost CM (almost IM) sharing.

1. Introduction

Firstly, throughout the paper by a meromorphic (resp. entire) function we always mean a meromorphic (resp. entire) function in the whole complex plane $\mathbb{C}$ and $M(\mathbb{C})$ denotes the field of meromorphic functions over $\mathbb{C}$. Let f and g be two non-constant meromorphic functions, and let $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share a counting multiplicity (CM) if $f - a$ and $g - a$ have the same zeros with the same multiplicities. Functions f and g are said to share a ignoring multiplicity (IM), if $f - a$ and $g - a$ have the same zeros with ignoring multiplicities. There also exist the extensions of CM or IM sharing namely almost CM (almost IM) sharing, which are defined in the next section.

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In 1920, Nevanlinna introduced his famous five value theorem. Later he also proved his four value uniqueness result to start a new era and these are the bases of uniqueness theory.

Next, by L-function we mean a Selberg class function with the Riemanian zeta $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ function as a prototype. The Selberg class $\mathcal{S}$ of L-function is the set of convergent Dirichlet series $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ of complex variable s satisfying four axioms given in [8].

As we know L-functions possess meromorphic continuations. It will be interesting to determine the number of shared values for which an L-function becomes identical with an arbitrary meromorphic function. In present days this has become a new trend. In this direction, we would like to mention first the following result due to Li [6], where considering two distinct complex values, Li proved the following uniqueness result.

Theorem 1. *Let f be a meromorphic function in $\mathbb{C}$ having finitely many poles, and let a and b be any two distinct finite complex values. If f and a non-constant L-function $\mathcal{L}$ share a CM and b IM, then $f = \mathcal{L}$.*

The above theorem really boosts up the researches to investigate the uniqueness problem of an L-function with a meromorphic function having finitely many poles, sharing different types of sets. In this direction, already a number of authors have dealt the case.

In 2016, considering the set sharing problem instead of value sharing Lin and Lin [7] obtained the following theorem.

Theorem 2 (see [7])**.** *Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles, $S_1, S_2 \subset \mathbb{C}$ be two distinct sets such that $S_1 \cap S_2 = \emptyset$ and $\#(S_i) \leq 2$, $i = 1, 2$, where $\#(S)$ denotes the cardinality of the set S. Suppose for a finite set $S = \{\alpha_i \mid i = 1, 2, \ldots, n\}$, $C(S)$ is defined by $C(S) = \frac{1}{n}\sum_{i=1}^{n} \alpha_i$. If f and a non-constant L-function $\mathcal{L}$ share S_1 CM and S_2 IM, then* (i) *$\mathcal{L} = f$ when $C(S_1) \neq C(S_2)$ and* (ii) *$\mathcal{L} = f$ or $\mathcal{L} + f = 2C(S_1)$ when $C(S_1) = C(S_2)$.*

Now before going to further discussions and stating the next results, we need some basic conceptions. Here the results of our paper are mainly based on Nevanlinna theory. For the convenience of the readers, next we will introduce some basics notations and results of Nevanlinna theory which will be used in the proofs of results. Let f be a meromorphic functions in $\mathbb{C}$, then the proximity function $m(r, f)$ and the counting functions $N(r, \infty; f)$ or $N(r, f)$ (counting multiplicity) and $\overline{N}(r, \infty; f)$ or $\overline{N}(r, f)$ (ignoring multiplicity) and the Nevanlinna characteristic function $T(r, f)$ are defined as (see [11])

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

$$N(r, \infty; f) = \int_0^r \frac{n(t, \infty; f) - n(0, \infty; f)}{t} dt + n(0, \infty; f) \log r,$$

$$\begin{aligned}
\overline{N}(r,\infty;f) &= \int_0^r \frac{\overline{n}(t,\infty;f)-\overline{n}(0,\infty;f)}{t}dt + \overline{n}(0,\infty;f)\log r,\\
N(r,a;f) &= \int_0^r \frac{n(t,a;f)-n(0,a;f)}{t}dt + n(0,a;f)\log r,\\
\overline{N}(r,a;f) &= \int_0^r \frac{\overline{n}(t,a;f)-\overline{n}(0,a;f)}{t}dt + \overline{n}(0,a;f)\log r,\\
T(r,f) &= m(r,f)+N(r,\infty;f),
\end{aligned}$$

respectively, where $\log^+ x = \max\{\log x, 0\}$ for all $x \geq 0$, $n(t,\infty;f)$ $(n(t,a;f))$ denotes the number of poles (zeros) of f ($f-a$) in the disc $|z|<t$, counting multiplicities and $\overline{n}(t,\infty;f)$ ($\overline{n}(t,a;f)$) denote the number of poles (zeros) of f ($f-a$) in the disc $|z|<t$, ignoring multiplicities for some $a \in \mathbb{C}$.

Also the order of f is defined as

$$\rho(f) := \limsup_{r\longrightarrow\infty} \frac{\log T(r,f)}{\log r},$$

and one says that f is of maximal type provided

$$\Gamma(f) := \limsup_{r\longrightarrow\infty} \frac{T(r,f)}{r^{\rho(f)}} = \infty,$$

when $0<\rho(f)<\infty$.

We recall the following results.
(i) The Nevanlinna First Fundamental Theorem: $T(r,f)=T(r,0;f)+O(1)$.
(ii) The Second Fundamental Theorem: $(q-2)T(r,f) \leq \sum_{i=1}^{q} \overline{N}(r,a_i;f) + S(r,f)$, where $a_1,a_2,\ldots,a_q \in \mathbb{C}\cup\{\infty\}$ and $S(r,f)=O(\log(rT(r,f)))$ when f is of infinite order, $r\longrightarrow\infty$, except possibly on a set of finite Lebesgue measure. When f has finite order, then $S(r,f)=O(\log r)$ for all r.

In 2016, in a new direction Han [5] proved much more general version of Theorem 1 and obtained the following result.

Theorem 3. *Let f be a meromorphic function in $\mathbb{C}$ of finite, non-zero order such that either* (i) *$\rho(f)$ is not an integer or* (ii) *$\rho(f)$ is an integer whereas $\Gamma(f)=\infty$. Take $a,b\in\mathbb{C}$. Assume that f and another non-constant meromorphic function g in $\mathbb{C}$ share the values $\{a\}$ CM and $\{b\}$ IM. When, in addition, both f and g have finitely many poles, then $f=g$.*

L-function can be treated as a meromorphic function and it can have only one pole at $z=1$. Also from p. 150 of [9], we know that for a non-constant L-function, $T(r,L)=O(r\log r)$, i.e., L is of maximal type. Hence Theorem 3 also holds if we choose f as a L-function.

Next, considering a special class of meromorphic functions having finitely many poles and finite non-integer order Chen ([1], [2]) proved the following uniqueness results.

Theorem 4 (see [1]). *Let $S_1 = \{\alpha_1\}$, $S_2 = \{\beta_1, \beta_2\}$ where α_1 and β_1, β_2 are distinct finite complex numbers satisfying $(\beta_1 - \alpha_1)^2 \neq (\beta_2 - \alpha_1)^2$. If two non-constant meromorphic functions f and g having finitely many poles, share S_1 CM, S_2 IM, and if the order of f is neither an integer nor infinite, then $f = g$.*

Theorem 5 (see [2]). *In the same situation as in Theorem 4, if f and g share S_1 IM and S_2 CM, then $f = g$.*

We observe that in Theorems 1–5 the elements of the sets were chosen from $\mathbb{C}$, i.e., the uniqueness results were found on the basis of value or set sharing in $\mathbb{C}$. Naturally the question arises "What will happen if we choose the sets in Theorems 1–5 as a subset of $M(\mathbb{C})$ containing some rational functions instead of values?" So it will be interesting to re-investigate the theorems considering sets of rational functions. In Theorems 4, 5 the elements α_1, β_1 and β_2 were chosen in such a way that they satisfy the condition $2\alpha_1 \neq \beta_1 + \beta_2$, so it will also be interesting to find the conclusion of Theorems 4, 5, when $2\alpha_1 = \beta_1 + \beta_2$ holds.

The purpose of the paper is to address the above two issues and provide fruitful solutions in this regard. In the present paper we have dealt with two different classes of meromorphic functions so as to improve Theorems 1–5 in terms of sharing of sets of rational functions.

Next we see that for some non-constant $f \in M(\mathbb{C})$ almost CM sharing is weaker than CM sharing. Perceiving this we have also relaxed the nature of the sharing of the sets in our results from strictly CM (IM) sharing to almost CM (IM) sharing, and thus have been able to refine the sharing notion of sets.

2. Definitions

Throughout this section we will consider f and g to be two arbitrary non-constant meromorphic functions. Before going to the main results we invoke the following definitions.

Definition 1. For a non-constant meromorphic function f, the set of all small functions of f is denoted by $S(f)$, i.e., $S(f) = \{a \in M(\mathbb{C}) : T(r, a) = S(r, f)\ \textit{as}\ r \longrightarrow \infty\}$. Clearly, every element in $\mathbb{C}$ belongs to $S(f)$ and $S(f) \subset M(\mathbb{C})$.

Definition 2. Let $a \in S(f) \cap S(g)$ and let $E(0, f-a)$ ($\overline{E}(0, f-a)$) denote the set of zeros of $f - a$, counted according to its multiplicity (ignoring multiplicity). If $E(0, f-a) = E(0, g-a)$ ($\overline{E}(0, f-a) = \overline{E}(0, g-a)$), then we say that $f - a$, $g - a$ share 0 CM(IM) or f, g share a CM (IM).

Definition 3. Let $S = \{a_1, a_2, \ldots, a_q\} \subset S(f) \cap S(g)$ and set $F = \prod_{i=1}^{q}(f - a_i)$ and $G = \prod_{i=1}^{q}(g - a_i)$. If $E(0, F) = E(0, G)$ ($\overline{E}(0, F) =$

$\overline{E}(0,G)$), then we say f and g share the set S CM (IM). Here if z_0 is zero of $f-a_i$ of multiplicity m then in $E(0,F)$ ($\overline{E}(0,F)$), z_0 is appearing m (one) times. Also if z_0 is a zero of $f-a_j$ for several a_j, $j=1,2,\ldots,q$, then z_0 appears m times where m is the sum of the multiplicities of zeros of the corresponding functions.

Definition 4. If $a\in\mathbb{C}$ and $f-a$ has at most finitely many zeros then a is said to be a generalized Picard exceptional value of f. If a is a non-constant small function of f and $f-a$ has at most finitely many zeros then 0 is said to be a generalized Picard exceptional value of $f-a$.

Definition 5. Let $a,b\in S(f)\cap S(g)$. By $N(r,0;f-a\mid g-b)$ ($\overline{N}(r,0;f-a\mid g-b)$) we mean the counting function (reduced counting function) of common zeros of $f-a$ and $g-b$ having same multiplicities (irrespective of their multiplicities).

And $\overline{N}_*(r,0;f-a\mid g-b)$ denotes the reduced counting function of common zeros of $f-a$ and $g-b$ having different multiplicities.

Definition 6. Let $S=\{a_1,a_2,\ldots,a_q\}\subset S(f)\cap S(g)$ and set $F=\prod_{i=1}^{q}(f-a_i)$ and $G=\prod_{i=1}^{q}(g-a_i)$. From the previous definition we have $N(r,0;F\mid G)$ ($\overline{N}(r,0;F\mid G)$) be the counting function (reduced counting function) of common zeros of F and G having the same multiplicities (irrespective of their multiplicities). Now if

$$N(r,0;F)+N(r,0;G)-2N(r,0;F\mid G)=S(r,F)+S(r,G)=S(r,f)+S(r,g),$$

then we say that f and g share S almost CM. On the other hand, if

$$\overline{N}(r,0;F)+\overline{N}(r,0;G)-2\overline{N}(r,0;F\mid G)=S(r,f)+S(r,g),$$

then we say that f and g share S almost IM.

3. Main theorems and some relevant examples

Assumption A. Let f and $g\in M(\mathbb{C})$ having finitely many poles in $\mathbb{C}$ and $\rho(f)$ is non-zero finite such that either (i) $\rho(f)$ is not an integer or (ii) $\rho(f)$ is an integer and f is of maximal type. In addition, we take a, b, c, d to be distinct rational functions.

Theorem 6. *Let Assumption A be satisfied. If f and g share $\{a\}$ CM and $\{b\}$ IM then $f=g$.*

Theorem 7. *Let Assumption A be satisfied. If f and g share the sets $S_1=\{a\}$ CM and $S_2=\{b,c\}$ IM then*
(i) *if $2a\neq b+c$, then $f=g$,*
(ii) *if $2a=b+c$, then $f=g$ or $f+g=2a$.*

Theorem 8. *Let Assumption A be satisfied. If f and g share the sets $S_1=\{a\}$ IM and $S_2=\{b,c\}$ CM and*

(i) *if* $b-a, c-a$ *are linearly dependent but* $2a \neq b+c$, *then* $f = g$,
(ii) *if* $2a = b+c$, *then* $f = g$ *or* $f+g = 2a$.

Theorem 9. *Let Assumption A be satisfied and* a, b, c, d *be distinct rational functions satisfying* $(c-a)^2(c-b)^2 \neq (d-a)^2(d-b)^2$. *If* f *and* g *share the sets* $S_1 = \{a, b\}$ *CM and* $S_2 = \{c, d\}$ *IM, then* $f = g$.

Corollary 1. *In the same situations as in Theorems 6–9, if* f *and* g *share the sets almost CM and almost IM instead of CM, IM then we can also get the same results.*

Note 3.1. Since $\rho(f)$ is finite, $S(r, f) = O(\log r)$ for all r, hence $S(f)$ contains rational functions. The rational functions a, b, c, d also belong to both $S(f)$ and $S(g)$. Thus the above definitions are also applicable here.

Remark 1. Choosing $a, b \in \mathbb{C}$; Theorem 6 becomes Theorem 3. Thus Theorem 6 is an improvement of Theorem 1 as well.

Remark 2. Theorem 7 improves Theorem 4. Here the values in those theorems can be replaced by some rational functions as well. In particular, when a, b, c are distinct finite values in $\mathbb{C}$, then $b-a$, $c-a$ are always linearly dependent, i.e., for some $k \neq 0$ we can write $b-a = k(c-a)$, and hence Theorem 8 is an improvement of Theorems 2, 5.

Remark 3. From Corollary 1, we can immediately relax the strict CM (IM) sharing to almost CM (almost IM) sharing in Theorems 1–5.

Choose $f = \sum_{n=1}^{\infty} \frac{z^n}{n^{kn}}$ for some positive integers k, n. Then from Lemma 4, it is easy to verify that $\rho(f) = \frac{1}{k}$ and it has finitely many poles in $\mathbb{C}$. Also Riemann zeta function ζ satisfies all the properties of Selberg class and hence it is an L-function of finite order and of maximal type. Clearly these two functions represent two different classes in $M(\mathbb{C})$. Here $S(r, f) = O(\log r)$ and hence any small function of f is a rational function.

In the following example we show that in the case (ii) of Theorems 7, 8, it can happen that $f + g = 2a$.

For the subsequent examples let us consider $f \in \{\zeta, \sum_{n=1}^{\infty} \frac{z^n}{n^{mn}}\}$. Clearly f satisfies the conditions of Theorems 6–9.

Example 1. Let us consider $S_1 = \{0\}$, $S_2 = \{r, -r\}$, where r is some rational function. Now let us consider $g = -f$. Clearly f and g share the sets S_1 and S_2 CM. Obviously $f \neq g$ but $f + g = 0$.

In view of Lemma 5 which will be proved afterwards, the following Examples 2, 3 show that Theorem 6 does not hold when g has infinitely many poles. On the other hand, Examples 4 and 5 demonstrate that Theorem 7 and Example 6 show that Theorem 8 will fail if g has infinitely many poles.

Example 2. Let us consider $S_1 = \{1\}$, $S_2 = \{-1\}$ and $g = \frac{1}{f}$. Clearly from Lemma 5, g has infinitely many poles, as f has infinitely many zeros. Here f, g share both S_i $(i = 1, 2)$ CM, but $f \neq g$.

Example 3. Let us consider $S_1 = \{0\}$, $S_2 = \{\frac{1}{Q}\}$ where Q is a rational function and choose $g = \frac{f}{2-Qf}$. Clearly, f, g share the set S_1, S_2 almost CM or CM according to whether Q has poles or not, but $f \neq g$.

Example 4. Let $S_1 = \{1\}$, $S_2 = \{r, \frac{1}{r}\}$ where r $(\neq 1)$ is some rational function. Then f and $g = \frac{1}{f}$ share S_1 CM and S_2 almost CM or CM according as f and r have some common poles or not, but $f \neq g$.

Example 5. Let us consider $S_1 = \{0\}$, $S_2 = \{Q, \frac{Q}{Q-1}\}$ $(Q \neq 1)$ and $g = \frac{f}{f-1}$. Then f, g share the set S_1 CM and S_2 almost CM or CM according to whether f and Q have some common poles or not, but $f \neq g$.

Example 6. Let us consider $S_1 = \{0\}$, $S_2 = \{P, \frac{cP}{1-c}\}$ and $g = \frac{cPf}{f-cP}$, where P is a rational function and $c \neq 0, 1$ a constant. Here f, g share the set S_1, S_2 almost CM or CM according to whether P has zero or not, but $f \neq g$.

Next example implies that in Theorem 9 the condition of having finitely many poles can not be dropped.

Example 7. Let us take $S_1 = \{r, \frac{1}{r}\}$, $S_2 = \{q, \frac{1}{q}\}$ where r, q are some rational functions such that $r \neq 1$ and $(r-q)^2(r-1/q)^2 \neq (1/r-q)^2(1/r-1/q)^2$. Then clearly f and $g = 1/f$ share both S_i $(i = 1, 2)$ almost CM or CM according to whether f and r, q have some common poles or not, but $f \neq g$.

Our next example will show that our results cease to hold for an arbitrary meromorphic function which does not satisfy the given conditions in Assumption A.

Example 8. The functions $f = e^z$ and $g = e^{-z}$ share $\{1\}$, $\{z, \frac{1}{z}\}$ CM but $f \neq g$. On the other hand considering the two sets as $\{0\}$, $\{1, -1\}$ CM, we see that f and g share the same sets CM but neither $f = g$ nor $f + g = 0$.

4. Lemmas

Lemma 1. *Let f be a meromorphic function and a, b be two rational functions. Then*

$$T(r, f) \leq \overline{N}(r, 0; f-a) + \overline{N}(r, 0; f-b) + \overline{N}(r, \infty; f) + S(r, f).$$

Proof. Let us consider a function $F = \frac{f-a}{b-a}$. Using the First Fundamental Theorem it can be shown that $T(r, F) = T(r, f) + S(r, f)$.

Now using the Second Fundamental Theorem we get

$$\begin{aligned} T(r,f) &= T(r,F)+S(r,f) \\ &\le \overline{N}(r,0;F)+\overline{N}(r,0;F-1)+\overline{N}(r,\infty;F)+S(r,F) \\ &\le \overline{N}(r,0;f-a)+\overline{N}(r,0;f-b)+\overline{N}(r,\infty;f)+S(r,f), \end{aligned}$$

hence we get the result. □

Lemma 2 (see [11], Lemma 1.22)**.** *Let f be a non-constant meromorphic function and let $k \geq 1$ be an integer. Then $m\left(r,\frac{f^{(k)}}{f}\right) = S(r,f)$. Further if $\rho(f) < \infty$, then*

$$m\left(r,\frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 3 (see [11], Theorem 1.14)**.** *Let $f(z)$, $g(z) \in M(\mathbb{C})$. Let the orders of f and g be $\rho(f)$ and $\rho(g)$, respectively. Then*

$$\rho(fg) \leq \max\{\rho(f),\rho(g)\}.$$

Lemma 4 (see [3], p. 288)**.** *Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in \mathcal{E}(\mathbb{C})$ be non-constant and of finite order. Then*

$$\rho(f) = \frac{1}{\liminf_{n\longrightarrow\infty}\frac{-\log|a_n|}{n\log n}}.$$

Lemma 5. *Assume f to be the same as in Theorem 6 and let $\alpha \in S(f)$ be a rational function. Then $f-\alpha$ has no generalized Picard exceptional value in $\mathbb{C}$.*

Proof. Let us assume that $f(z)-\alpha(z)$ has finitely many zeros. Also let $a_1, a_2, \ldots, a_t$ be the zeros of $f(z)-\alpha(z)$ with multiplicity $k_1, k_2, \ldots, k_t$, respectively. Since f has finitely many poles and $N(r,\infty;\alpha) = O(\log r)$, let $b_1, b_2, \ldots, b_s$ be the poles of $f-\alpha$ with multiplicity $l_1, l_2, \ldots, l_s$, respectively. Then by Hadamard Factorization Theorem (see [10], p. 250) we have,

$$\frac{(z-b_1)^{l_1}(z-b_2)^{l_2}\ldots(z-b_s)^{l_s}(f(z)-\alpha(z))}{(z-a_1)^{k_1}(z-a_2)^{k_2}\ldots(z-a_t)^{k_t}} = e^{p(z)}, \tag{4.1}$$

for some polynomial $p(z)$, and from above we have

$$f(z) = \alpha(z) + \frac{(z-a_1)^{k_1}(z-a_2)^{k_2}\ldots(z-a_t)^{k_t}e^{p(z)}}{(z-b_1)^{l_1}(z-b_2)^{l_2}\ldots(z-b_s)^{l_s}}. \tag{4.2}$$

Therefore, from above we conclude that

$$T(r,f) \leq O(r^{deg(p(z))}) + O(\log r). \tag{4.3}$$

Now from (4.1)–(4.3) and *Lemma 3*, we have $\rho(f) = deg(p(z))$, which implies that $\rho(f)$ is an integer and f is not of maximal type. Hence we arrive at a contradiction. Therefore $f - \alpha$ has infinitely many zeros. $\square$

Lemma 6. *Let f be a non-constant meromorphic function of finite order and a, b be two distinct rational functions. Suppose*

$$L_{a,b}(f) = \begin{vmatrix} f-a & b-a \\ f'-a' & b'-a' \end{vmatrix}.$$

Then $m\left(r, \frac{L_{a,b}(f)}{f-x}\right) = S(r,f)$ for $x = a, b$.

Proof. Here

$$\begin{aligned} L_{a,b}(f) &= (f'-a')(f-b)-(f-a)(f'-b') \\ &= (f-a)(b'-a')-(f'-a')(b-a) \\ &= (f-b)(b'-a')-(f'-b')(b-a). \end{aligned}$$

With the help of Lemma 2, $m\left(r, \frac{L_{a,b}(f)}{f-a}\right) = m\left(r, \frac{L_{a,b}(f)}{f-b}\right) = S(r,f)$ follows immediately. $\square$

5. Proofs of theorems

Proof of Theorem 6. It is given that f, g share $\{a\}$ CM and $\{b\}$ IM. First, we will show that$\rho(f) = \rho(g)$. To this end, in view of *Lemma 1* we see that

$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f-a)+\overline{N}(r,0;f-b)+\overline{N}(r,\infty;f)+S(r,f) \\ &\le \overline{N}(r,0;g-a)+\overline{N}(r,0;g-b)+O(\log r) \\ &\le 2T(r,g)+O(\log r), \end{aligned} \tag{5.1}$$

which implies $\rho(f) \le \rho(g)$. Proceeding similarly again we get

$$T(r,g) \le 2T(r,f)+O(\log r), \tag{5.2}$$

which implies $\rho(g) \le \rho(f)$.

We obtain $\rho(f) = \rho(g)$ hence $S(r,f) = S(r,g) = O(\log r)$. Now let us define the function

$$G = \frac{f-a}{g-a}.$$

Since f, g share $\{a\}$ CM and have finitely many poles, we can get a rational function Q such that GQ is a zero free entire function. Then

$$GQ = \frac{Q(f-a)}{g-a} = e^{\phi}, \tag{5.3}$$

where ϕ is an entire function. Since $\rho(f)$ $(= \rho(g))$ is finite and $T(r,a) = O(\log r)$, from 3 we obtain $\rho(e^{\phi}) \le \rho(f)$, hence ϕ is a polynomial of finite degree.

Now let z_0 be a zero of $f-b$ such that it is not a common zero of $f-a$ and $f-b$, that is to say $a(z_0)\neq b(z_0)$. From (5.3) we see that z_0 is also a zero of $\frac{e^{\phi}}{Q}-1$.

If z_0 is a common zero of $f-a$ and $f-b$, i.e., $a(z_0)=b(z_0)$, then $z_0\in\overline{E}(0,a-b)$.

Hence,

$$\overline{N}(r,0;f-b\mid f-a)\leq\overline{N}(r,0;a-b)\leq T(r,a-b)+O(1)=O(\log r).$$

Therefore from the above discussion and noting that Q is a rational function we have

$$\overline{N}(r,0;f-b)\quad\leq\quad\overline{N}(r,1;e^{\phi}/Q)+O(\log r)\leq T(r,e^{\phi})+O(\log r). \tag{5.4}$$

Next, introduce the following auxiliary function:

$$\Delta=\left(\frac{L_{a,b}(f)}{(f-a)(f-b)}-\frac{L_{a,b}(g)}{(g-a)(g-b)}\right)(f-g). \tag{5.5}$$

We claim that $\Delta=0$. Suppose that this is not the case. We will derive a contradiction below.

Let z_0 be a zero of $f-b$. Then it is easy to verify that z_0 is not a pole of Δ. The only poles of Δ occur at the poles of f and g which are finitely many in number. Hence,

$$\overline{N}(r,\infty;\Delta)\leq O(\log r).$$

Now using Lemma 6 we have

$$\begin{aligned}
m(r,\Delta)\quad &=\quad m\left(r,\frac{L_{a,b}(f)(f-g)}{(f-a)(f-b)}-\frac{L_{a,b}(g)(f-g)}{(g-a)(g-b)}\right)\\
&\leq\quad m\left(r,\frac{L_{a,b}(f)}{f-b}\right)+m\left(r,1-\frac{g-a}{f-a}\right)+m\left(r,\frac{L_{a,b}(g)}{g-b}\right)\\
&\quad\quad +m\left(r,\frac{f-a}{g-a}-1\right)+O(1)\\
&\leq\quad m\left(r,1-\frac{e^{\phi}}{Q}\right)+m\left(r,\frac{Q}{e^{\phi}}-1\right)+O(\log r)\\
&\leq\quad O(T(r,e^{\phi}))+O(\log r).
\end{aligned}$$

Therefore from above $T(r,\Delta)\leq O(T(r,e^{\phi}))+O(\log r)$.

Next let z_1 be a zero of $f-a$ such that $a(z_1)\neq b(z_1)$. Then from (5.5) it is easy to verify that z_1 is also a zero of Δ. The case $a(z_1)=b(z_1)$ implies $z_1\in\overline{E}(0,b-a)$ and $\overline{N}(r,0;b-a)=O(\log r)$.

Therefore we have

$$\begin{aligned}
\overline{N}(r,0;f-a)\quad &\leq\quad \overline{N}(r,0;\Delta)+O(\log r)\\
&\leq\quad T(r,\Delta)+O(\log r)
\end{aligned}$$

$$\leq \quad O(T(r, e^{\phi})) + O(\log r). \tag{5.6}$$

Now using Lemma 1 and (5.4), (5.6) we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f-a) + \overline{N}(r,0;f-b) + \overline{N}(r,\infty;f) + O(\log r) \\ &\leq O(T(r,e^{\phi})) + O(\log r) \\ &\leq O(r^{deg(\phi)}) + O(\log r). \end{aligned} \tag{5.7}$$

Now from $\rho(e^{\phi}) \leq \rho(f)$ and from (5.7) we have $\rho(e^{\phi}) = \rho(f)$.

If $\rho(f)$ is not an integer, then from $\rho(e^{\phi}) = \rho(f)$ we get a contradiction.

If $\rho(f) \geq 1$ is an integer, then as (5.7) yields $\Gamma(f) = O(1)$, we get a contradiction against the maximal type assumption of f.

Hence our claim is proved and we have $\Delta = 0$. Now let $f \neq g$, then we have

$$\frac{L_{a,b}(f)}{(f-a)(f-b)} - \frac{L_{a,b}(g)}{(g-a)(g-b)} = 0.$$

Now let z_1 be a zero of $f-b$ of multiplicity p and zero of $g-b$ of multiplicity q. Then from the above relation we get $p = q$. Since z_1 is arbitrary, we get

$$N(r,0;f-b) + N(r,0;g-b) - 2N(r,0;f-b \mid g-b) = 0.$$

In view of Definition 5 we have $\overline{N}_*(r,0;f-b \mid g-b) = 0$. That is to say f and g share b CM.

Now considering an auxiliary function $e^{\psi} = \frac{Q(f-b)}{(g-b)}$, for some polynomial ψ and rational Q and then arguing exactly in the same way as in (5.3)−(5.4), we deduce that $\overline{N}(r,0;f-a) \leq O(T(r,e^{\psi}))$.

Finally, using Lemma 1 we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f-a) + \overline{N}(r,0;f-b) + \overline{N}(r,\infty;f) + O(\log r) \\ &\leq O(T(r,e^{\phi})) + O(T(r,e^{\psi})) + O(\log r) \\ &\leq O(r^{d}) + O(\log r), \end{aligned} \tag{5.8}$$

where $d = \max\{deg(\phi), deg(\psi)\}$.

Clearly (5.8) leads to a contradiction. Therefore we must have $f = g$. $\square$

Proof of Theorem 7. It is given that f, g share $S_1 = \{a\}$ CM and $S_2 = \{b, c\}$ IM. Then using The Second Main Theorem for small functions (see [12], Corollary 1) and proceeding similarly as in (5.1), (5.2) we get $\rho(f) = \rho(g)$. Wve introduce the auxiliary function

$$F = \frac{f-a}{g-a}.$$

Now we can have a rational function U such that $\frac{U(f-a)}{g-a}$ has neither a pole nor a zero in $\mathbb{C}$. Such a U does exist since f, g have only finitely many poles and in view of the assumption that f and g share a CM, a possible zero or pole of $\frac{f-a}{g-a}$ may only come from a pole of g or f. Thus, UF is an entire function without any zero. Hence, there is an entire function η such that

$$UF = \frac{U(f-a)}{g-a} = e^{\eta}. \tag{5.9}$$

Since $f,\ g$ have finite order, by Lemma 3 η is a polynomial such that $deg(\eta) \leq \rho(f)\ (= \rho(g))$.

Now let z_0 be a zero of $(f-b)(f-c)$ such that $a(z_0) \neq b(z_0), c(z_0)$. Then we have

$$F(z_0) = 1 \text{ or } \frac{b(z_0)-a(z_0)}{c(z_0)-a(z_0)} \text{ or } \frac{c(z_0)-a(z_0)}{b(z_0)-a(z_0)}.$$

Therefore z_0 is a zero of

$$\tau = (F-1)\left(F - \frac{b-a}{c-a}\right)\left(F - \frac{c-a}{b-a}\right).$$

Again let z_0 be a common zero of $f-b$ and $f-a$, then z_0 is a a zero of $b-a$. Hence $\overline{N}(r,0;f-b \mid f-a) = O(\log r)$.

Similarly it can be shown that $\overline{N}(r,0;f-a \mid f-c) = O(\log r)$.

Therefore using Lemma 1 we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f-b) + \overline{N}(r,0;f-c) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq \overline{N}(r,0;\tau) + O(\log r) \leq O(T(r,F) + O(\log r) \\ &\leq O(T(r,e^{\eta}/U)) + O(\log r) \\ &\leq O(r^{deg(\eta)}) + O(\log r). \end{aligned}$$

Next resorting to the same analysis as done immediately after (5.7), we can have a contradiction.

Therefore we must have $\tau = 0$.

First let $2a \neq b+c$. We have to consider the following cases.

Case 1. Let $F = 1$. Then we have $f = g$.

Case 2. Let $F = \frac{b-a}{c-a}$ i.e., $\frac{f-a}{g-a} = \frac{b-a}{c-a}$. Now let z_0 be a zero of $f-b$ such that $a(z_0) \neq b(z_0), c(z_0)$. Then the set sharing property of f and g yields that z_0 is a zero of $g-b$ or $g-c$. If z_0 is a zero of $g-b$, then from the given relation $\frac{f-a}{g-a} = \frac{b-a}{c-a}$, we get that z_0 is a zero of $b-c$. Again $\overline{N}(r,0;(a-b)(a-c)) = O(\log r)$. Hence we have $\overline{N}(r,0;f-b \mid g-b) \leq \overline{N}(r,0;b-c) + O(\log r) \leq O(\log r)$.

As from Lemma 5 we know that $\overline{N}(r,0;f-b) \neq O(\log r)$, it follows that $\overline{N}(r,0;f-b \mid g-c) \neq O(\log r)$. Proceeding similarly we can show that $\overline{N}(r,0;f-c \mid g-c) = O(\log r)$ and $\overline{N}(r,0;f-c \mid g-b) \neq O(\log r)$.

Now let us consider $z_1 \in \overline{E}(0,f-c) \cap \overline{E}(0,g-b)$ where $a(z_1)$, $b(z_1)$, $c(z_1)$ are all distinct. Clearly from the above relation we get that z_1 is a zero of $b+c-2a$. Again if $a(z_1)$, $b(z_1)$, $c(z_1)$ are not all distinct, then z_1 is a zero of $(b-a)(b-c)(c-a)$ and $\overline{N}(r,0;(b-a)(b-c)(c-a)) = O(\log r)$. Hence

finally we have $\overline{N}(r,0;f-c)=\overline{N}(r,0;f-c\mid g-c)+\overline{N}(r,0;f-c\mid g-b)\leq O(\log r)+\overline{N}(r,0;b+c-2a)\leq O(\log r)$, a contradiction.
Case 3. Let $F=\frac{c-a}{b-a}$. Proceeding similarly as above we can discard this case.
Next let $2a=b+c$. Then $\frac{b-a}{c-a}=-1=\frac{c-a}{b-a}$. Hence from $\tau=0$ we will get either $f=g$ or $f+g=2a$. □

Proof of Theorem 8. It is given that f and g share $\{b,c\}$ CM. Then using The Second Main Theorem for small functions (see [12], Corollary 1) and proceeding similarly as done in (5.1), (5.2) we get $\rho(f)=\rho(g)$.

Now let us define

$$H=\frac{(f-b)(f-c)\mathcal{U}}{(g-b)(g-c)},$$

where $\mathcal{U}$ is a rational function such that H has no zero and pole. Then there exist an entire function $q(z)$ such that

$$H=\frac{(f-b)(f-c)\mathcal{U}}{(g-b)(g-c)}=e^{q}.$$

Since $\rho(f)$, $\rho(g)$ is finite then clearly from Lemma 3 we get $\rho(e^{q})\leq\rho(f)=\rho(g)$ and so $q(z)$ is a polynomial of finite degree.
Case 1. Suppose $f\not\equiv g$ and $2a\not\equiv b+c$. Now we claim that $f+g\not\equiv b+c$, because if $f+g\equiv b+c$, then we have

$$z_0\in\overline{E}(0,f-a)\implies z_0\in\overline{E}(0,b+c-2a)\implies\overline{N}(r,0;f-a)=O(\log r)$$

which implies '0' is a generalized exceptional value of $f-a$, a contradiction with Lemma 5. Therefore we must have $f+g\not\equiv b+c$.

Now

$$\frac{e^{q}}{\mathcal{U}}-1=\frac{(f-g)(f+g-b-c)}{(g-b)(g-c)},$$

thus $\frac{e^{q}}{\mathcal{U}}-1\not\equiv 0$.

Let z_0 be a zero of $f-a$. If $a(z_0)\neq b(z_0),c(z_0)$ then clearly z_0 is a zero of $\frac{e^{q}}{\mathcal{U}}-1$. If $a(z_0)=b(z_0)$ or $c(z_0)$, then z_0 is a zero of $(b-a)(c-a)$ and we have $\overline{N}(r,0;(b-a)(c-a))=O(\log r)$.

Therefore from the above discussion we get

$$\overline{N}(r,0;f-a)\leq\overline{N}\left(r,0;\frac{e^{q}}{\mathcal{U}}-1\right)+O(\log r)\leq O(r^{deg(q)})+O(\log r).\tag{5.10}$$

Let us consider the following two functions:

$$\Delta_0=\left(\frac{L_{a,b}(f)}{(f-a)(f-b)(f-c)}-\frac{L_{a,b}(g)}{(g-a)(g-b)(g-c)}\right)(f-g)(f+g-b-c),$$

$$\Delta_1=\left(\frac{L_{a,b}(f)}{(f-a)(f-b)(f-c)}+\frac{L_{a,c}(g)}{(g-a)(g-b)(g-c)}\right)(f-g)(f+g-b-c).$$

First we claim that at least one of $\Delta_0,\ \Delta_1$ is identically equal to zero. On the contrary, suppose $\Delta_0 \not\equiv 0$ and $\Delta_1 \not\equiv 0$, which implies $f \not\equiv g$. Now

$$\begin{aligned}
& m(r,\Delta_0) \\
= \;& m\left(r, \frac{(f-g)(f+g-b-c)}{(f-b)(f-c)}.\frac{L_{a,b}(f)}{f-a} - \frac{(f-g)(f+g-b-c)}{(g-b)(g-c)}.\frac{L_{a,b}(g)}{g-a}\right) \\
\leq \;& m\left(r, \frac{\mathcal{U}}{e^q}-1\right) + m\left(r, \frac{L_{a,b}(f)}{(f-a)}\right) + m\left(r, \frac{e^q}{\mathcal{U}}-1\right) + m\left(r, \frac{L_{a,b}(g)}{(g-a)}\right) + O(1) \\
\leq \;& O(T(r, e^q/\mathcal{U})) + O(\log r) \leq O(r^{deg(q)}) + O(\log r).
\end{aligned}$$

It is easy to verify that a point, say, z_0 which is either a zero of $f-a$ or $f-b$ or $f-c$, provided that all of $a(z_0)$, $b(z_0)$, $c(z_0)$ are distinct, is not a pole of Δ_0.

Therefore, $N(r,\infty;\Delta_0) = O(\log r)$. Hence we have

$$T(r,\Delta_0) \leq O(r^{deg(q)}) + O(\log r).$$

Proceeding similarly, we can deduce that $T(r,\Delta_1) \leq O(r^{deg(q)}) + O(\log r)$.

Now let z_0 be a common zero of $f-b$ and $g-b$ of multiplicity k but $b(z_0) \neq c(z_0), a(z_0)$. Then we conclude that z_0 is also a zero of Δ_0. We have

$$\overline{N}(r,0;f-b \mid g-b) \leq \overline{N}(r,0;\Delta_0) + O(\log r) \leq O(r^{deg(q)}) + O(\log r). \quad (5.11)$$

Let z_1 be a common zero of $f-b$ and $g-c$ and $a(z_1), b(z_1), c(z_1)$ are all distinct. Since f and g share $\{b,c\}$ CM, the multiplicity of a common zero say z_1 of $f-b$ and $g-c$ ($b(z_1) \neq c(z_1)$) is the same. So it is easy to verify that z_1 is also a zero of Δ_1.

Hence we have

$$\overline{N}(r,0;f-b \mid g-c) \leq \overline{N}(r,0;\Delta_1) + O(\log r) \leq O(r^{deg(q)}) + O(\log r). \quad (5.12)$$

Using Lemma 1, from (5.10), (5.11) and (5.12) we have

$$\begin{aligned}
T(r,f) &\leq \overline{N}(r,0;f-a) + \overline{N}(r,0;f-b) + \overline{N}(r,\infty;f) + S(r,f) \\
&\leq \overline{N}(r,0;f-a) + \overline{N}(r,0;f-b \mid g-b) + \overline{N}(r,0;f-b \mid g-c) \\
&\quad + O(\log r) \\
&\leq O(r^{deg(q)}) + O(\log r). \qquad (5.13)
\end{aligned}$$

Next following the same analysis as just after (5.7) we can get a contradiction again. Hence our claim is proved. Therefore at least one of $\Delta_0,\ \Delta_1$ is identically zero.

Now let us consider the following cases.

Case 1.1. Let us suppose that $\Delta_0 = 0$. Then by $f \not\equiv g$ and $2a \not\equiv b+c$ we have

$$\frac{L_{a,b}(f)}{(f-a)(f-b)(f-c)} - \frac{L_{a,b}(g)}{(g-a)(g-b)(g-c)} = 0. \quad (5.14)$$

Here f and g share $\{a\}$ IM. Now let us suppose that z_0 is a zero of $f-a$ of multiplicity p and a zero of $g-a$ of multiplicity q and $a(z_0), b(z_0), c(z_0)$ are all distinct. Then considering the coefficient of $\frac{1}{z-z_0}$ from (5.14) we must have $p=q$. Since z_0 is arbitrary, it follows that the number of zeros of $f-a$ and $g-a$ with different multiplicity is finite. Hence

$$N(r,0;f-a)+N(r,0;g-a)-2N(r,0;f-a \mid g-a)=O(\log r).$$

Therefore, $f-a$ and $g-a$ share $\{0\}$ almost CM. Then we can find a rational function $\mathcal{Q}$ such that $F_0=\frac{(f-a)\mathcal{Q}}{(g-a)}$ is a zero free entire function. Hence we can write it in the form

$$F_0=\frac{(f-a)\mathcal{Q}}{(g-a)}=e^{p(z)}$$

for some polynomial $p(z)$.

Now adopting the same method as done in Theorem 7 we will get $f=g$.

Case 1.2. Let us consider $\Delta_1=0$. Then from the assumption $f\neq g$ and $2a\neq b+c$ we have

$$\hat{\Delta}_1=\frac{L_{a,b}(f)}{(f-a)(f-b)(f-c)}+\frac{L_{a,c}(g)}{(g-a)(g-b)(g-c)} \quad = \quad 0. \tag{5.15}$$

Let $S=E(0,(b-a)(c-a)(b-c)(b+c-2a))$. Let z_0 be a zero of $f-b$ of multiplicity k but $z_0\notin S$. Now according to the definition of set sharing for f and g, z_0 is a zero of $g-b$ or $g-c$ of multiplicity k.

Let z_0 be a zero of $g-b$. Now it is given that $b-a, c-a$ are linearly dependent. Here since $z_0\notin S$ and $\hat{\Delta}_1(=0)$ has no pole, considering the coefficient of $\frac{1}{z-z_0}$ we must have $2a=b+c$, a contradiction. Hence z_0 is a zero of $g-c$.

Similarly it can be shown that for some $z_1\notin S$, $z_1\in E(0,g-c)$ implies $z_1\in E(0,f-b)$. Hence $\overline{E}(0,g-c)\backslash S=\overline{E}(0,f-b)\backslash S$. Immediately we have $\overline{E}(0,g-b)\backslash S=\overline{E}(0,f-c)\backslash S$. Since f and g share $\{b,c\}$ CM, any zero z_2, a zero of $g-b$ of multiplicity p, is also a zero of $f-c$ of order p. Now $\overline{N}(r,0;(b-a)(c-a)(b-c)(b+c-2a))=O(\log r)$, therefore from Lemma 5, there exists $z'\notin S$ which is a zero of $g-b$. Again since $\hat{\Delta}_1$ has no pole and $z'\notin S$, from (5.15), considering the coefficient of $\frac{1}{z-z'}$, we must have $2a=b+c$, a contradiction.

Hence, $\overline{E}(0,g-b)\backslash S=\overline{E}(0,f-c)\backslash S=\emptyset$. It follows that $\overline{N}(r,0;g-b)=\overline{N}(r,0;f-c)=O(\log r)$, which again contradicts Lemma 5.

Therefore from the above discussion when $2a\neq b+c$ we must have $f=g$.

Case 2. If $2a=b+c$, then $(f-b)(f-c)=f^2-2af+bc$.

Now let us consider the following function:

$$\delta=\frac{((f-b)(f-c))'}{(f-b)(f-c)}-\frac{((g-b)(g-c))'}{(g-b)(g-c)}.$$

Since f and g share $\{b,c\}$ CM, any zero of $(f-b)(f-c)$, $(g-b)(g-c)$ can not be a pole of δ. Thus

$$\overline{N}(r,\infty;\delta) \le \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) \le O(\log r).$$

Also it is easy to verify that

$$\overline{N}(r,0;f-a) \le \overline{N}(r,0;\delta) + O(\log r) \le T(r,\delta) \le O(\log r),$$

which contradicts Lemma 5. Therefore one must have $\delta = 0$. Now from $\delta = 0$, integrating we get $(f-b)(f-c) = C(g-b)(g-c)$ for some constant C. First we note that the common zeros of $f-a$ and $(f-b)(f-c)$ are zeros of $(a-b)(a-c)$ and $\overline{N}(r,0;(a-b)(a-c)) = O(\log r)$. Since f and g share $\{a\}$ IM, from Lemma 5 it follows that there are infinitely many zeros of $f-a$ which are not the zeros of $(f-b)(f-c)$.

Considering such a zero of $f-a$, which is not a zero of $(f-b)(f-c)$, we will have $C=1$ and this implies either $f=g$ or $f+g=2a$. □

Proof of Theorem 9. Let

$$G_0 = \frac{(f-a)(f-b)}{(g-a)(g-b)}.$$

Now since f, g share $\{a,b\}$ CM and f, g have finitely many poles, we can find a rational function $\hat{Q}$ such that $\hat{Q}G_0$ has no zero and no pole. Then we can write

$$\hat{Q}G_0 = \frac{(f-a)(f-b)\hat{Q}}{(g-a)(g-b)} = e^v. \tag{5.16}$$

Using The Second Main Theorem for small functions (see [12]), proceeding similarly as done in (5.1), (5.2) we get $\rho(f) = \rho(g)$. Hence by Lemma 3 we get that v is a polynomial with $deg(v) \le \rho(f)$ $(= \rho(g))$.

It is given that f and g share $\{c,d\}$ IM. Let us consider z_0 to be a zero of $(f-c)(f-d)$ but $z_0 \notin S = \overline{E}(0,(b-a)(b-c)(b-d)(c-d)(c-a)(d-a))$. Clearly z_0 is a zero of Ψ, where Ψ is defined by

$$\Psi = (G_0 - 1)\left(G_0 - \frac{(c-a)(c-b)}{(d-a)(d-b)}\right)\left(G_0 - \frac{(d-a)(d-b)}{(c-a)(c-b)}\right).$$

If $z_0 \in S$, then from $\overline{N}(r,0;(b-a)(b-c)(b-d)(c-d)(c-a)(d-a)) = O(\log r)$ and from the above finally we get

$$\begin{aligned}
\overline{N}(r,0;f-c) + \overline{N}(r,0;f-d) &\le \overline{N}(r,0;\Psi) + O(\log r) \\
&\le O(T(r,G_0)) + O(\log r) \\
&\le O(T(r,e^v/\hat{Q})) + O(\log r) \le O(r^{deg(v)}) \\
&\quad + O(\log r).
\end{aligned} \tag{5.17}$$

Now from Lemma 1 and (5.17) we have

$$T(r,f) \le \overline{N}(r,0;f-c) + \overline{N}(r,0;f-d) + \overline{N}(r,\infty;f)$$

$$\leq \quad O(r^{deg(v)}) + O(\log r). \tag{5.18}$$

Here from the discussion after (5.7) we get a contradiction again. Therefore $\Psi = 0$.

Now let us consider the following cases.

Case 1. If $G_0 = 1$, then we have $f = g$ or $f + g = a + b$. Since f and g share $\{c, d\}$ IM, from $f + g = a + b$ we must have either $a + b = c + d$ or $a + b = 2c = 2d$, which leads to a contradiction. Hence we will have $f = g$.

Case 2. $G_0 = \frac{(c-a)(c-b)}{(d-a)(d-b)}$.

Let z_0 be a zero of $f - d$ and $z_0 \notin S$. Since f and g share $\{c, d\}$ IM, z_0 may be a zero of $g - c$ or $g - d$. Let z_0 be a zero of $g - d$, then from the given relation we have that z_0 is a zero of $(c - d)(c + d - a - b)$. Hence $\big(\overline{E}(0, f - d) \cap \overline{E}(0, g - d)\big) \setminus S \subseteq \overline{E}(0, (c-d)(c+d-a-b))$ implies $\overline{N}(r, 0; f - d \mid g - d) = O(\log r)$. Therefore from Lemma 5, $\overline{N}(r, 0; f - d \mid g - c) \neq O(\log r)$.

Now let z_1 ($\notin S$) be a zero of $f - d$ and $g - c$. Then from the given relation we have that z_1 is a zero $(c - a)^2(c - b)^2 - (d - a)^2(d - b)^2$. Hence $\big(\overline{E}(0, f - d) \cap \overline{E}(0, g - c)\big) \setminus S \subseteq \overline{E}(0, (c - a)^2(c - b)^2 - (d - a)^2(d - b)^2)$.

Now from the above discussion and the line just before (5.17) we have

$$\begin{aligned}
\overline{N}(r, 0; f - d) &= \overline{N}(r, 0; f - d \mid g - d) + \overline{N}(r, 0; f - d \mid g - c) \\
&\leq \overline{N}(r, 0; (c - d)(c + d - a - b)) + \overline{N}(r, 0; (c - a)^2(c - b)^2 \\
&\quad -(d - a)^2(d - b)^2) + O(\log r) \\
&\leq O(\log r),
\end{aligned}$$

which contradicts Lemma 5.

Proceeding in the same way as in **Case 2**, we can discard the third case: $G_0 = \frac{(d-a)(d-b)}{(c-a)(c-b)}$. This completes the proof of the theorem. □

Proof of Corollary 1. Here we will prove the corollary only for Theorem 9. For the remaining theorems, it can be proved easily in the same manner.

Let us consider the case when f and g share $\{a, b\}$ almost CM and $\{c, d\}$ almost IM. Denote the functions $F = (f - a)(f - b)$, $G = (g - a)(g - b)$ and $F_o = (f - c)(f - d)$, $G_o = (g - c)(g - d)$.

Now let us define the function

$$\Psi = \frac{F'}{F} - \frac{G'}{G}.$$

We note that the zeros of F and G with different multiplicities and poles of F, G are responsible for poles of Ψ, i.e.,

$$\begin{aligned}
\overline{N}(r, \infty; \Psi) = N(r, \infty; \Psi) &\leq N(r, 0; F) + N(r, 0; G) - 2N(r, 0; F \mid G) \\
&\quad +\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) \\
&\leq O(\log r).
\end{aligned}$$

Now, $T(r,\Psi)=m(r,\Psi)+N(r,\infty;\Psi)=O(\log r)$, and this implies that Ψ is a rational function having simple poles. Then we can write it as

$$\Psi=\frac{F'}{F}-\frac{G'}{G}=P+\sum_{i=1}^{m}\frac{a_i}{z-b_i}, \tag{5.19}$$

where P is a polynomial. Now integrating both sides of (5.19) we get

$$\frac{F}{G}=\frac{(f-a)(f-b)}{(g-a)(g-b)}=\prod_{i=1}^{m}(z-b_i)^{a_i}e^{\int P}=Q_oe^{p}, \tag{5.20}$$

where p is a polynomial of finite degree and $Q_o=\prod_{i=1}^{m}(z-b_i)^m$.

Also here f and g share $\{c,d\}$ almost IM, therefore $\overline{N}(r,0;F_o)-\overline{N}(r,0;F_o\mid G_o)=O(\log r)$. Now from Lemma 5 and from the above discussion we must have $\overline{N}(r,0;F_o)\neq O(\log r)$. Here from (5.20) we have

$$\begin{aligned}\overline{N}(r,0;(f-c)(f-d)) &\leq \overline{N}(r,0;(Q_oe^p-1)(Q_oe^p-x)(Q_oe^p-1/x))\\ &\quad +O(\log r),\end{aligned} \tag{5.21}$$

where $x=\frac{(c-a)(c-b)}{(d-a)(d-b)}$.

Now from (5.21) we get

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,0;f-c)+\overline{N}(r,0;f-d)+\overline{N}(r,\infty;f)\\ &\leq O(T(r,e^p))+O(\log r).\end{aligned}$$

Now proceeding similarly as done after (5.18) in Theorem 9 the rest of the proof can be carried out to get the desired result.

Adopting the same procedure we can get the other results also from strict CM (IM) sharing to almost CM (almost IM) sharing.

□

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References

[1] J. F. Chen, *Uniqueness of meromorphic functions sharing two finite sets*, Open Math. **15** (2017), 1244–1250.

[2] J. F. Chen, *Meromorphic functions sharing unique range sets with one or two elements*, Georgian Math. J. (2019), https://doi.org/10.1515/gmj-2019-2073.

[3] J. B. Conway, *Functions of one Complex Variable*, Grad. Texts Math. **11**, Springer, New York, 1973.

[4] M. L. Fang, *Unicity theorem or meromorphic function and its differential polynomial*, Adv. Math. (PRC) **24** (1995), 244–249.

[5] Q. Han, *Some uniqueness results related to L-functions*, Boll. Unione Mat. Ital. **10** (2017), 503–515.

[6] B. Q. Li, *A result on value distribution of L-functions*, Proc. Amer. Math. Soc. **138** (2010), 2071–2077.

[7] P. Lin and W. Lin, *Value distribution of L-functions concerning sharing sets*, Filomat **30** (2016), 3795–3806.

[8] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, in: Proccedings of the Amalfi Conference on Analytic Number Theory, (Maiori, 1989), Univ. Salerno, Salerno (1992), 367–385.

[9] J. Steuding, *Value Distribution of L-Functions*, Lect. Notes Math. **1877**, Springer, Berlin, 2007.

[10] E. C. Titchmarsh, *The Theory of Functions, 2nd ed.*, Oxford Univ. Press, Oxford, 1939.

[11] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Math. Appl. **557**, Kluwer Academic Press, Dordrecht, 2003.

[12] K. Yamaonoi, *The second main theorem for small functions and related problems*, Acta Math **192** (2004), 225–294.

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