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Coefficients bounds for a family of bi-univalent functions defined by Horadam polynomials

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Abstract. In the present paper, we determine upper bounds for the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for a certain family of holomorphic and bi-univalent functions defined by using the Horadam polynomials. Also, we solve Fekete–Szegö problem of functions belonging to this family. Further, we point out several special cases of our results.

1. Introduction

Denote by $\mathcal{A}$ the collection of holomorphic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, assume that S stands for the sub-collection of the set $\mathcal{A}$ containing functions in U satisfying (1) which are univalent in U.

According to the Koebe One-Quarter Theorem [7] every function $f \in S$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z$ $(z \in U)$ and $f(f^{-1}(w)) = w$ $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ stand for the class of bi-univalent functions in U given by (1). In fact, Srivastava et al. [20] have actually revived the study of holomorphic and bi-univalent functions in recent years, it was followed

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by such works as those by Bulut [6], Adegani and et al. [2], Güney et al. [9], Srivastava and Wanas [21] and others (see, for example [8, 19, 22]). We notice that the class Σ is not empty. For example, the functions z, $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log\frac{1+z}{1-z}$ are members of Σ. Until now, the coefficient estimate problem for each of the following Taylor–Maclaurin coefficients $|a_n|$ $(n = 3, 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem.

With a view to recalling the principle of subordination between holomorphic functions, let the functions f and g be holomorphic in U. We say that the function f is subordinate to g, if there exists a Schwarz function ω holomorphic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$ such that $f(z) = g(\omega(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. It is well known that (see [17]), if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Recently, Hörçum and Koçer [13] considered the Horadam polynomials $h_n(r)$, which are given by the following recurrence relation (see also Horadam and Mahon [12]):

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (r \in \mathbb{R}, n \in \mathbb{N}\setminus\{1,2\}; \mathbb{N} = \{1,2,3,\cdots\}), \tag{3}$$

with $h_1(r) = a$ and $h_2(r) = br$, for some real constants a, b, p and q. The characteristic equation of repetition relation (3) is $t^2 - prt - q = 0$. This equation has two real roots $x = \frac{pr+\sqrt{p^2r^2+4q}}{2}$ and $y = \frac{pr-\sqrt{p^2r^2+4q}}{2}$.

Remark 1. By selecting the particular values of a, b, p and q, the Horadam polynomial $h_n(r)$ reduces to several polynomials. Some of them are illustrated below.

(1) Taking $a = b = p = q = 1$, we obtain the Fibonacci polynomials $F_n(r)$.
(2) Taking $a = 2$ and $b = p = q = 1$, we attain the Lucas polynomials $L_n(r)$.
(3) Taking $a = q = 1$ and $b = p = 2$, we have the Pell polynomials $P_n(r)$.
(4) Taking $a = b = p = 2$ and $q = 1$, we get the Pell–Lucas polynomials $Q_n(r)$.
(5) Taking $a = b = 1$, $p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $T_n(r)$ of the first kind.
(6) Taking $a = 1$, $b = p = 2$ and $q = -1$, we have the Chebyshev polynomials $U_n(r)$ of the second kind.

These polynomials, the families of orthogonal polynomials and other special polynomials as well as their generalizations are potentially important in a variety of disciplines in many sciences, especially in mathematics, statistics and physics. For more information associated with these polynomials see [11, 12, 14].

The generating function of the Horadam polynomials $h_n(r)$ (see [13]) is given by

$$\Pi(r,z)=\sum_{n=1}^{\infty}h_n(r)z^{n-1}=\frac{a+(b-ap)rz}{1-prz-qz^2}. \tag{4}$$

In fact, Srivastava et al. [18] have already applied the Horadam polynomials in a similar context involving analytic and bi-univalent functions. It was followed by such works as those by Abirami et al. [1], Al-Hawary et al. [10], Wanas and Yalçin [23].

2. Main Results

We begin this section by defining the family $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$ as follows.

Definition 1. For $0\le\gamma\le 1$, $0\le\lambda\le 1$, $0\le\delta\le 1$ and $r\in\mathbb{R}$, a function $f\in\Sigma$ is said to be in the family $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$ if it satisfies the subordinations

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left[(1-\delta)\frac{zf'(z)}{f(z)}+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\lambda}\prec\Pi(r,z)+1-a$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left[(1-\delta)\frac{wg'(w)}{g(w)}+\delta\left(1+\frac{wg''(w)}{g'(w)}\right)\right]^{\lambda}\prec\Pi(r,w)+1-a,$$

where a is real constant and the function $g=f^{-1}$ is given by (2).

Remark 2. It should be remarked that the family $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$ is a generalization of well-known families considered earlier. These families are:

(1) for $\gamma=0$ and $\lambda=1$, the family $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$ reduces to the family $M_\Sigma(\delta,r)$ which was introduced by Magesh et al. [16];
(2) for $\lambda=0$ and $\gamma=1$, the family $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$ reduces to the family $S^*_\Sigma(r)$ which was studied by Srivastava et al. [18];
(3) for $\gamma=0$ and $\lambda=\delta=1$, the family $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$ reduces to the family $\mathcal{K}_\Sigma(r)$ which was considered by Magesh et al. [16];
(4) for $\gamma=0$, $\lambda=a=1$, $b=p=2$, $q=-1$ and $r\longrightarrow t$, the family $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$ reduces to the family $H_\Sigma(\delta,t)$ which was given by Altınkaya and Yalçin [3];
(5) for $\lambda=0$, $\gamma=a=1$, $b=p=2$, $q=-1$ and $r\longrightarrow t$, the family $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$ reduces to the family $S_\Sigma(t)$ which was introduced by Altınkaya and Yalçin [4];
(6) for $\gamma=0$, $\lambda=a=1$, $b=p=2$, $q=-1$, $r\longrightarrow t$ and $\Pi(t,z)=\left(\frac{1}{1-2tz+z^2}\right)^{\alpha}$, $0<\alpha\le 1$, the family $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$ reduces to the family $M_\Sigma(\alpha,\delta)$ which was investigated by Liu and Wang [15];

(7) for $\lambda = 0$, $\gamma = a = 1$, $b = p = 2$, $q = -1$, $r \longrightarrow t$ and $\Pi(t,z) = \left(\frac{1}{1-2tz+z^2}\right)^{\alpha}$, $0 < \alpha \leq 1$, the family $\mathcal{F}_{\Sigma}(\gamma,\lambda,\delta,r)$ reduces to the family $S^*_{\Sigma}(\alpha)$ which was studied by Brannan and Taha [5].

Theorem 1. *For $0 \leq \gamma \leq 1$, $0 \leq \lambda \leq 1$, $0 \leq \delta \leq 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the family $\mathcal{F}_{\Sigma}(\gamma,\lambda,\delta,r)$. Then*

$$|a_2| \leq \frac{|br|\sqrt{2|br|}}{\sqrt{\left|\left[\left(\gamma(\gamma+1)+\Upsilon(\lambda,\delta,\gamma)\right)b - 2p\Phi^2(\lambda,\delta,\gamma)\right]br^2 - 2qa\Phi^2(\lambda,\delta,\gamma)\right|}}$$

and

$$|a_3| \leq \frac{|br|}{2\left(\gamma+\lambda(2\delta+1)\right)} + \frac{b^2r^2}{2\Phi^2(\lambda,\delta,\gamma)},$$

where

$$\begin{aligned} \Upsilon(\lambda,\delta,\gamma) &= \lambda(\delta+1)\left[2(\gamma+1)+(\lambda-1)(\delta+1)\right], \\ \Phi(\lambda,\delta,\gamma) &= \gamma+\lambda(\delta+1). \end{aligned} \tag{5}$$

Proof. Let $f \in \mathcal{F}_{\Sigma}(\gamma,\lambda,\delta,r)$. Then there are two holomorphic functions $u, v : U \longrightarrow U$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \quad (z \in U) \tag{6}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \quad (w \in U), \tag{7}$$

with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, $z, w \in U$ such that

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left[(1-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\lambda} = \Pi(r,u(z)) + 1 - a$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left[(1-\delta)\frac{wg'(w)}{g(w)} + \delta\left(1+\frac{wg''(w)}{g'(w)}\right)\right]^{\lambda} = \Pi(r,v(w)) + 1 - a.$$

Or, equivalently

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left[(1-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\lambda} = 1 + h_1(r) + h_2(r)u(z) + h_3(r)u^2(z) + \cdots \tag{8}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left[(1-\delta)\frac{wg'(w)}{g(w)} + \delta\left(1+\frac{wg''(w)}{g'(w)}\right)\right]^{\lambda} = 1 + h_1(r) + h_2(r)v(w) + h_3(r)v^2(w) + \cdots. \tag{9}$$

Combining (6), (7), (8) and (9) yields

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left[(1-\delta)\frac{zf'(z)}{f(z)}+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\lambda}$$
$$=1+h_2(r)u_1z+\left[h_2(r)u_2+h_3(r)u_1^2\right]z^2+\cdots \quad (10)$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left[(1-\delta)\frac{wg'(w)}{g(w)}+\delta\left(1+\frac{wg''(w)}{g'(w)}\right)\right]^{\lambda}$$
$$=1+h_2(r)v_1w+\left[h_2(r)v_2+h_3(r)v_1^2\right]w^2+\cdots. \quad (11)$$

It is quite well-known that if $|u(z)|<1$ and $|v(w)|<1$, $z,w\in U$, then

$$|u_i|\le 1 \quad \text{and} \quad |v_i|\le 1 \text{ for all } i\in\mathbb{N}. \quad (12)$$

Comparing the corresponding coefficients in (10) and (11), after simplifying, we have

$$(\gamma+\lambda(\delta+1))\,a_2=h_2(r)u_1, \quad (13)$$

$$2\left(\gamma+\lambda(2\delta+1)\right)a_3$$
$$+\frac{1}{2}\left[\gamma(\gamma-1)+\lambda(\delta+1)\left(2\gamma+(\lambda-1)(\delta+1)\right)-2\left(\gamma+\lambda(3\delta+1)\right)\right]a_2^2$$
$$=h_2(r)u_2+h_3(r)u_1^2, \quad (14)$$

$$-\left(\gamma+\lambda(\delta+1)\right)a_2=h_2(r)v_1 \quad (15)$$

and

$$2\left(\gamma+\lambda(2\delta+1)\right)(2a_2^2-a_3)$$
$$+\frac{1}{2}\left[\gamma(\gamma-1)+\lambda(\delta+1)\left(2\gamma+(\lambda-1)(\delta+1)\right)-2\left(\gamma+\lambda(3\delta+1)\right)\right]a_2^2$$
$$=h_2(r)v_2+h_3(r)v_1^2. \quad (16)$$

It follows from (13) and (15) that

$$u_1=-v_1 \quad (17)$$

and

$$2\left(\gamma+\lambda(\delta+1)\right)^2a_2^2=h_2^2(r)(u_1^2+v_1^2). \quad (18)$$

If we add (14) to (16), we find that

$$\left[\gamma(\gamma+1)+\lambda(\delta+1)\left(2(\gamma+1)+(\lambda-1)(\delta+1)\right)\right]a_2^2$$
$$=h_2(r)(u_2+v_2)+h_3(r)(u_1^2+v_1^2). \quad (19)$$

Substituting the value of $u_1^2+v_1^2$ from (18) in the right hand side of (19), we deduce that

$$a_2^2=\frac{h_2^3(r)(u_2+v_2)}{h_2^2(r)\left(\gamma(\gamma+1)+\Upsilon(\lambda,\delta,\gamma)\right)-2h_3(r)\Phi^2(\lambda,\delta,\gamma)}. \quad (20)$$

where $\Upsilon(\lambda,\delta,\gamma)$ and $\Phi(\lambda,\delta,\gamma)$ are given by (5).
Further computations using (3), (12) and (20) give

$$|a_2| \leq \frac{|br|\sqrt{2|br|}}{\sqrt{\left|\left[\left(\gamma(\gamma+1)+\Upsilon(\lambda,\delta,\gamma)\right)b-2p\Phi^2(\lambda,\delta,\gamma)\right]br^2-2qa\Phi^2(\lambda,\delta,\gamma)\right|}}.$$

Next, if we subtract (16) from (14), we can easily see that

$$4\left(\gamma+\lambda(2\delta+1)\right)\left(a_3-a_2^2\right)=h_2(r)(u_2-v_2)+h_3(r)(u_1^2-v_1^2). \tag{21}$$

In view of (17) and (18), we get from (21)

$$a_3=\frac{h_2(r)(u_2-v_2)}{4\left(\gamma+\lambda(2\delta+1)\right)}+\frac{h_2^2(r)(u_1^2+v_1^2)}{2\Phi^2(\lambda,\delta,\gamma)}.$$

Thus applying (3), we obtain

$$|a_3| \leq \frac{|br|}{2\left(\gamma+\lambda(2\delta+1)\right)}+\frac{b^2r^2}{2\Phi^2(\lambda,\delta,\gamma)}.$$

This completes the proof of Theorem 1. □

In the next theorem, we discuss the "Fekete–Szegö problem" for the family $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$.

Theorem 2. *For* $0\leq\gamma\leq1$, $0\leq\lambda\leq1$, $0\leq\delta\leq1$ *and* $r,\mu\in\mathbb{R}$, *let* $f\in\mathcal{A}$ *be in the family* $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$. *Then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{|br|}{2(\gamma+\lambda(2\delta+1))}, \\ for\ \ |\mu-1| \leq \frac{\left|\left[\left(\gamma(\gamma+1)+\Upsilon(\lambda,\delta,\gamma)\right)b-2p\Phi^2(\lambda,\delta,\gamma)\right]br^2-2qa\Phi^2(\lambda,\delta,\gamma)\right|}{4b^2r^2(\gamma+\lambda(2\delta+1))}; \\ \frac{2|br|^3|\mu-1|}{\left|\left[\left(\gamma(\gamma+1)+\Upsilon(\lambda,\delta,\gamma)\right)b-2p\Phi^2(\lambda,\delta,\gamma)\right]br^2-2qa\Phi^2(\lambda,\delta,\gamma)\right|}, \\ for\ \ |\mu-1| \geq \frac{\left|\left[\left(\gamma(\gamma+1)+\Upsilon(\lambda,\delta,\gamma)\right)b-2p\Phi^2(\lambda,\delta,\gamma)\right]br^2-2qa\Phi^2(\lambda,\delta,\gamma)\right|}{4b^2r^2(\gamma+\lambda(2\delta+1))}. \end{cases}$$

Proof. It follows from (20) and (21) that

$$a_3-\mu a_2^2=\frac{h_2(r)(u_2-v_2)}{4\left(\gamma+\lambda(2\delta+1)\right)}+(1-\mu)\,a_2^2=\frac{h_2(r)(u_2-v_2)}{4\left(\gamma+\lambda(2\delta+1)\right)}$$
$$+\frac{h_2^3(r)(u_2+v_2)(1-\mu)}{h_2^2(r)\left(\gamma(\gamma+1)+\Upsilon(\lambda,\delta,\gamma)\right)-2h_3(r)\Phi^2(\lambda,\delta,\gamma)}$$
$$=h_2(r)\left[\left(\psi(\mu,r)+\frac{1}{4\left(\gamma+\lambda(2\delta+1)\right)}\right)u_2+\left(\psi(\mu,r)-\frac{1}{4\left(\gamma+\lambda(2\delta+1)\right)}\right)v_2\right],$$

where

$$\psi(\mu,r)=\frac{h_2^2(r)(1-\mu)}{h_2^2(r)\left(\gamma(\gamma+1)+\Upsilon(\lambda,\delta,\gamma)\right)-2h_3(r)\Phi^2(\lambda,\delta,\gamma)}.$$

According to (3), we find that

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{|br|}{2(\gamma+\lambda(2\delta+1))}, & 0 \leq |\psi(\mu,r)| \leq \frac{1}{4(\gamma+\lambda(2\delta+1))}; \\ \\ 2\,|br|\,|\psi(\mu,r)|\,, & |\psi(\mu,r)| \geq \frac{1}{4(\gamma+\lambda(2\delta+1))}. \end{cases}$$

After some computations, we have proved the statement of Theorem 2. □

Putting $\mu = 1$ in Theorem 2, we obtain the following result:

Corollary 1. *For $0 \leq \gamma \leq 1$, $0 \leq \lambda \leq 1$, $0 \leq \delta \leq 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the family $\mathcal{F}_\Sigma(\gamma,\lambda,\delta,r)$. Then*

$$\left|a_3-a_2^2\right| \leq \frac{|br|}{2\left(\gamma+\lambda(2\delta+1)\right)}.$$

Remark 3. Our theorems give the known results for special families which are described in (1) − (5) of Remark 2.

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