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On the bicomplex Gaussian Fibonacci and Gaussian Lucas numbers

Bahar Kuloğlu and Engİn Özkan

Abstract. We give the bicomplex Gaussian Fibonacci and the bicomplex Gaussian Lucas numbers and establish the generating functions and Binet's formulas related to these numbers. Also, we present the summation formula, matrix representation and Honsberger identity and their relationship between these numbers. Finally, we show the relationships among the bicomplex Gaussian Fibonacci, the bicomplex Gaussian Lucas, Gaussian Fibonacci, Gaussian Lucas and Fibonacci numbers.

1. Introduction

Number sequences and their many generalizations have been the subject of study by scientists [10, 11, 12]. Some generalizations of number sequences are also formed with bicomplex numbers introduced by Segre in 1892 [15, 2, 7, 8]. The bicomplex numbers are defined by four bases elements $1, i, j, ij$ where i, j, ij satisfy the following properties:

$$i^2 = -1,\ j^2 = -1 \text{ and } ij = ji.$$

A bicomplex number can be expressed in the following form:

$$a = a_0 + ia_1 + ja_2 + ija_3 = (a_0 + ia_1) + j(a_2 + ia_3).$$

For more details about the bicomplex numbers one can see in [1, 3, 6, 14, 16].

The bicomplex Fibonacci and Lucas numbers are introduced as:

$$BF_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}ij,$$

$$BL_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}ij,$$

where F_n and L_n are the n-th Fibonacci and the n-th Lucas number, respectively, i and j are imaginary units [6].

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Gaussian Fibonacci and Gaussian Lucas numbers and their polynomials are widely studied by many researchers [4, 5, 9, 13]. These numbers are defined by the following recurrences [4]:

$$GF_n = GF_{n-1} + GF_{n-2}, \quad n \geq 2,$$

with $GF_0 = i$, $GF_1 = 1$, and

$$GL_n = GL_{n-1} + GL_{n-2}, \quad n \geq 2,$$

with $GL_0 = 2 - i$, $GL_1 = 1 + 2i$. The n-th term of Gaussian Fibonacci and Gaussian Lucas numbers is given, respectively, by

$$GF_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}, \tag{1}$$

$$GL_n = \alpha^n + \beta^n + i(\alpha^{n-1} + \beta^{n-1}), \tag{2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.

Now, we bring some important identities related to Gaussian Fibonacci and Gaussian Lucas numbers [4]:

(a) $GL_n = GF_{n+1} + GF_{n-1}$, $n \geq 1$,
(b) $GF_{n+1}^2 + GF_n^2 = F_{2n}(1 + 2i)$,
(c) $GF_n GL_n = F_{2n-1}(1 + 2i)$,
(d) $GF_{n+1}GF_{p+1} + GF_n GF_p = F_{n+p}(1 + 2i)$,
(e) $GL_n^2 - 5GF_n^2 = (-1)^n 4(2 - i)$,
(f) $\sum_{j=0}^{n} GF_j = GF_{n+2} - 1$,
(g) $\sum_{j=1}^{n} GF_{2j-1} = GF_{2n} - 1$, $n \geq 1$,
(h) $\sum_{j=0}^{n} GF_{2j} = GF_{2n+1} - 1$.

After all this, we can introduce the bicomplex Gaussian Fibonacci and bicomplex Gaussian Lucas numbers.

2. The bicomplex Gaussian Fibonacci numbers

The n-th bicomplex Gaussian Fibonacci number is defined for $n \geq 0$ by

$$BGF_n = GF_n + iGF_{n+1} + jGF_{n+2} + ijGF_{n+3}, \tag{3}$$

where GF_n is the n-th Gaussian Fibonacci number, i and j are imaginary units ($i^2 = -1$, $j^2 = -1$, $(ij)^2 = 1$).

Here we can get the following relation:

$$BGF_n = BGF_{n-1} + BGF_{n-2}, \tag{4}$$

where $BGF_0 = 2i + 3ij$ and $BGF_1 = i + 4ij$.

Let BGF_n and BGF_r be two bicomplex Gaussian Fibonacci numbers. Then the sum, difference and product of them are defined by

$$\begin{aligned} BGF_n \pm BGF_r &= (GF_n \pm GF_r) + i\,(GF_{n+1} \pm GF_{r+1}) \\ &\quad + j\,(GF_{n+2} \pm GF_{r+2}) + ij(GF_{n+3} \pm GF_{r+3}), \end{aligned}$$

$$\begin{aligned} BGF_n \quad \times \quad & BGF_r \\ = \quad & (GF_nGF_r - GF_{n+1}GF_{r+1} - GF_{n+2}GF_{r+2} + GF_{n+3}GF_{r+3}) \\ & +i\,(GF_nGF_{r+1} + GF_{n+1}GF_r - GF_{n+2}GF_{r+3} - GF_{n+3}GF_{r+2}) \\ & +j\,(GF_nGF_{r+2} - GF_{n+1}GF_{r+3} + GF_{n+2}GF_r - GF_{n+3}GF_{r+1}) \\ & +ij\,(GF_nGF_{r+3} + GF_{n+1}GF_{r+2} + GF_{n+2}GF_{r+1} + GF_{n+3}GF_r)\,. \end{aligned}$$

We now give some properties related to the bicomplex Gaussian Fibonacci numbers.

Theorem 1. *We have the following relations:*

$$\begin{aligned} {BGF_n}^2 + {BGF_{n+1}}^2 \quad = \quad & (1+2i)[2BGF_{2n} - F_{2n} - F_{2n+2} - F_{2n+4} + F_{2n+6} \\ & -2(iF_{2n+5} + jF_{2n+4} - ijF_{2n+3})], \end{aligned}$$

$$\begin{aligned} {BGF_{n+1}}^2 - {BGF_{n-1}}^2 \quad = \quad & (1+2i)[2BGF_{2n-1} - F_{2n-1} - F_{2n+1} - F_{2n+3} \\ & +F_{2n+5} - 2(iF_{2n+4} + jF_{2n+3} - ijF_{2n+2})], \end{aligned}$$

$$\begin{aligned} BGF_n \quad - \quad & i\;BGF_{n+1} + jBGF_{n+2} - ijBGF_{n+3} \\ = \quad & GF_n + 2j\,(GF_{n+2} + GF_{n+4}) + GF_{n+2} - GF_{n+4} - GF_{n+6}, \end{aligned}$$

$$\begin{aligned} BGF_n \quad - \quad & i\;BGF_{n+1} - jBGF_{n+2} - ijBGF_{n+3} = GF_n + GF_{n+2} \\ + \quad & GF_{n+4} - GF_{n+6} + 2iGF_{n+5} + 2jGF_{n+4} - 2ijGF_{n+3}. \end{aligned}$$

Proof. Using equality (3), we obtain

$$\begin{aligned} & {BGF_n}^2 + {BGF_{n+1}}^2 = \left({GF_n}^2 + {GF_{n+1}}^2\right) - \left({GF_{n+1}}^2 + {GF_{n+2}}^2\right) \\ & - \left({GF_{n+2}}^2 + {GF_{n+3}}^2\right) + \left({GF_{n+3}}^2 + {GF_{n+4}}^2\right) \\ & +2i\,(GF_nGF_{n+1} + GF_{n+1}GF_{n+2} + jGF_nGF_{n+3} + jGF_{n+1}GF_{n+4}) \\ & +2i\,(jGF_{n+1}GF_{n+2} + jGF_{n+2}GF_{n+3} - GF_{n+2}GF_{n+3} - GF_{n+3}GF_{n+4}) \\ & +2j(-GF_{n+1}GF_{n+3} - GF_{n+2}GF_{n+4} + GF_nGF_{n+2} + GF_{n+1}GF_{n+3}). \end{aligned}$$

Using identity (d) in Section 1, we have

$$\begin{aligned} & {BGF_n}^2 + {BGF_{n+1}}^2 = 1 + 2i\,(F_{2n} - F_{2n+2} - F_{2n+4} + F_{2n+6}) \\ & \quad + (1+2i)\,2i\,(F_{2n+1} + 2jF_{2n+3} - F_{2n+5}) + (1+2i)\,2j(F_{2n+2} - F_{2n+4}) \\ & = 1 + 2i\,(F_{2n} - F_{2n+2} - F_{2n+4} + F_{2n+6}) \\ & \quad + (1+2i)\,(2\,(iF_{2n+1} + jF_{2n+2} + ijF_{2n+3})) \\ & \quad -2(iF_{2n+5} + jF_{2n+4} - ijF_{2n+3}). \end{aligned}$$

From the definition of bicomplex Fibonacci numbers, we obtain the first relation

$$\begin{aligned} {BGF_n}^2 + {BGF_{n+1}}^2 = (1+2i)[& 2BGF_{2n} - F_{2n} - F_{2n+2} - F_{2n+4} + F_{2n+6} \\ & - 2(iF_{2n+5} + jF_{2n+4} - ijF_{2n+3})]. \end{aligned}$$

For the second relation we have

$$
\begin{aligned}
& BGF_{n+1}{}^2 - BGF_{n-1}{}^2 = \left(GF_{n+1}{}^2 - GF_{n-1}{}^2\right) \\
- & \left(GF_{n+2}{}^2 - GF_n{}^2\right) - \left(GF_{n+3}{}^2 - GF_{n+1}{}^2\right) + \left(GF_{n+4}{}^2 - GF_{n+2}{}^2\right) \\
+ & 2i\left(GF_{n+1}GF_{n+2} - GF_{n-1}GF_n - \left(GF_{n+3}GF_{n+4} - GF_{n+1}GF_{n+2}\right)\right) \\
+ & 2j\left(GF_{n+1}GF_{n+3} - GF_{n-1}GF_{n+1} - \left(GF_{n+2}GF_{n+4} - GF_nGF_{n+2}\right)\right) \\
+ & 2ij\left(GF_{n+1}GF_{n+4} - GF_{n-1}GF_{n+2} + \left(GF_{n+2}GF_{n+3} - GF_nGF_{n+1}\right)\right).
\end{aligned}
$$

Using identity (d) in Section 1, we get

$$
\begin{aligned}
& BGF_{n+1}{}^2 - BGF_{n-1}{}^2 = (1+2i)\left(F_{2n-1} - F_{2n+1} - F_{2n+3} + F_{2n+5}\right) \\
& +2(1+2i)\left(iF_{2n} - iF_{2n+4} + jF_{2n+1} - jF_{2n+3} + ijF_{2n+3} + ijF_{2n+2}\right) \\
= \; & 2(1+2i)\left(iF_{2n} + jF_{2n+1} + ijF_{2n+2} - iF_{2n+4} - jF_{2n+3} + ijF_{2n+2}\right).
\end{aligned}
$$

From the definition of bicomplex Fibonacci numbers we obtain

$$
\begin{aligned}
BGF_{n+1}{}^2 - BGF_{n-1}{}^2 = (1+2i)[& 2BGF_{2n-1} - F_{2n-1} - F_{2n+1} - F_{2n+3} \\
& + F_{2n+5} - 2(iF_{2n+4} + jF_{2n+3} - ijF_{2n+2})].
\end{aligned}
$$

Other relations can be proved similarly. □

The generating function for the bicomplex Gaussian Fibonacci numbers is given in the following theorem.

Theorem 2. *Let $GGf(t)$ be a generating function for the bicomplex Gaussian Fibonacci numbers. Then we have*

$$GGf(t) = \frac{2i + 3ij + t(ij - i)}{1 - t - t^2}.$$

Proof. We get

$$GGf(t) = \sum_{n=0}^{\infty} BGF_n t^n = BGF_0 + BGF_1 t^1 + \cdots + BGF_n t^n + \ldots$$

$$tGGf(t) = BGF_0 t + BGF_1 t^2 + \cdots + BGF_n t^{n+1} + \ldots$$

and

$$t^2 GGf(t) = BGF_0 t^2 + BGF_1 t^3 + \cdots + BGF_n t^{n+2} + \ldots$$

So, we obtain

$$GGf(t)\left(1 - t - t^2\right) = BGF_0 + t(BGF_1 - BGF_0).$$

From initial conditions for bicomplex Gaussian Fibonacci numbers, we have

$$GGf(t)\left(1 - t - t^2\right) = 2i + 3ij + t(ij - i),$$

which gives the result. □

Theorem 3. *Binet's formula of bicomplex Gaussian Fibonacci numbers is given by*

$$BGF_n = i\frac{\alpha^{n-1}\widehat{\alpha} - \beta^{n-1}\widehat{\beta}}{\alpha-\beta},$$

where $\widehat{\alpha} = 1+\alpha^2+j\alpha^2+j\alpha^4$, $\widehat{\beta} = 1+\beta^2+j\beta^2+j\beta^4$.

Proof. Using equalities (1) and (3), we get

$$\begin{aligned} & BGF_n = GF_n + iGF_{n+1} + jGF_{n+2} + ijGF_{n+3} \\ = \ & \left(\frac{\alpha^n-\beta^n}{\alpha-\beta} + i\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right) + i\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} + i\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) \\ & +j\left(\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta} + i\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right) \\ & +ij\left(\frac{\alpha^{n+3}-\beta^{n+3}}{\alpha-\beta} + i\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\right) \\ = \ & \frac{i\alpha^{n-1}(1+\alpha^2+j\alpha^2+j\alpha^4)}{\alpha-\beta} - \frac{i\beta^{n-1}(1+\beta^2+j\beta^2+j\beta^4)}{\alpha-\beta}. \end{aligned}$$

Here, taking $1+\alpha^2+j\alpha^2+j\alpha^4 = \widehat{\alpha}$, $1+\beta^2+j\beta^2+j\beta^4 = \widehat{\beta}$, we obtain the desired result. □

Theorem 4. *Let* m *and* n *be two positive integers. The Honsberger identity for the bicomplex Gaussian Fibonacci numbers is given by*

$$\begin{aligned} & BGF_m \times BGF_n + BGF_{m+1} \times BGF_{n+1} \\ & \quad = (1+2i)(2BGF_{m+n} - F_{m+n} - F_{m+n+2} - F_{m+n+4} + F_{m+n+6} \\ & \qquad - 2(iF_{m+n+5} + jF_{m+n+4} - ijF_{m+n+3})). \end{aligned}$$

Proof. Using equality (3), we obtain

$$\begin{aligned} & BGF_m \times BGF_n + BGF_{m+1} \times BGF_{n+1} \\ & = GF_mGF_n + GF_{m+1}GF_{n+1} - (GF_{m+1}GF_{n+1} + GF_{m+2}GF_{n+2}) \\ & \quad - (GF_{m+2}GF_{n+2} + GF_{m+3}GF_{n+3}) + GF_{m+3}GF_{n+3} + GF_{m+4}GF_{n+4} \\ & \quad + i\left(GF_mGF_{n+1} + GF_{m+1}GF_{n+2} + GF_{m+1}GF_n + GF_{m+2}GF_{n+1}\right) \\ & \quad - i\left((GF_{m+2}GF_{n+3} + GF_{m+3}GF_{n+4}) - (GF_{m+3}GF_{n+2} + GF_{m+4}GF_{n+3})\right) \\ & \quad + j\left(GF_mGF_{n+2} + GF_{m+1}GF_{n+3} + GF_{m+2}GF_n + GF_{m+3}GF_{n+1}\right) \\ & \quad - j\left((GF_{m+1}GF_{n+3} + GF_{m+2}GF_{n+4}) - (GF_{m+3}GF_{n+1} + GF_{m+4}GF_{n+2})\right) \\ & \quad + ij\left(GF_mGF_{n+3} + GF_{m+1}GF_{n+4} + GF_{m+1}GF_{n+2} + GF_{m+2}GF_{n+3}\right) \\ & \quad + ij\left((GF_{m+2}GF_{n+1} + GF_{m+3}GF_{n+2}) + (GF_{m+3}GF_n + GF_{m+4}GF_{n+1})\right). \end{aligned}$$

Using identity (d) in Section 1, we get

$$BGF_m \times BGF_n + BGF_{m+1} \times BGF_{n+1}$$

$$
\begin{aligned}
&= (1+2i)\left(F_{m+n} - F_{m+n+2} - F_{m+n+4} + F_{m+n+6}\right) \\
&\quad +2\,(1+2i)\,i\left(F_{m+n+1} - F_{m+n+5}\right) \\
&\quad +2\,(1+2i)\,j\left(F_{m+n+2} - F_{m+n+4}\right) + 4ij(1+2i)(F_{m+n+3}) \\
&= (1+2i)\left(F_{m+n} - F_{m+n+2} - F_{m+n+4} + F_{m+n+6}\right. \\
&\quad +iF_{m+n+1} + iF_{m+n+1} \\
&\quad -2iF_{m+n+5} + jF_{m+n+2} + jF_{m+n+2} - 2jF_{m+n+4} + ijF_{m+n+3} \\
&\quad \left. +ijF_{m+n+3} + 2ijF_{m+n+3}\right) \\
&= (1+2i)(2BGF_{m+n} - F_{m+n} - F_{m+n+2} - F_{m+n+4} + F_{m+n+6} \\
&\quad -2(iF_{m+n+5} + jF_{m+n+4} - ijF_{m+n+3}).
\end{aligned}
$$

□

Theorem 5. *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{p=0}^{n} BGF_p = BGF_{n+2} - 4ij - i,$
(b) $\sum_{p=0}^{n} BGF_{2p+1} = BGF_{2n+2} - 2i - 3ij,$
(c) $\sum_{p=0}^{n} BGF_{2p} = BGF_{2n+1} + i - ij.$

Proof. Using equality (3), we get

$$
\begin{aligned}
\sum_{p=0}^{n} GF_p + i\sum_{p=0}^{n} GF_{p+1} &+ j\sum_{p=0}^{n} GF_{p+2} + ij\sum_{p=0}^{n} GF_{p+3} \\
&= (GF_0 + \cdots + GF_n) \\
&+ i(GF_1 + \cdots + GF_{n+1}) \\
&+ j(GF_2 + \cdots + GF_{n+2}) \\
&+ ij(GF_3 + \cdots + GF_{n+3}).
\end{aligned}
$$

Using identity (f) in Section 1 and equality (3), we obtain

$$
\begin{aligned}
\sum_{p=0}^{n} BGF_p &= GF_{n+2} + iGF_{n+3} + jGF_{n+4} + ijGF_{n+5} - 1 \\
&\quad -i - (iGF_0 + jGF_1 + ijGF_2) - j \\
&\quad -jGF_0 - ij - ijGF_0 - ijGF_1 \\
&= BGF_{n+2} - 1 - i - \left(i^2 + j + ij\,(1+i)\right) \\
&\quad -j - ij - ij - i^2 j - ij \\
&= BGF_{n+2} - 4ij - i.
\end{aligned}
$$

This proves (a). Proofs of (b) and (c) are done similarly using identities (g) and (h) in Section 1.

□

3. Bicomplex Gaussian Lucas numbers

The n-th bicomplex Gaussian Lucas number is defined for $n \geq 0$ by

$$BGL_n = GF_n + iGL_{n+1} + jGL_{n+2} + ijGL_{n+3}, \tag{5}$$

where GL_n is the n-th Gaussian Lucas number and i and j are imaginary units. Here we can get the following relation:

$$BGL_n = BGL_{n-1} + BGL_{n-2}, \tag{6}$$

where $BGL_0 = 5ij$ and $BGL_1 = 5i + 10ij$.

The first three theorems below can be proved similarly to bicomplex Gaussian Fibonacci numbers.

Theorem 6. *Let $GGl\,(t)$ be a generating function for the bicomplex Gaussian Lucas numbers. Then we have*

$$GGl\,(t) = \frac{5ij + t(10ij + 5i)}{1 - t - t^2}.$$

Theorem 7. *Binet's formula of the bicomplex Gaussian Lucas number is given by*

$$BGL_n = i(\alpha^{n-1}\widehat{\alpha} + \beta^{n-1}\widehat{\beta})$$

where $\widehat{\alpha} = 1 + \alpha^2 + j\alpha^2 + j\alpha^4$, $\widehat{\beta} = 1 + \beta^2 + j\beta^2 + j\beta^4$.

Theorem 8. *For $n \geq 0$, we have the following formula:*

$$\sum_{p=0}^{n} BGL_p = BGF_{n+2} - 5i - 9ij.$$

The following theorems show the relationships between the bicomplex Gaussian Fibonacci, the bicomplex Gaussian Lucas, Gaussian Fibonacci, Gaussian Lucas and Fibonacci numbers.

Theorem 9. *For $n \geq 0$, we have the following formula:*

$$\begin{aligned} BGF_n \times BGL_n &= (1+2i)\left[F_{2n-1} - F_{2n+1} - F_{2n+3} + F_{2n+5}\right] \\ &\quad + i\left(GF_n GL_{n+1} + GF_{n+1}GL_n - GF_{n+2}GL_{n+3} - GF_{n+3}GL_{n+2}\right) \\ &\quad + j\left(GF_n GL_{n+2} - GF_{n+1}GL_{n+3} + GF_{n+2}GL_n - GF_{n+3}GL_{n+1}\right) \\ &\quad + ij\left(GF_n GL_{n+3} + GF_{n+1}GL_{n+2} + GF_{n+2}GL_{n+1} + GF_{n+3}GL_n\right). \end{aligned}$$

Proof. We have

$$\begin{aligned} &BGF_n \times BGL_n \\ &\quad = (GF_n + iGF_{n+1} + jGF_{n+2} + ijGF_{n+3}) \\ &\qquad \times (GL_n + iGL_{n+1} + jGL_{n+2} + ijGL_{n+3}) \\ &\quad = GF_n GL_n - GF_{n+1}GL_{n+1} - GF_{n+2}GL_{n+2} + GF_{n+3}GL_{n+3} \\ &\qquad + i\left(GF_n GL_{n+1} + GF_{n+1}GL_n - GF_{n+2}GL_{n+3} - GF_{n+3}GL_{n+2}\right) \end{aligned}$$

$$+j\left(GF_nGL_{n+2}-GF_{n+1}GL_{n+3}+GF_{n+2}GL_n-GF_{n+3}GL_{n+1}\right)$$
$$+ij\left(GF_nGL_{n+3}+GF_{n+1}GL_{n+2}+GF_{n+2}GL_{n+1}+GF_{n+3}GL_n\right).$$

Using identity (c) in Section 1, we get the result. □

Theorem 10. *It holds, $BGF_{n+1}+BGF_{n-1}=BGL_n$, $n\geq 1$.*

Proof. Using identity (a) in Section 1 and equality (3), we obtain

$$\begin{aligned}BGF_{n+1} &+ BGF_{n-1}\\ &= GF_{n+1}+iGF_{n+2}+jGF_{n+3}+ijGF_{n+4}+GF_{n-1}\\ &\quad +iGF_n+jGF_{n+1}+ijGF_{n+2}\\ &= GF_{n+1}+GF_{n-1}+i\left(GF_{n+2}+GF_n\right)+j\left(GF_{n+3}+GF_{n+1}\right)\\ &\quad +ij(GF_{n+4}+GF_{n+2})\\ &= GL_n+iGL_n+jGL_{n+2}+ijGL_{n+3}=BGL_n.\end{aligned}$$

□

Theorem 11. *For $n\geq 0$, we have*

$$\begin{aligned}BGL_n^2-5BGF_n^2=2(&iGL_nGL_{n+1}-5iGF_nGF_{n+1}+jGL_nGL_{n+2}\\ &-5jGF_nGF_{n+2}+ijGL_nGL_{n+3}-5ijGF_nGF_{n+3}\\ &+ijGL_{n+1}GL_{n+2}-5ijGF_{n+1}GF_{n+2}\\ &-jGL_{n+1}GL_{n+3}+5jGF_{n+1}GF_{n+3}-jGL_{n+2}GL_{n+3}\\ &+5jGF_{n+2}GF_{n+3}).\end{aligned}$$

Proof. We have

$$\begin{aligned}BGL_n^2-5BGF_n^2 &= (GL_n+iGL_{n+1}+jGL_{n+2}+ijGL_{n+3})^2\\ &\quad -5(GF_n+iGF_{n+1}+jGF_{n+2}+ijGF_{n+3})^2\\ &= GL_n^2-5GF_n^2-\left(GL_{n+1}^2-5GF_{n+1}^2\right)\\ &\quad -\left(GL_{n+2}^2-5GF_{n+2}^2\right)+GL_{n+3}^2-5GF_{n+3}^2\\ &\quad +2(iGL_nGL_{n+1}-5iGF_nGF_{n+1}+jGL_nGL_{n+2}\\ &\quad -5jGF_nGF_{n+2}+ijGL_nGL_{n+3}-5ijGF_nGF_{n+3}\\ &\quad +ijGL_{n+1}GL_{n+2}-5ijGF_{n+1}GF_{n+2}-jGL_{n+1}GL_{n+3}\\ &\quad +5jGF_{n+1}GF_{n+3}-jGL_{n+2}GL_{n+3}+5jGF_{n+2}GF_{n+3}).\end{aligned}$$

Using (e) in Section 1, we get

$$\begin{aligned}BGL_n^2-5BGF_n^2 &= (-1)^n4\left(2-i\right)-(-1)^{n+1}4\left(2-i\right)\\ &\quad -(-1)^{n+2}4\left(2-i\right)+(-1)^{n+3}4\left(2-i\right)\\ &\quad +2(iGL_nGL_{n+1}-5iGF_nGF_{n+1}+jGL_nGL_{n+2}\\ &\quad -5jGF_nGF_{n+2}+ijGL_nGL_{n+3}\end{aligned}$$

$$
\begin{aligned}
&- 5ijGF_nGF_{n+3} + ijGL_{n+1}GL_{n+2} - 5ijGF_{n+1}GF_{n+2} \\
&- jGL_{n+1}GL_{n+3} + 5jGF_{n+1}GF_{n+3} - jGL_{n+2}GL_{n+3} \\
&+ 5jGF_{n+2}GF_{n+3}),
\end{aligned}
$$

which reduces to the result of theorem. □

4. Matrices related to bicomplex Gaussian Fibonacci and Lucas numbers

We define the following square matrices:

$$A_{GF} = \begin{pmatrix} BGF_3 & BGF_2 \\ BGF_2 & BGF_1 \end{pmatrix} \text{ and } A_{GL} = \begin{pmatrix} BGL_3 & BGL_2 \\ BGL_2 & BGL_1 \end{pmatrix}.$$

The matrices A_{GF} and A_{GL} can be called bicomplex Gaussian Fibonacci numbers matrix and bicomplex Gaussian Lucas numbers matrix, respectively.

Theorem 12. *For $n \geq 0$, we have the following matrix identity:*

$$A_{GF}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} BGF_{n+3} & BGF_{n+2} \\ BGF_{n+2} & BGF_{n+1} \end{pmatrix},$$

where $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is the Fibonacci Q-matrix.

Proof. Let us prove the theorem by induction on n. If $n = 0$, then the result is clear. Now, we assume it is true for $n = k$, that is

$$A_{GF}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} BGF_{k+3} & BGF_{k+2} \\ BGF_{k+2} & BGF_{k+1} \end{pmatrix}.$$

If we use equality (5), then by induction hypothesis we get

$$A_{GF}A^{k+1} = (A_{GF}A^k) = \begin{pmatrix} BGF_{k+3} & BGF_{k+2} \\ BGF_{k+2} & BGF_{k+1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} BGF_{k+3} + BGF_{k+2} & BGF_{k+3} \\ BGF_{k+2} + BGF_{k+1} & BGF_{k+2} \end{pmatrix} = \begin{pmatrix} BGF_{k+4} & BGF_{k+3} \\ BGF_{k+3} & BGF_{k+2} \end{pmatrix}.$$

□

Theorem 13. *For $n \geq 0$ we have the following matrix identity:*

$$A_{GL}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} BGL_{n+3} & BGL_{n+2} \\ BGL_{n+2} & BGL_{n+1} \end{pmatrix}.$$

Proof. Let us prove the theorem by induction on n. If $n = 0$ then the result is clear. Now, we assume it is true for $n = k$, that is

$$A_{GL}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} BGL_{k+3} & BGL_{k+2} \\ BGL_{k+2} & BGL_{k+1} \end{pmatrix}.$$

If we use equality (6) then by induction hypothesis, we get

$$A_{GL}A^{k+1} = (A_{GL}A^k) = \begin{pmatrix} BGL_{k+3} & BGL_{k+2} \\ BGL_{k+2} & BGL_{k+1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} BGL_{k+3} + BGL_{k+2} & BGL_{k+3} \\ BGL_{k+2} + BGL_{k+1} & BGL_{k+2} \end{pmatrix} = \begin{pmatrix} BGL_{k+4} & BGL_{k+3} \\ BGL_{k+3} & BGL_{k+2} \end{pmatrix}.$$

□

5. Conclusion

In this paper, we defined the bicomplex Gaussian Fibonacci and the bicomplex Gaussian Lucas numbers. Some properties, including the Binet formula, generating function, summation formula, relationships between these numbers and Honsberger identities, matrix representation, were given.

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Graduate School of Natural and Applied Sciences, Erzincan Binali Yildirim University, Erzincan, Turkey
E-mail address: bahar_kuloglu@hotmail.com

Department of Mathematics, Erzincan Binali Yildirim University, Faculty of Arts and Sciences, Erzincan, Turkey
E-mail address: eozkan@erzincan.edu.tr or eozkanmath@gmail.com