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Fekete's lemma for componentwise subadditive functions of two or more real variables

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Abstract. We prove an analogue of Fekete's subadditivity lemma for functions of several real variables which are subadditive in each variable taken singularly. This extends both the classical case for subadditive functions of one real variable, and a similar result for functions of integer variables. While doing so, we prove that the functions with the property mentioned above are bounded in every closed and bounded subset of their domain. The arguments expand on those in Chapter 6 of E. Hille's 1948 textbook.

1. Introduction

A real-valued function f defined on a semigroup $(S, \cdot)$ is *subadditive* if

$$f(x \cdot y) \leqslant f(x) + f(y) \tag{1}$$

for every $x, y \in S$. Examples of subadditive functions include the absolute value of a complex number; the ceiling of a real number (smallest integer not smaller than it); the cardinality of a a finite subset of a given set; and the length of a word over an alphabet. Subadditive functions have applications in many fields including information theory [9], economics, and combinatorics.

A classical result in mathematical analysis, *Fekete's lemma* [3] states that, if f is a real-valued subadditive function of one positive integer or positive real variable, then $f(x)/x$ converges, for $x \to +\infty$, to its greatest lower bound. This simple fact has a huge number of applications in many fields, including symbolic dynamics (cf. [9, Chapter 4]) and the theory of neural networks (see [5]). Reusing a metaphor from [1], Fekete's lemma says that for a sequence of independent observations, the *average information per observation* converges to its greatest lower bound.

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Given the importance and ubiquity of Fekete's lemma, we wonder if similar results may hold for functions of many variables. Oddly, the mathematical literature seems to contain generalizations where, in almost all cases, the function in the limit actually depends again on a single variable, which is sometimes a real number, sometimes a finite set; many of these are closer to a corollary than to an extension. So maybe we could look for a different *type* of limit, or even for a different *flavor* of subadditivity.

With the aim of understanding which of the above could be feasible, we note that, if $S = S_1 \times S_2$ is a product semigroup, we can also consider the case of a function which is subadditive in *each variable, however given the other*. That is, instead of requiring $f(x_1 y_1, x_2 y_2) \leqslant f(x_1, x_2) + f(y_1, y_2)$ for every x_1, x_2, y_1, and y_2, we could demand that:

(1) $f(x_1 y_1, x_2) \leqslant f(x_1, x_2) + f(y_1, x_2)$ for every x_1, x_2, and y_1; and
(2) $f(x_1, x_2 y_2) \leqslant f(x_1, x_2) + f(x_1, y_2)$ for every x_1, x_2, and y_2.

The two requirements above, even together, do not imply subadditivity as a function defined on the product semigroup, nor does the latter imply the former: see Example 3.4. Oddly again, this multivariate "componentwise subadditivity" seems not to have been addressed very often in the literature.

In this paper, we state and prove an extension of Fekete's lemma to componentwise subadditive functions of $d \geqslant 2$ real variables. We state a special case as an example, leaving the full statement to Section 5.

Proposition 1.1. *Let f be a function of two positive real variables which is subadditive in each of them, however given the other. For every $\delta > 0$ there exists $R > 0$ such that, if both $x_1 > R$ and $x_2 > R$, then*

$$\frac{f(x_1, x_2)}{x_1 \cdot x_2} < \inf_{x_1, x_2 > 0} \frac{f(x_1, x_2)}{x_1 \cdot x_2} + \delta \,.$$

In addition,

$$\lim_{x_1 \to +\infty} \lim_{x_2 \to +\infty} \frac{f(x_1, x_2)}{x_1 \cdot x_2} = \lim_{x_2 \to +\infty} \lim_{x_1 \to +\infty} \frac{f(x_1, x_2)}{x_1 \cdot x_2} = \inf_{x_1, x_2 > 0} \frac{f(x_1, x_2)}{x_1 \cdot x_2} \,.$$

That is: if the componentwise subadditive function $f(x, y)$ is considered as a *net* on the *directed set* of pairs of positive reals with the *product ordering* where $(x_1, x_2) \leqslant (y_1, y_2)$ if and only if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$, then the *simultaneous limit* on this directed set of the net $\dfrac{f(x_1, x_2)}{x_1 \cdot x_2}$ is its greatest lower bound. This is a generalization of the original statement where: the functions depend on multiple *independent* real variables; both notions of subadditivity and limit are extended; and the original lemma is a special case for $d = 1$. The double limit is also remarkable, because multiple limits need not commute, let alone coincide with a simultaneous limit.

A similar statement for functions defined on d-tuples of positive integers (instead of reals) was proved in [1]; see also [10] for an application. The argument presented there, however, relies on a hidden hypothesis of boundedness on compact subsets, which comes for free in the integer setting (where compact subsets are precisely the finite subsets) but must be proved in the new one, and *cannot* be inferred from boundedness in each variable however given the others (see Example 4.3). By adapting the proof of [6, Theorem 6.4.1] we obtain the following result: componentwise subadditive functions defined on suitable regions of $\mathbb{R}^d$ are indeed bounded on compact subsets. For $d = 2$ and positive variables the statement goes as follows.

Proposition 1.2. *Under the hypotheses of Proposition 1.1, the function f is bounded on $[a, b] \times [c, d]$ for every $0 < a < b$ and $0 < c < d$.*

The paper is organized as follows. Section 2 provides the theoretical background. In Section 3 we introduce componentwise subadditivity and explain how it is different from subadditivity in the product semigroup. In Section 4 we adapt the argument from [6, Theorem 6.4.1] to prove that componentwise subadditive functions of d real variables are bounded on compact subsets of $\mathbb{R}^d$. In Section 5 we state, prove, and discuss the main theorem; boundedness will have a crucial role in the proof. Section 6 is a discussion on how the beautiful *Ornstein–Weiss lemma* [11], an important result on subadditive functions defined on finite subsets of groups of a certain class which includes $\mathbb{Z}$ and $\mathbb{R}$, *is not* an extension of Fekete's lemma.

2. Background

Throughout the paper, the subsets of $\mathbb{R}^d$ and the real-valued functions of real variables are presumed to be Lebesgue measurable. We also let real-valued functions take value either $+\infty$ or $-\infty$, but not both. We assume that the reader is familiar with the notions of directed set, net, subnet, and upper limit, lower limit, and limit of a net.

We denote by $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_-$ the sets of real numbers, positive real numbers, and negative real numbers, respectively. Similarly, we denote by $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{Z}_-$ the sets of integers, positive integers, and negative integers, respectively. All these sets are considered as additive semigroups (groups in the case of $\mathbb{R}$ and $\mathbb{Z}$). If m and n are integers and $m \leqslant n$, we denote the *slice* $\{m, m+1, \ldots, n-1, n\} = [m, n] \cap \mathbb{Z}$ as $[m:n]$. If X is a set, we denote by $\mathcal{PF}(X)$ the set of its finite subsets. For an integer $d \geqslant 0$ we denote by $\mathsf{2}^d$ the set of binary words of length d.

The *ordered product* of a family $\{(X_i, \leqslant_i)\}_{i \in I}$ of ordered sets is the ordered set $(X, \leqslant_\Pi)$ where $X = \prod_{i \in I} X_i$ and the *product ordering* $\leqslant_\Pi$ is defined as

$$x \leqslant_\Pi y \iff x_i \leqslant y_i \text{ for every } i \in I\,. \tag{2}$$

If each $(X_i, \leqslant_i)$ is a directed set, then so is $(X, \leqslant_\Pi)$. If $d \geqslant 1$ and $w \in 2^d$, the *orthant* denoted by w is the directed set

$$\mathcal{R}_w = (\mathbb{R}_w, \leqslant_\Pi) = \prod_{i=1}^{d} (X_i, \leqslant_i) \tag{3}$$

where $(X_i, \leqslant_i) = (\mathbb{R}_+, \leqslant)$ if $w_i = 0$ and $(X_i, \leqslant_i) = (\mathbb{R}_-, \geqslant)$ if $w_i = 1$. For example, $\mathcal{R}_{10}$ is the open second quadrant of the Cartesian plane, with $(x_1, x_2) \leqslant (y_1, y_2)$ if and only if $x_1 \geqslant y_1$ and $x_2 \leqslant y_2$. In particular, the *main orthant* of $\mathbb{R}^d$, corresponding to $w = 0^d$, is $\mathcal{R}_+^d = \left(\mathbb{R}_+^d, \leqslant_\Pi\right)$. Note that, if $f : \mathbb{R}_+^d \to \mathbb{R}$ is a net on $\mathcal{R}_+^d$ and $\{x_{i,n}\}_{n \geqslant 1}$, $i \in [1:d]$, are sequences of positive reals such that $\lim_{n\to\infty} x_{i,n} = +\infty$ for every $i \in [1:d]$, then $g(n) = f(x_{1,n}, \ldots, x_{d,n})$ is a subnet of f. Consequently, if f converges to $L \in \mathbb{R}$ in $\mathcal{R}_+^d$, then $g(n)$ converges to L for $n \to \infty$.

3. Componentwise subadditivity

In the literature, subadditivity is most often studied in functions of a single variable, which sometimes may be vector rather than scalar. But in some cases, it is of interest to consider functions of d independent variables, which are subadditive when considered as functions of only one of those, but however given the remaining ones.

Definition 3.1. Let $S_1, \ldots, S_d$ be semigroups, let $S = \prod_{i=1}^d S_i$, and let $f : S \to \mathbb{R}$. Given $i \in [1:d]$, we say that f is *subadditive in x_i independently of the other variables* if, however given $x_j \in S_j$ for every $j \in [1:d] \setminus \{i\}$, the function $x_i \mapsto f(x_1, \ldots, x_i, \ldots, x_d)$ is subadditive on S_i. We say that f is *componentwise subadditive* if it is subadditive in each variable independently of the others.

Example 3.2. If $f_1 : S_1 \to \mathbb{R}$ and $f_2 : S_2 \to \mathbb{R}$ are both subadditive and nonnegative, then $f : S_1 \times S_2 \to \mathbb{R}$ defined by $f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$ is componentwise subadditive.

If one between f_1 and f_2 takes negative values, then f might not be componentwise subadditive. For example, $f(x_1) = -x_1$ is subadditive on $\mathbb{R}_+$, because it is linear, and $f_2(x_2) = \sqrt{x_2}$ is also subadditive on $\mathbb{R}_+$, because it is nondecreasing and $x_2 + y_2 < \left(\sqrt{x_2} + \sqrt{y_2}\right)^2$ for every $x_2, y_2 > 0$; but for any fixed $x_1 > 0$, the function $x_2 \mapsto -x_1\sqrt{x_2}$ is not subadditive on $\mathbb{R}_+$.

Example 3.3. (cf. [1, Section 3]) Let d be a positive integer and let A be a finite set with $a \geqslant 2$ elements, considered as a discrete space. The *translation* by $v \in \mathbb{Z}^d$ is the function $\sigma_v : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ defined by $\sigma_v(c)(x) = c(x + v)$ for every $x \in \mathbb{Z}^d$. A d-dimensional *subshift* on A is a subset X of $A^{\mathbb{Z}^d}$ which is closed in the product topology and invariant by translation, that is, if $c \in X$, then $\sigma_v(c) \in X$ for every $v \in \mathbb{Z}^d$. Given d positive integers $n_1, \ldots, n_d$, an *allowed pattern* of sides $n_1, \dots, n_d$ for X is a function $p : \prod_{i=1}^d [1:n_i] \to A$ such that there exists $c \in X$ for which the restriction of c to $\prod_{i=1}^d [1:n_i]$ coincides with p. Let $\mathcal{A}_X(n_1, \dots, n_d)$ be the number of allowed patterns for X of sides $n_1, \dots, n_d$. Then

$$f(n_1, \dots, n_d) = \log_a \mathcal{A}_X(n_1, \dots, n_d) \text{ for every } n_1, \dots, n_d \in \mathbb{Z}_+ \quad (4)$$

is componentwise subadditive, because every allowed pattern of sides $n_1 + m_1, n_2, \dots, n_d$ can be obtained by joining an allowed pattern of sides $n_1, n_2, \dots, n_d$ with an allowed pattern of sides $m_1, n_2, \dots, n_d$, but joining two such allowed patterns does not necessarily produce an allowed pattern; similarly for the other $d-1$ coordinates. This works because X is invariant by translations.

Componentwise subadditivity is very different from subadditivity with respect to the operation of the product semigroup. Already with $d = 2$, if $f : S_1 \times S_2 \to \mathbb{R}$ is subadditive, then for every $x_1, y_1 \in S_1$ and $x_2, y_2 \in S_2$ we have

$$f(x_1 y_1, x_2 y_2) \leqslant f(x_1, x_2) + f(y_1, y_2)\,, \quad (5)$$

while if f is componentwise subadditive, then for every $x_1, y_1 \in S_1$ and $x_2, y_2 \in S_2$ we have the more complex upper bound:

$$f(x_1 y_1, x_2 y_2) \leqslant f(x_1, x_2) + f(x_1, y_2) + f(y_1, x_2) + f(y_1, y_2)\,. \quad (6)$$

If f is nonnegative, then (5) implies (6), which however is weaker than the conditions of Definition 3.1; if f is nonpositive, then (6) implies (5). In general, however, neither implies the other.

Example 3.4. By our discussion in Example 3.2, the function $f(x_1, x_2) = \sqrt{x_1 x_2}$ is componentwise subadditive on $\mathbb{R}^2_+$. However, f is not subadditive, because $f(3,3) = 3 > 2\sqrt{2} = f(1,2) + f(2,1)$.

Example 3.5. The function (4) of Example 3.3 is not, in general, subadditive. For example, for $d = 2$ and $X = A^{\mathbb{Z}^2}$, every pattern is allowed, so $f(n_1, n_2) = n_1 n_2$; but if n_1, n_2, m_1, and m_2 are all positive, then $(n_1 + m_1)(n_2 + m_2) > n_1 n_2 + m_1 m_2$.

Although componentwise subadditivity is very different from subadditivity in the product semigroup, Fekete's lemma can tell us something important for the case of positive integer or real variables.

Lemma 3.6. *Let $S = \prod_{i=1}^d S_i$ with each S_i being either $\mathbb{R}_+$ or $\mathbb{Z}_+$, and let $f : S^d \to \mathbb{R} \cup \{-\infty\}$ be componentwise subadditive. Having fixed $k \in [1:d-1]$, let $i, j_1, \dots, j_k \in [1:d]$ be pairwise different. However fixing the values of the remaining variables, the function $h : S_i \to \mathbb{R} \cup \{-\infty\}$ defined* *by the multiple limit*

$$h(x_i) = \lim_{x_{j_1}\to+\infty} \cdots \lim_{x_{j_k}\to+\infty} \frac{f(x_1,\ldots,x_d)}{\prod_{i=1}^k x_{j_i}} \tag{7}$$

is subadditive.

Proof. It is sufficient to prove the claim for $k = 1$; the general case follows by repeated application. To simplify notation, let $j = j_1$. Fix the values of x_s for $s \in [1\!:\!d] \setminus \{i, j\}$. By hypothesis, for every $x_i \in S_i$ the function $x_j \mapsto f(x_1,\ldots,x_d)$ is subadditive, so by Fekete's lemma $h(x_i) = \lim_{x_j\to\infty} \frac{f(x_1,\ldots,x_d)}{x_j}$ exists. But for every $x_i, x_i', x_j > 0$ it is:

$$\frac{f(\ldots,x_i+x_i',\ldots)}{x_j} \leqslant \frac{f(\ldots,x_i,\ldots)}{x_j} + \frac{f(\ldots,x_1',\ldots)}{x_j},$$

so it must be $h(x_1 + x_1') \leqslant h(x_1) + h(x_1')$, too. Note that the proof relies on x_j being positive. □

The following observation is crucial for the next sections; we leave the proof to the reader.

Proposition 3.7. *Let $w = w_1 \ldots w_d$ be a binary word of length d and let $f : \mathbb{R}_w \to \mathbb{R}$. For every $i \in [1\!:\!d]$, let $x_{w,i} = (-1)^{w_i} x_i \in \mathbb{R}_+$, and let $f_w : \mathbb{R}_+^d \to \mathbb{R}$ be defined by $f_w(x_{w,1},\ldots,x_{w,d}) = f(x_1,\ldots,x_d)$. The following are equivalent:*

(1) *$f(x_1,\ldots,x_d)$ is componentwise subadditive in $\mathbb{R}_w$;*
(2) *$f_w(x_{w,1},\ldots,x_{w,d})$ is componentwise subadditive in $\mathbb{R}_+^d$.*

The same holds if $\mathbb{R}_w$ and $\mathbb{R}_+^d$ are replaced with $\mathbb{Z}_w = \mathbb{R}_w \cap \mathbb{Z}^d$ and $\mathbb{Z}_+^d$, respectively.

4. Componentwise subadditive functions of d real variables are bounded on compacts

In [1] the following is proved:

Proposition 4.1 (Fekete's lemma in $\mathbb{Z}_+^d$; [1, Theorem 1])**.** *Let $\mathcal{U} = \left(\mathbb{Z}_+^d, \leqslant_\Pi\right)$ and let $f : \mathbb{Z}_+^d \to \mathbb{R}$ be componentwise subadditive. Then*

$$\lim_{(x_1,\ldots,x_d)\in\mathcal{U}} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d} = \inf_{x_1,\ldots,x_d\in\mathbb{Z}_+} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d}. \tag{8}$$

Example 4.2. With the notation of Example 3.3 and $\mathcal{U}$ as in Proposition 4.1, the value:

$$h(X) = \lim_{(x_1,\ldots,x_d)\in\mathcal{U}} \frac{\log_a \mathcal{A}_X(x_1,\ldots,x_d)}{x_1\cdots x_d} \tag{9}$$

is well defined, and is called the *entropy* of the subshift X. For $d = 1$ this coincides with [9, Definition 4.1.1].

We try to reuse the argument from [1] to prove Proposition 1.1. Fix $s, t > 0$. Every $x > 0$ large enough has a unique writing $x = qs + r$ with q a positive integer and $r \in [s, 2s)$, and every $y > 0$ large enough has a unique writing $y = mt + p$ with m a positive integer and $p \in [t, 2t)$. By componentwise subadditivity,

$$\frac{f(x,y)}{xy} \leqslant \frac{qm}{xy} f(s,t) + \frac{q}{xy} f(s,p) + \frac{m}{xy} f(r,t) + \frac{1}{xy} f(r,p) \,.$$

Consider the four summands on the right-hand side. Clearly, $\lim_{x \to +\infty} q/x = 1/s$ and $\lim_{y \to +\infty} m/y = 1/t$. Therefore, the first summand converges to $f(s,t)/st$ for $(x,y) \in \mathcal{R}^2_+$.

Now, by [6, Theorem 6.4.1], a subadditive function of one positive real variable is bounded in every compact subset of $\mathbb{R}_+$. Then $p \mapsto f(s,p)$ is bounded on $[t, 2t]$ and $r \mapsto f(r,t)$ is bounded on $[s, 2s]$. Consequently, the second and third summand vanish for $(x,y) \in \mathcal{R}^2_+$.

But the fourth summand presents a problem. What we know, is that $x \mapsto f(x,y)$ is bounded in $[s, 2s]$ for every $y \in [t, 2t]$, and $y \mapsto f(s,y)$ is bounded in $[t, 2t]$ for every $x \in [s, 2s]$. This is, in general, *strictly less* than f being bounded in $[s, 2s] \times [t, 2t]$, which is what we actually need to show that the fourth summand vanishes when x and y both grow arbitrarily large!

Example 4.3 (suggested by Arthur Rubin)**.** Let $h : \mathbb{R}_+ \to \mathbb{R}$ be defined as $h(t) = n$ if $t = m/n$ with $m, n \in \mathbb{Z}_+$ and $\gcd(m,n) = 1$, and $h(t) = 0$ if t is irrational. Then $f : \mathbb{R}^2_+ \to \mathbb{R}$ defined by $f(x,y) = \min(h(x), h(y))$ satisfies the following conditions:

(1) for every $x \in [1,2]$, the function $y \mapsto f(x,y)$ is bounded in $[1,2]$;
(2) for every $y \in [1,2]$, the function $x \mapsto f(x,y)$ is bounded in $[1,2]$.

However, f is not bounded in $[1,2] \times [1,2]$, because $f(1 + 1/n, 1 + 1/n) = n$ for every $n \in \mathbb{Z}_+$. On the other hand, $h(4) = 1$ and $h(\pi) = h(4 - \pi) = 0$, so f is neither subadditive nor componentwise subadditive in $\mathbb{R}^2_+$.

We could overcome this issue if a result of boundedness such as the one in [6, Theorem 6.4.1] held for componentwise subadditive functions. Luckily, it is so, and we can follow the same idea of Hille's proof. Given $f : \mathbb{R}^d_+ \to \mathbb{R}$ and $t_1, \ldots, t_d \in \mathbb{R}_+$, let

$$V_{t_1,\ldots,t_d,k} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d_+ \mid 0 < x_i < t_i \forall i \in [1\,{:}\,d] \,, f(x_1, \ldots, x_d) \geqslant k\} \,. \tag{10}$$

Under our hypothesis that f is measurable, so is (10).

The next statement is the cornerstone of our argument. For Lemma 4.4 and Theorem 4.5, the symbol μ and the word "measure" denote the d-dimensional Lebesgue measure.

Lemma 4.4. *Let $f : \mathbb{R}_+^d \to \mathbb{R}$ be componentwise subadditive. Then, for every $t_1, \ldots, t_d \in \mathbb{R}_+$,*

$$\mu\left(V_{t_1,\ldots,t_d,\frac{f(t_1,\ldots,t_d)}{2^d}}\right) \geqslant \frac{t_1 \cdots t_d}{2^d}\,. \tag{11}$$

Proof. Call V the set on the left-hand side of (11). For every $i \in [1\!:\!d]$, given $x_i \in (0, t_i)$, let $y_i^{(0)} = x_i$ and $y_i^{(1)} = t_i - x_i$. Then, for every $w \in \mathbf{2}^d$, the transformation

$$(x_1, \ldots, x_d) \mapsto \left(y_1^{(w_1)}, \ldots, y_d^{(w_d)}\right)$$

is a measure-preserving continuous involution, hence the set

$$V_w = \left\{\left(y_1^{(w_1)}, \ldots, y_d^{(w_d)}\right) \mid (x_1, \ldots, x_d) \in V\right\}$$

is measurable and satisfies $\mu(V_w) = \mu(V)$. Note that $V = V_{0^d}$.

By repeatedly applying subadditivity, once in each variable, we arrive at

$$f(t_1, \ldots, t_d) \leqslant \sum_{w \in \mathbf{2}^d} f\left(y_1^{(w_1)}, \ldots, y_d^{(w_d)}\right)\,. \tag{12}$$

For example, for $d = 2$ we have

$$\begin{aligned} f(t_1, t_2) &\leqslant f(x_1, t_2) + f(t_1 - x_1, t_2) \\ &\leqslant f(x_1, x_2) + f(x_1, t_2 - x_2) \\ &\quad + f(t_1 - x_1, x_2) + f(t_1 - x_1, t_2 - x_2)\,. \end{aligned}$$

For (12) to hold, at least one of the 2^d summands on the right-hand side must be no smaller than $\dfrac{f(t_1, \ldots, t_d)}{2^d}$. Then $\bigcup_{w \in \mathbf{2}^d} V_w = \prod_{i=1}^d (0, t_i)$, so

$$t_1 \cdots t_d \leqslant \sum_{w \in \mathbf{2}^d} \mu(V_w) = 2^d\,\mu(V)\,.$$

□

From Lemma 4.4 the next theorem follows.

Theorem 4.5. *Let $w \in \mathbf{2}^d$ and let $f : \mathbb{R}_w \to \mathbb{R}$ be componentwise subadditive. Then f is bounded in every compact subset of $\mathbb{R}_w$.*

Proof. Thanks to Proposition 3.7, it is sufficient to prove the claim for $w = 0^d$ (thus, $\mathbb{R}_w = \mathbb{R}_+^d$) and for every compact hypercube of the form $H = [a, b]^d$ with $0 < a < b$. We proceed by contradiction, following the argument from [6, Theorem 6.4.1].

First, suppose that f is unbounded from above in H. Then, for every $n \geqslant 1$ and $i \in [1\!:\!d]$, there exists $x_{i,n} \in [a, b]$ such that $f(x_{1,n}, \ldots, x_{d,n}) \geqslant 2^d n$. Let $W_{t_1,\ldots,t_d}$ be the set in (11). By construction, for every $n \geqslant 1$ we have

$$W_{x_{1,n},\ldots,x_{d,n}} \subseteq V_{b,\ldots,b,n}\,,$$

and, by Lemma 4.4,

$$\mu\left(W_{x_{1,n},\ldots,x_{d,n}}\right) \geqslant \frac{x_{1,n}\cdots x_{d,n}}{2^d} \geqslant \left(\frac{a}{2}\right)^d .$$

Now, the sets $V_{b,\ldots,b,n}$ are measurable and form a nonincreasing sequence, so $V = \bigcap_{n\geqslant 0} V_{b,\ldots,b,n}$ is measurable and $\mu(V) \geqslant (a/2)^d$; in particular, V cannot be empty. But for $(x_1,\ldots,x_d) \in V$ we must have $f(x_1,\ldots,x_d) \geqslant n$ for every $n \geqslant 1$: which is impossible.

Next, suppose that f is unbounded from below in H. Then, for every $n \geqslant 1$ and $i \in [1:d]$, there exists $x_{i,n} \in [a,b]$ such that $f(x_{1,n},\ldots,x_{d,n}) \leqslant -n$. We may assume that $\lim_{n\to\infty} x_{i,n} = x_i \in [a,b]$ exists for every $i \in [1:d]$. Let $s = \min(a,1)$, $t = b+4$, and $J = [s,t]^d$, then every point $(z_1,\ldots,z_d)$ where each z_i belongs to either $[a,b]$ or $[1,4]$ belongs to J. Let now $y_i \in [1,4]$ for every $i \in [1:d]$ and

$$M = \sup\{f(z_1,\ldots,z_d) \mid (z_1,\ldots,z_d) \in J\} ,$$

which is a real number because of the previous point. By applying subadditivity in each variable, for such $y_1,\ldots,y_d$ and n we obtain

$$f(y_1 + x_{1,n},\ldots,y_d + x_{d,n}) \leqslant (2^d-1)M - n ,$$

because $-n$ is an upper bound for $f(x_{1,n},\ldots,x_{d,n})$ and M is an upper bound for the other 2^d-1 summands. For example, for $d=2$ we have

$$\begin{aligned} f(y_1 + x_{1,n}, y_2 + x_{2,n}) &\leqslant f(y_1,y_2) + f(y_1,x_{2,n}) \\ &\quad + f(x_{1,n},y_2) + f(x_{1,n},x_{2,n}) \\ &\leqslant 3M - n . \end{aligned}$$

But for every n such that $|x_{i,n} - x_i| \leqslant 1$ we have $[x_i+2, x_i+3] \subseteq [x_{i,n}+1, x_{i,n}+4]$. Calling

$$K = \prod_{i=1}^{d} [x_i+2, x_i+3] \subseteq J ,$$

for every n large enough every element of K can be written in the form $(y_1 + x_{1,n},\ldots,y_d + x_{d,n})$ for suitable $y_1,\ldots,y_d \in [1,4]$. For every $(z_1,\ldots,z_d) \in K$ we must then have $f(z_1,\ldots,z_d) \leqslant (2^d-1)M - n$ for every n large enough, which is impossible. □

In turn, Theorem 4.5 allows us to prove the following result.

Theorem 4.6. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be componentwise subadditive. Then f is bounded in every compact subset of $\mathbb{R}^d$.*

Proof. It is sufficient to show finitely many open sets $V_1, \ldots, V_n$ such that f is bounded on the compacts of each V_i and

$$\mathbb{R}^d = \left(\bigcup_{w \in 2^d} \mathbb{R}_w \right) \cup \left(\bigcup_{i=1}^{n} V_i \right) .$$

We give the argument for $d = 3$, the ideas for arbitrary $d \geqslant 1$ are similar. Let $I = [-1/2, 1/2]$ and $U = [-3/2, -1/2] \cup [1/2, 3/2]$.

We start by proving that f is bounded in every compact subset of the open set

$$Z_{00} = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y > 0\} = \mathbb{R}_{000} \cup \mathbb{R}_{001} \cup D_{00} ,$$

where $D_{00} = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y > 0, z = 0\}$ is the first quadrant of the XY plane. To do this, we only need to show that f is bounded in every set of the form $H = [a, b] \times [a, b] \times I$. Let $V = [a, b] \times [a, b] \times U$. If $(x, y, z) \in H$, then $(x, y, z - 1)$ and $(x, y, z + 1)$ are both in V. Let T and t be an upper bound and a lower bound for f in V, respectively. Then, for every $(x, y, z) \in H$,

$$f(x, y, z) \leqslant f(x, y, z - 1) + f(x, y, 1) \leqslant 2T$$

and

$$f(x, y, z) \geqslant f(x, y, z + 1) - f(x, y, 1) \geqslant t - T .$$

By similar arguments, f is bounded in every compact subset of every subset of $\mathbb{R}^3$ which is the union of two adjacent orthants and the corresponding "quadrant". As for each open orthant there are three which border it by one "quadrant", there are $\dfrac{8 \cdot 3}{2} = 12$ such subsets.

We now show that f is bounded in every compact subset of the open "upper demispace" $Z_0 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$. To do so, it will suffice to show that f is bounded in every set of the form $L = I \times I \times [a, b]$ with $0 < a < b$. Let $W = U \times U \times [a, b]$ and let S and s be an upper bound for f in W, respectively. Then, for every $(x, y, z) \in L$,

$$\begin{aligned} f(x, y, z) &\leqslant f(x - 1, y, z) + f(1, y, z) \\ &\leqslant f(x - 1, y - 1, z) + f(x - 1, 1, z) + f(1, y - 1, z) + f(1, 1, z) \\ &\leqslant 4S \end{aligned}$$

and

$$\begin{aligned} f(x, y, z) &\geqslant f(x + 1, y, z) - f(1, y, z) \\ &\geqslant f(x + 1, y + 1, z) - f(x + 1, 1, z) - f(1, y, z) \\ &\geqslant s - 2S . \end{aligned}$$

Similarly, f is bounded in each of the other five open "demispaces".

To conclude the proof, we only need to show that f is bounded in $K = I \times I \times I$. Let $E = U \times U \times U$ and let M and m be an upper bound and a lower bound for f in E, respectively. Then, for every $(x, y, z) \in K$,

$$\begin{aligned} f(x,y,z) &\leqslant f(x-1,y,z) + f(1,y,z) \\ &\leqslant f(x-1,y-1,z) + f(x-1,1,z) + f(1,y-1,z) + f(1,1,z) \\ &\leqslant f(x-1,y-1,z-1) + f(x-1,z-1,1) \\ &\quad + f(x-1,1,z-1) + f(x-1,1,1) \\ &\quad + f(1,y-1,z-1) + f(1,y-1,1) \\ &\quad + f(1,1,z-1) + f(1,1,1) \\ &\leqslant 8M \end{aligned}$$

and

$$\begin{aligned} f(x,y,z) &\geqslant f(x+1,y,z) - f(1,y,z) \\ &\geqslant f(x+1,y+1,z) - f(x+1,1,z) - f(1,y,z) \\ &\geqslant f(x+1,y+1,z+1) - f(x+1,y+1,1) \\ &\quad - f(x+1,1,z) - f(1,y,z) \\ &\geqslant m - 3M\,. \end{aligned}$$

□

Note that the argument of Lemma 4.4 also works if f is subadditive, rather than componentwise subadditive. In this case, however, the denominator in (11) and in the proof is 2 rather than 2^d. A more complex variant of it can then be stated, where f is a function of k variables x_i, each taking values in an orthant of $\mathbb{R}^{d_i}_+$, and the denominator would then be 2^k. From this, a generalization of Theorem 4.5 to the case of componentwise functions of k variables, the ith of which takes values in $\mathbb{R}^{d_i}_+$, can be derived.

5. Fekete's lemma for componentwise subadditive functions of d real variables

We can now state and prove the main result of this paper.

Theorem 5.1 (Fekete's lemma in $\mathbb{R}^d_+$)**.** *Let $d \geqslant 1$ and let $f : \mathbb{R}^d_+ \to \mathbb{R}$ be componentwise subadditive. Then*

$$\lim_{(x_1,\ldots,x_d)\in\mathcal{R}^d_+} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d} = \inf_{x_1,\ldots,x_d\in\mathbb{R}_+} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d}\,, \tag{13}$$

which can be $-\infty$. In addition, for every permutation σ of $[1:d]$,

$$\lim_{x_{\sigma(1)}\to+\infty} \cdots \lim_{x_{\sigma(d)}\to+\infty} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d} = \lim_{(x_1,\ldots,x_d)\in\mathcal{R}^d_+} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d}\,, \tag{14}$$

regardless of any of the limits being finite or (negatively) infinite.

The proof of (13) is similar to that of [1, Theorem 1], with an important change. The proof of (14) relies on (13), the original Fekete's lemma, and the following Lemma 5.2, whose proof we leave to the reader.

Lemma 5.2. *Let u be a real-valued function depending on $d \geqslant 1$ variables x_i, no matter of what type. Then, for every permutation σ of $[1:d]$,*

$$\inf_{x_{\sigma(1)}} \dots \inf_{x_{\sigma(d)}} u(x_1, \dots, x_d) = \inf_{x_1, \dots, x_d} u(x_1, \dots, x_d) . \tag{15}$$

Proof of Theorem 5.1. Fix $t_1, \dots, t_d \in \mathbb{R}_+$. For every $i \in [1:d]$ and $x_i \geqslant 2t_i$ there exist unique $q_i \in \mathbb{Z}_+$ and $r_i \in [t_i, 2t_i)$ such that $x_i = q_i t_i + r_i$. For every $i \in [1:d]$ let $y_i^{(0)} = r_i$ and $y_i^{(1)} = t_i$. By repeatedly applying subadditivity, once per each variable, we find that

$$f(x_1, \dots, x_d) \leqslant \sum_{w \in 2^d} q_1^{w_1} \cdots q_d^{w_d} \cdot f\left(y_1^{(w_1)}, \dots, y_d^{(w_d)}\right) . \tag{16}$$

Now, on the right-hand side of (16), each occurrence of f has k arguments chosen from the t_i's and $d-k$ chosen from the r_i's, is multiplied by the q_i's corresponding to the t_i's, and is bounded from above by the constant

$$M = \sup\{f(y_1, \dots, y_d) \mid y_i \in [t_i, 2t_i] \; \forall i \in [1:d]\} ,$$

which exists because of Theorem 4.5. Such boundedness is crucial for the proof, and was ensured for free in the case of positive integer variables from [1], but had to be proved for positive real variables. From now on, the proof of (13) is identical to the proof of the equality of (4) and (5) from [1].

Now, by Lemma 3.6, for every choice of $i, j_1, \dots, j_k \in [1:d]$ all different, and however fixed the remaining variables, the function (7) is subadditive. Then (14) follows from Lemma 5.2 by repeated application of (13):

$$\begin{aligned}
& \lim_{x_{\sigma(1)} \to +\infty} \dots \lim_{x_{\sigma(d)} \to +\infty} \frac{f(x_1, \dots, x_d)}{x_1 \cdots x_d} \\
= & \inf_{x_{\sigma(1)} > 0} \lim_{x_{\sigma(2)} \to +\infty} \dots \lim_{x_\sigma(d) \to \infty} \frac{f(x_1, \dots, x_d)}{x_1 \cdots x_d} \\
= & \dots \\
= & \inf_{x_{\sigma(1)} > 0} \dots \inf_{x_{\sigma(d)} > 0} \frac{f(x_1, \dots, x_d)}{x_1 \cdots x_d} \\
= & \inf_{x_1, \dots, x_d > 0} \frac{f(x_1, \dots, x_d)}{x_1 \cdots x_d} \\
= & \lim_{(x_1, \dots, x_d) \in \mathcal{R}_+^d} \frac{f(x_1, \dots, x_d)}{x_1 \cdots x_d} .
\end{aligned}$$

□

From Theorem 5.1 and Proposition 3.7 the next result follows.

Theorem 5.3. *Let $d \geqslant 1$, let $w, w' \in 2^d$ and let $f : \mathbb{R}_w \to \mathbb{R}$ be componentwise subadditive.*

(1) *If w contains evenly many 1s, then*

$$\lim_{(x_1,\ldots,x_d)\in\mathcal{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d} = \inf_{(x_1,\ldots,x_d)\in\mathbb{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d} \tag{17}$$

is not $+\infty$, but can be $-\infty$.

(2) *If w contains oddly many 1s, then*

$$\lim_{(x_1,\ldots,x_d)\in\mathcal{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d} = \sup_{(x_1,\ldots,x_d)\in\mathbb{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d} \tag{18}$$

is not $-\infty$, but can be $+\infty$.

(3) *Suppose now w contains evenly many 1s, w' differs from w in exactly one coordinate, and f is defined and componentwise subadditive in $\mathbb{R}_w \cup \mathbb{R}_{w'} \cup U_{w,w'}$, where*

$$U_{w,w'} = \{x \in \mathbb{R}^d \mid x_i = 0, (x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_d) \in \mathbb{R}_w \cup \mathbb{R}_{w'}\}$$

is the boundary between $\mathbb{R}_w$ and $\mathbb{R}_{w'}$. Then

$$\lim_{(x_1,\ldots,x_d)\in\mathcal{R}_{w'}} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d} \leqslant \lim_{(x_1,\ldots,x_d)\in\mathcal{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d}; \tag{19}$$

consequently, both limits are finite.

For $d = 1$ we recover [6, Theorem 6.6.1]. To prove Theorem 5.3, we make use of the following result, whose proof we leave to the reader.

Lemma 5.4. *Let S be a semigroup and $f : S \to \mathbb{R}$ be a subadditive function. If S is a monoid with the identity e, then $f(e) \geqslant 0$. If, in addition, S is a group, then $f(x) + f(x^{-1}) \geqslant 0$ for every $x \in S$.*

Proof of Theorem 5.3. For $x = (x_1,\ldots,x_d) \in \mathcal{R}_w$ and $i \in [1:d]$ let $x_w = (x_{w,1},\ldots,x_{w,d})$ and f_w be defined as in Proposition 3.7. If w contains evenly many 1s, then $x_1\cdots x_d = x_{w,1}\cdots x_{w,d}$ and

$$\begin{aligned}
\lim_{x\in\mathcal{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d} &= \lim_{x_w\in\mathcal{R}_+^d} \frac{f_w(x_{w,1},\ldots,x_{w,d})}{x_{w,1}\cdots x_{w,d}} \\
&= \inf_{x_w\in\mathbb{R}_+^d} \frac{f_w(x_{w,1},\ldots,x_{w,d})}{x_{w,1}\cdots x_{w,d}} \\
&= \inf_{x\in\mathbb{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d}.
\end{aligned}$$

If w contains oddly many 1s, then $x_1\cdots x_d = -x_{w,1}\cdots x_{w,d}$ and

$$\lim_{x\in\mathcal{R}_w} \frac{f(x_1,\ldots,x_d)}{x_1\cdots x_d} = -\lim_{x_w\in\mathcal{R}_+^d} \frac{f_w(x_{w,1},\ldots,x_{w,d})}{x_{w,1}\cdots x_{w,d}}$$

$$= \; -\inf_{x_w \in \mathbb{R}_+^d} \frac{f_w(x_{w,1}, \ldots, x_{w,d})}{x_{w,1} \cdots x_{w,d}}$$
$$= \; \sup_{x \in \mathbb{R}_w} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \, .$$

Suppose now w has evenly many 1s and w' differs from w only in component i, and f is defined and componentwise subadditive in $\mathbb{R}_w \cup \mathbb{R}_{w'} \cup U_{w,w'}$. Then, for every $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_d$, the function $x_i \mapsto f(x_1, \ldots, x_i, \ldots, x_d)$ is subadditive on $\mathbb{R}$. By Lemma 5.4, for every $x_i \in \mathbb{R}$,

$$f(x_1, \ldots, x_i, \ldots, x_d) + f(x_1, \ldots, -x_i, \ldots, x_d) \geqslant 0 \, .$$

Then

$$\lim_{x \in \mathcal{R}_w} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} - \lim_{x' \in \mathcal{R}_{w'}} \frac{f(x'_1, \ldots, x'_d)}{x'_1 \cdots x'_d}$$
$$= \lim_{x \in \mathcal{R}_w} \frac{f(x_1, \ldots, x_i, \ldots, x_d)}{x_1 \cdots x_d} + \lim_{x \in \mathcal{R}_w} \frac{f(x_1, \ldots, -x_i, \ldots, x_d)}{x_1 \cdots x_d}$$
$$= \lim_{x \in \mathcal{R}_w} \frac{f(x_1, \ldots, x_i, \ldots, x_d) + f(x_1, \ldots, -x_i, \ldots, x_d)}{x_1 \cdots x_d}$$

is nonnegative. The last passage is valid because the two limits on the second line are either finite or $-\infty$. □

As every subnet of a convergent net converges to the same limit, we get the following corollary.

Corollary 5.5. *Let $f : \mathbb{R}_+^d \to \mathbb{R}$ be componentwise subadditive and let T be either $\mathbb{R}_+$ or $\mathbb{Z}_+$. For every $i \in [1{:}d]$ let $x_i(t) : T \to \mathbb{R}_+$ satisfy $\lim_{t \to +\infty} x_i(t) = +\infty$. Then*

$$\lim_{t \to +\infty} \frac{f(x_1(t), \ldots, x_d(t))}{x_1(t) \cdots x_d(t)} = \inf_{x_1, \ldots, x_d > 0} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \, , \tag{20}$$

and also

$$\inf_{t \in T} \frac{f(x_1(t), \ldots, x_d(t))}{x_1(t) \cdots x_d(t)} = \inf_{x_1, \ldots, x_d > 0} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \, . \tag{21}$$

In particular,

$$\lim_{n \to \infty} \frac{f(n, \ldots, n)}{n^d} = \inf_{n \geqslant 1} \frac{f(n, \ldots, n)}{n^d} = \inf_{x_1, \ldots, x_d > 0} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \, . \tag{22}$$

Sketch of proof. We only remark that (21) follows from (13) and

$$\inf_{x_1, \ldots, x_d > 0} \frac{f(x_1, \ldots, x_d)}{x_1 \cdots x_d} \;\leqslant\; \liminf_{t \to \infty} \frac{f(x_1(t), \ldots, x_d(t))}{x_1(t) \cdots x_d(t)} \, .$$

□

Note that, in general, even if f is componentwise subadditive on $\mathbb{R}_+^d$, $t \mapsto f(t, \ldots, t)$ is not subadditive on $\mathbb{R}_+$: a simple example is $f(x_1, x_2) = x_1 \cdot x_2$. This provides further evidence that Theorems 5.1 and 5.3 are not special cases of [6, Theorem 6.6.1].

A real-valued function defined on a semigroup $(S, \cdot)$ is *superadditive* if it satisfies $f(x \cdot y) \geqslant f(x) + f(y)$ for every $x, y \in S$. As f is superadditive if and only if $-f$ is subadditive, an analogue of Theorem 5.1 holds for componentwise superadditive functions, provided one swaps the roles of inf and sup and those of $-\infty$ and $+\infty$. If f is superadditive in some variables and subadditive in other variables, however, Theorem 5.1 does not hold.

Example 5.6. The function $f : \mathbb{R}_+^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1^2\sqrt{x_2}$ is superadditive in x_1 and subadditive in x_2, and $f(x_1, x_2)/x_1 x_2 = x_1/\sqrt{x_2}$. But $\lim_{(x_1,x_2)\in\mathcal{R}_+^2} \frac{f(x_1,x_2)}{x_1 x_2}$ does not exist, because for every $y, R > 0$ there exist $x_1, x_2 > R$ such that $x_1/\sqrt{x_2} = y$. Also, $\lim_{x_1\to\infty} \lim_{x_2\to\infty} \frac{f(x_1,x_2)}{x_1 \cdot x_2} = 0$ but $\lim_{x_2\to\infty} \lim_{x_1\to\infty} \frac{f(x_1,x_2)}{x_1 \cdot x_2} = +\infty$.

As a final remark for this section, the following statement appears in the literature as an extension to arbitrary dimension of [6, Theorem 6.1.1].

Proposition 5.7 (cf. [8, Theorem 16.2.9])**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be subadditive in the variable $\mathbf{x} \in \mathbb{R}^d$. Then, for every $\mathbf{x} \in \mathbb{R}^d$, the following limit exists:*

$$L_{\mathbf{x}} = \lim_{t\to+\infty} \frac{f(t\mathbf{x})}{t}\,.$$

This, however, is not so much an extension than a *corollary.* If $f : \mathbb{R}^d \to \mathbb{R}$ satisfies $f(\mathbf{x} + \mathbf{y}) \leqslant f(\mathbf{x}) + f(\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, then obviously $g_{\mathbf{x}}(t) = f(t\mathbf{x})$ satisfies $g_{\mathbf{x}}(s + t) \leqslant g_{\mathbf{x}}(s) + g_{\mathbf{x}}(t)$ for every $s, t > 0$, and $L_{\mathbf{x}}$ is simply the limit of $g_{\mathbf{x}}(t)/t$ according to [6, Theorem 6.1.1]. On the other hand, Theorem 5.3 is an extension.

6. A comparison with the Ornstein–Weiss lemma

A group G is *amenable* if there exist a directed set $\mathcal{U} = (U, \preccurlyeq)$ and a net $\{F_x\}_{x\in U}$ of finite nonempty subsets of G such that:

$$\lim_{x\in\mathcal{U}} \frac{|gF_x \setminus F_x|}{|F_x|} = 0 \;\text{ for every } g \in G\,. \tag{23}$$

A net such as in (23) is called a *(left) Følner net* on the group G, from the Danish mathematician Erling Følner who introduced them in [4]. Every abelian group is amenable: for a proof, see [2, Chapter 4].

Proposition 6.1 (Ornstein–Weiss lemma; cf. [11])**.** *Let G be an amenable group and let $f : \mathcal{PF}(G) \to \mathbb{R}$ be a function which*

(1) *is subadditive with respect to set union, that is,* $f(A \cup B) \leqslant f(A) + f(B)$ *for every* $A, B \in \mathcal{PF}(G)$*; and*
(2) *satisfies* $f(A) = f(gA)$ *for every* $A \in \mathcal{PF}(G)$ *and* $g \in G$.

Then, for every directed set $\mathcal{U} = (U, \preccurlyeq)$ *and every left Følner net* $\mathcal{F} = \{F_x\}_{x \in U}$ *on* G,

$$L = \lim_{x \in \mathcal{U}} \frac{f(F_x)}{|F_x|} \tag{24}$$

exists, and does not depend on the choice of $\mathcal{U}$ *and* $\mathcal{F}$.

The Ornstein–Weiss lemma says that, for "well behaved" functions on amenable groups, a notion of *asymptotic average* is well defined. A detailed proof of Proposition 6.1 is given by F. Krieger in [7].

Example 6.2. Let G be an amenable group and let A be a finite set with $a \geqslant 2$ elements. The *shift* by $g \in G$ is the function $\sigma_g : A^G \to A^G$ defined by $\sigma_g(c)(x) = c(g \cdot x)$ for every $c \in A^G$ and $x \in G$. The notions of subshift and of allowed pattern with support $S \in \mathcal{PF}(G)$ are extended naturally from those of Example 3.3. Calling $\mathcal{A}_X(S)$ the number of allowed patterns for X with support S, and convening that the unique *empty pattern* $e : \emptyset \to A$ appears in every configuration, we have for every $S, T \in \mathcal{PF}(G)$,

$$\mathcal{A}_X(S) \leqslant \mathcal{A}_X(S \cup T) \leqslant \mathcal{A}_X(S) \cdot \mathcal{A}_X(T \setminus S) \leqslant \mathcal{A}_X(S) \cdot \mathcal{A}_X(T)\,.$$

Indeed, every allowed pattern on S (resp., $T \setminus S$) can be extended to at least one allowed pattern on $S \cup T$ (resp., T) but joining an allowed pattern over S and an allowed pattern over $T \setminus S$ does not necessarily yield an allowed pattern on $S \cup T$. Hence, $f(S) = \log_a \mathcal{A}_X(S)$ is subadditive on $\mathcal{PF}(G)$, and clearly satisfies $f(gS) = f(S)$ for every $g \in G$ and $S \in \mathcal{PF}(G)$. The *entropy* of X can then be defined as:

$$h(X) = \lim_{x \in \mathcal{U}} \frac{\log_a \mathcal{A}_X(F_x)}{|F_x|} \tag{25}$$

where $\mathcal{U} = (U, \preccurlyeq)$ is an arbitrary directed set and $\{F_x\}_{x \in U}$ is an arbitrary Følner net on G.

As the sets $E_{x_1,\ldots,x_d} = \prod_{i=1}^d [1 : x_i]$ with $x_1, \ldots, x_d \in \mathbb{Z}_+$ constitute a Følner net on $\mathbb{Z}^d$, defining the entropy of a d-dimensional subshift according to either Example 4.2 or Example 6.2 yields the same result. Nevertheless, the Ornstein–Weiss lemma does not generalize Fekete's lemma, nor it is possible to prove the latter from the former, as the limit (24) is only ensured to exist, not to coincide with any specific value. In addition, even if $f : \mathbb{Z} \to \mathbb{R}$ is subadditive, the "natural" conversion

$$g(A) = \begin{cases} f(|A|) & \text{if } A \neq \emptyset\,, \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

is invariant by translations, but needs not be subadditive on $\mathcal{PF}(\mathbb{Z})$, the main reason being that $|A \cup B|$ needs not equal $|A| + |B|$. Moreover, while invariance by translations is essential in the Ornstein–Weiss lemma, a translate of a subadditive function needs not be subadditive.

Example 6.3. The function $f(n) = n \bmod 2$ is easily seen to be subadditive on $\mathbb{Z}$. But the function g defined from f by (26) is not subadditive on $\mathcal{PF}(\mathbb{Z})$, because if $U = \{1, 2\}$ and $V = \{2, 3\}$, then $g(U \cup V) = 1$ and $g(U) = g(V) = 0$. Note that $h(n) = f(n + 1)$ is not subadditive, because $h(1) = 0$ but $h(2) = 1$.

7. Conclusions

We have discussed an extension of the notion of subadditivity in the case of many independent variables. In this context, we have proved a nontrivial extension of the classical Fekete's lemma to the case of functions of $d \geqslant 1$ real variables, which recovers the original statement for $d = 1$, and which is more general than other extensions already present in the literature. While doing so, we have also proved that these componentwise subadditive functions satisfy the important property of being bounded on compact subsets, the case $d = 1$ being already known from the literature.

We believe that our results can be of interest for researchers in economics, optimization, theory of dynamical systems, and mathematical analysis.

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