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# Natural vibrations of curved nano-beams and nano-arches

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Abstract. The natural vibrations of curved nano-beams and nano-arches are studied. The nano-arches under consideration have piecewise constant thickness; these are weakened with stable cracks located at re-entrant corners of the steps. A method of determination of natural frequencies is developed making use of the method of weightless rotating spring. The aim of the paper is to assess the sensitivity of the eigenfrequencies on the geometrical and physical parameters of the nano-arch. The results of the calculations favourably compare with similar works of other researchers.

## 1. Introduction

The main ideas of the non-local theory of elasticity were initially formulated by Eringen [7] and Eringen and Edelen [8] several decades ago. However, the rapid progress in the non-local theories got its start with the wide use of nanomaterials (see Thai [27], Thai et al. [28], Reddy [21]). Comprehensive reviews of papers dedicated to the mechanical behaviour of nano-beams, nano-plates and nano-arches can be found in the review papers by Faghidian [9, 10], Farajpour, Ghayesh, and Farokhi [11], also by Wang and Arash [31]. It is interesting to remark that the non-local theory was initially formulated in the integral form and later it was reformulated by Eringen in the differential form making use of a specific kernel function. Since the differential form is more simple it is widely used in the analysis of nanostructures.

A general approach to the application of the non-local theories of the elasticity in the bending, buckling and vibration of nano-beams was developed by Aydogdu [2]. Resorting to the results obtained by Nayfeh and Emam [20]

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a comparative study on the buckling and post-buckling analysis of nano-beams was undertaken by Emam [6]. The formulations corresponding to the classical Euler–Bernoulli approach, the first order Timoshenko theory and higher order shear deformation Reddy theories are analysed in detail. The comparison of the classical bending theory and the shear deformation theories is carried out by Reddy [21, 22], as well.

Vibrations of nano-beams containing cracks and other defects are studied by Roostai and Haghpanahi [24], Loya [19], also by Lellep and Lenbaum [17], Loghmani and Yazdi [18], Hossain and Lellep [12, 13]. In the paper [13] the effect of the temperature is taken into account. In the paper by Jiang and Wang [14] analytical solutions are developed for nano-beams subjected to axial and thermal stresses.

The vibrations of curved beams or arches and the detection of damages in these structures is studied by Cerri and Ruta [3], Viola, Artioli, and Dilena [29] with the help of vibrational analysis. Viola and Tornabene [30] undertake the analysis of circular arches of variable cross-section. The effects of cracks are taken into account in the paper by Karaagac et al. [15].

In the present paper the natural frequencies of circular nano-beams are defined, and the sensitivity of eigenfrequencies on the geometrical and material parameters of a nano-arch is studied.

## 2. Formulation of the problem

Let us consider the dynamic behaviour of a nano-arch of radius $R$. For determination of the positions of central points of the arch the polar angle $\varphi$ is introduced (Figure 1). Assume that the angles $\varphi = 0$ and $\varphi = \beta$ correspond to the edges of the nano-arch, respectively.

It is assumed that the thickness of the nano-arch is defined as

$$h = \begin{cases} h_0, & \varphi \in [0, \alpha), \\ h_1, & \varphi \in (\alpha, \beta]. \end{cases} \tag{1}$$

In (1), $h_0, h_1$ are given numbers and $\alpha \in (0, \beta)$. The width $b$ of the nano-arch is assumed to be a constant. The aim of the study is to determine the eigenfrequencies of stepped nano-beams weakened with defects at the re-entrant corners of steps. It is assumed that at the cross section $\varphi = \alpha$, a crack with length $c$ is located. The defect is treated as a stable crack; no attention will be paid to the extension of the crack during vibrations.

However, the sensitivity of the eigenfrequency with respect to the position of the crack and other parameters will be clarified.

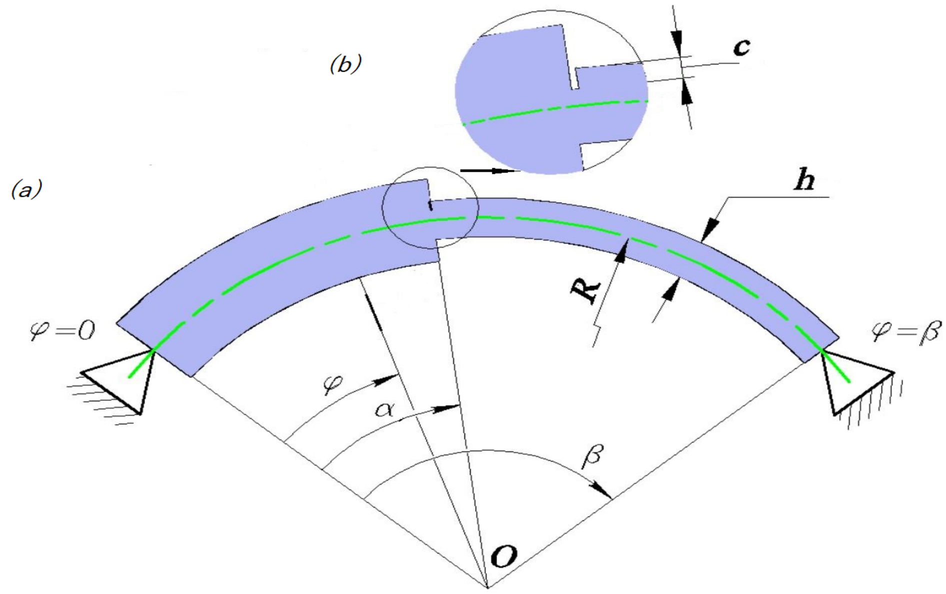


FIGURE 1. Stepped arch.

## 3. Governing equations and main hypotheses

The system of governing equations includes the equilibrium equations and constitutive and geometrical relations with boundary conditions. The equilibrium equations for an element of the arch or curved beam can be presented as (see Soedel [25])

$$\begin{aligned} \frac{\partial N}{\partial s} + \frac{Q}{R} + p_u &= \bar{\rho} h \ddot{U}, \\ \frac{\partial Q}{\partial s} - \frac{N}{R} + p &= \bar{\rho} h \ddot{W}, \\ \frac{\partial M}{\partial s} - Q &= 0. \end{aligned} \tag{2}$$

In (2), $N$, $M$ and $Q$ denote the membrane force, bending moment, and shear force, respectively. Here $s$ is the curvilinear coordinate related to the current angle as

$$s = R\varphi.$$

In (2), $U$, $W$ stand for the displacements in the circumferential and normal directions, respectively and $p_u$ and $p$ are the intensities of the distributed loading in these directions. Here $h$ stands for the thickness of the arch and $\bar{\rho}$ is the mass per unit length of the arch. Let $b$ be the width of the arch. In

(2), dots denote the differentiation with respect to time $t$. This means that

$$\ddot{U} = \frac{\partial^2 U}{\partial t^2}, \quad \ddot{W} = \frac{\partial^2 W}{\partial t^2}.$$

The strain-displacement equations can be taken according to Soedel [25] as

$$\varepsilon = \frac{1}{R}W + \frac{\partial U}{\partial s}, \tag{3}$$

and

$$\varkappa = -\frac{\partial^2 U}{\partial s^2} + \frac{\partial U}{\partial s}\frac{1}{R}. \tag{4}$$

In (3), (4) $\varepsilon$ stands for the relative extension of the curved element and $\varkappa$ is the curvature of the middle line of the arch.

It is assumed herein that the vibrations of the arch are in-plane motions only. According to the Hook's law in the classical theory of the elasticity one has

$$M_c = EI\varkappa, \tag{5}$$

where $M_c$ denotes the bending moment in the classical theory. Here $E$ is the Young's modulus and $I$ is the moment of inertia of the cross section of the arch. Evidently, in the case of a rectangular cross section

$$I = \frac{1}{12}bh^3. \tag{6}$$

As $h$ is a piecewise continuous function of the current angle, the second moment (6) has the constant values $I_0$ and $I_1$ in the regions $(0, \alpha)$ and $(\alpha, \beta)$, respectively. It is assumed that the material of the arch obeys the constitutive equations of the non-local theory of elasticity (see Eringen [7], Reddy [21]). In the non-local theory the stresses at a given point of the body depend on the strains at all points of the body. One of the simplest physical relations of this type can be presented as (see Reddy [21], Emam [6])

$$\sigma_{ij} - \eta\nabla^2\sigma_{ij} = \sigma_{ij}^c, \tag{7}$$

where $\sigma_{ij}$ denotes the components of the non-local stress tensor and $\sigma_{ij}^c$ stands for the classical elastic stress tensor. Here $\eta = (e_0 a)^2$ stands for a material constant, $a$ is the dimension of the lattice of the material and $e_0$ is a physical constant to be defined experimentally.

Instead of the stress components $\sigma_{ij}$ in (7) one can use the generalized stresses used in the equilibrium equations (2). However, it is reasonable to adapt the system (2) to the current problem.

Let us assume that $p_u = 0$, $U(\varphi, t)$ is small and that $\ddot{U}$ is also negligible. In this case the system (2) can be rewritten as

$$\begin{aligned} N^{'} &= -Q, \\ Q^{'} &= N + R(\rho h \ddot{W} - p), \\ M^{'} &= RQ, \end{aligned} \tag{8}$$

where prime denotes the differentiation with respect to the current angle $\varphi$. It can be seen from (8) that in the case when $M(\varphi)$ and $N(\varphi)$ vanish at the same cross section one has

$$M = -RN. \tag{9}$$

Equations (8) and (9) result in

$$M^{''} + M + R^2(p - \rho h \ddot{W}) = 0. \tag{10}$$

Taking in (7) instead of $\sigma_{ij}$ the moment $M$ yields

$$M - \eta M^{''} = M_c. \tag{11}$$

Combining (11) and (5) with (10) one obtains

$$M = \frac{1}{1+\eta}[\frac{-EI}{R^2}(W^{''} + W) + h\eta\rho R^2 \ddot{W}]. \tag{12}$$

Substituting the bending moment from (12) into (10) where $p = 0$ one obtains the equation

$$-\frac{EI}{R^2}(W^{IV} + 2W^{''} + W) + \rho h R^2(\eta \ddot{W}^{''} - \ddot{W}) = 0. \tag{13}$$

Equation (13) is a fourth order equation. The boundary conditions for (13) are in the case of simply supported arches

$$\begin{aligned} W(0, t) &= 0, \\ W(\beta, t) &= 0, \end{aligned} \tag{14}$$

and

$$\begin{aligned} M(0, t) &= 0, \\ M(\beta, t) &= 0. \end{aligned} \tag{15}$$

## 4. Solution of the governing equations

Making use of the method of separation of variables it is assumed that

$$W(\varphi, t) = X(\varphi)T(t), \tag{16}$$

where $X(\varphi)$ is a function of the coordinate $\varphi$ and $T(t)$ is a function depending on time $t$ only.

Differentiating (16) one can easily find that

$$\begin{aligned} \ddot{W} &= X(\varphi)\ddot{T}(t), \\ W'' &= X''(\varphi)T(t), \\ W^{IV} &= X^{IV}(\varphi)T(t), \\ \ddot{W}'' &= X''(\varphi)\ddot{T}(t). \end{aligned} \tag{17}$$

The substitution of (17) in (13) leads to the differential equations

$$\ddot{T} + \omega^2 T = 0, \tag{18}$$

and

$$-\frac{EI}{R^2}(X^{IV} + 2X'' + X) - \rho h R^2 \omega^2 (\eta X'' - X) = 0. \tag{19}$$

In (18) $\omega$ stands for the frequency of natural vibrations of the arch. The equation (19) must be solved separately in regions $(0, \alpha)$ and $(\alpha, \beta)$ respectively, since $h$ is defined by (1) so that it has different values in these sections. Evidently, the equation (18) has periodical solutions. One of these can be taken as

$$T = \bar{A}\sin(\omega t),$$

where $\bar{A}$ is a constant. Equation (19) can be presented in the form

$$X'''' + (2 + K_j\eta)X'' + (1 - K_j)X = 0, \tag{20}$$

where

$$K_j = \begin{cases} K_0, & \varphi \in (0, \alpha), \\ K_1, & \varphi \in (\alpha, \beta). \end{cases} \tag{21}$$

In (20), (21)

$$K_j = \frac{\omega^2 \rho h_j R^4}{EI_j},$$

where $j = 0, 1$. Here

$$I_0 = \frac{bh_0^3}{12}, \quad I_1 = \frac{bh_1^3}{12}.$$

In order to solve the linear fourth order equation (20) one has to solve the characteristic equation

$$\lambda_j^4 + (2 + K_j\eta)\lambda_j^2 + 1 - K_j = 0. \tag{22}$$

It is easy to recheck that

$$\lambda_j = \pm\left(\frac{1}{2}\left(-2 - K_j\eta \pm \sqrt{(2 + K_j\eta)^2 - 4(1 - K_j)}\right)\right)^{\frac{1}{2}}. \tag{23}$$

Thus the general solution of the (19) has the form

$$X_j = C_1 \cosh \mu_j\varphi + C_2 \sinh \mu_j\varphi + C_3 \cos \nu_j\varphi + C_4 \sin \nu_j\varphi, \tag{24}$$

where

$$\begin{aligned} \mu_j &= \left(-\frac{1}{2}(2+K_j\eta)+\frac{1}{2}\sqrt{(K_j\eta)^2+4K_j\eta+4K_j}\right)^{\frac{1}{2}}, \\ \nu_j &= \left(\frac{1}{2}(2+K_j\eta)+\frac{1}{2}\sqrt{(K_j\eta)^2+4K_j\eta+4K_j}\right)^{\frac{1}{2}}. \end{aligned} \tag{25}$$

It is expected in (22)–(25) that $j=0$, if $\varphi \in (0,\alpha)$, and $j=1$ if $\varphi \in (\alpha,\beta)$. The boundary conditions (14) and (15) together with (16) admit to assert that the boundary requirements for $X_j(t)$ are

$$\begin{aligned} X_j(0) &= 0, \\ X_j^{''}(0) &= 0, \end{aligned} \tag{26}$$

and

$$\begin{aligned} X_j(\beta) &= 0, \\ X_j^{''}(\beta) &= 0. \end{aligned} \tag{27}$$

Note that (26) and (27) hold in the case of an arch simply supported at both edges.

## 5. Local flexibility due to the crack

It is recognized long time ago that cracks and other defects are the sources of additional structural compliance. Dimarogonas [5], Chondros et al [4] also Kukla [16], suggested to treat the slope of deflection $W^{'}$ as a discontinuous variable and to define a new variable

$$\theta = W^{'}(\alpha+0,t) - W^{'}(\alpha-0,t),$$

so that

$$\theta = CM(\alpha,t),$$

where $C$ can be considered as the additional compliance, and $M$ stands for the moment of internal forces at $\varphi=\alpha$. Evidently, $C$ can be an appropriate matrix, as well. In this case instead of $M$ one has the vector of corresponding internal forces applied at the same cross section.

It is known in the linear elastic fracture mechanics that the energy release rate due to the crack can be calculated as

$$G = \frac{M^2}{2b}\frac{dC}{dc}. \tag{28}$$

In (28) $b$ is the width of the crack of rectangular cross section and $M$ is the moment applied at this cross section and $c$ stands for the length of the crack.

On the other hand, the stress intensity factor can be defined as (see Anderson [1])

$$K = \sigma\sqrt{\pi c}F(s),$$

where $s = \frac{c}{h}$, $F(s)$ is the shape function, and

$$\sigma = \frac{6M}{bh^2}.$$

The quantity $K$ is coupled with the energy release rate as

$$G = \frac{K^2}{E'}. \tag{29}$$

In (29) $E' = E$ for the plane stress state and $E' = \frac{E}{1-\nu^2}$ for plane strain state. In the literature (see Anderson [1], Dimarogonas [5], Tada et al. [26]) one can find different forms of shape functions. In the present paper we are following Dimarogonas [5], Rizos et al. [23], and using the function $F(s)$ in the form

$$F(s) = 1.93 - 3.07s + 14.53s^2 - 25.11s^3 + 25.8s^4.$$

## 6. Numerical results

Numerical results are presented for nano-arches simply supported at both ends in Figures 2–7. Here the nano-arches have a single step and the material constants are $E = 7 \times 10^{11}\ Pa$, $\nu = 0.3$, $\eta = 1\ nm$ and $R = 30\ nm$. It is assumed that $\beta = 1\ rad$, if $\beta$ is not specified in the legend. The wall thickness is $h_0 = 10\ nm$, $h_1 = 20\ nm$, except in Figure 2.

In Figure 2 by considering $h_0 = 10\ nm$ the natural frequency of the nano-arch versus the thickness $h_1$ is depicted for different values of the radius $R$. It can be seen from Figure 2 that the natural frequency increases if the thickness of the arch increases. However, if the radius of the arch increases then the natural frequency decreases.

In Figures 3 and 4 the natural frequency as a function of the step location $\alpha$ is presented. In Figure 3 the nano-arch with the central angle $\beta = 1\ rad$ is treated. Different curves in Figure 3 correspond to the crack extensions $s = 0$, $s = 0.1$, $s = 0.2$, $s = 0.3$, $s = 0.4$ respectively. It can be seen from Figure 3 that the lowest values of the natural frequency correspond to the arch without any cracks.

In Figure 4 similar results are presented for different values of the central angle $\beta$ (here $s = 0.7$).

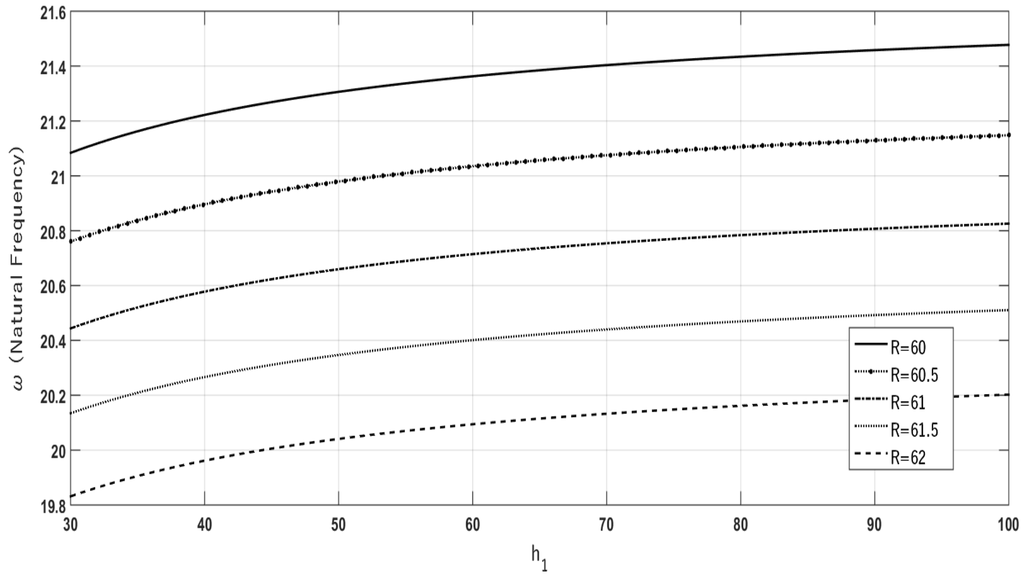


FIGURE 2. Natural frequency vs thickness of the nano-arch.

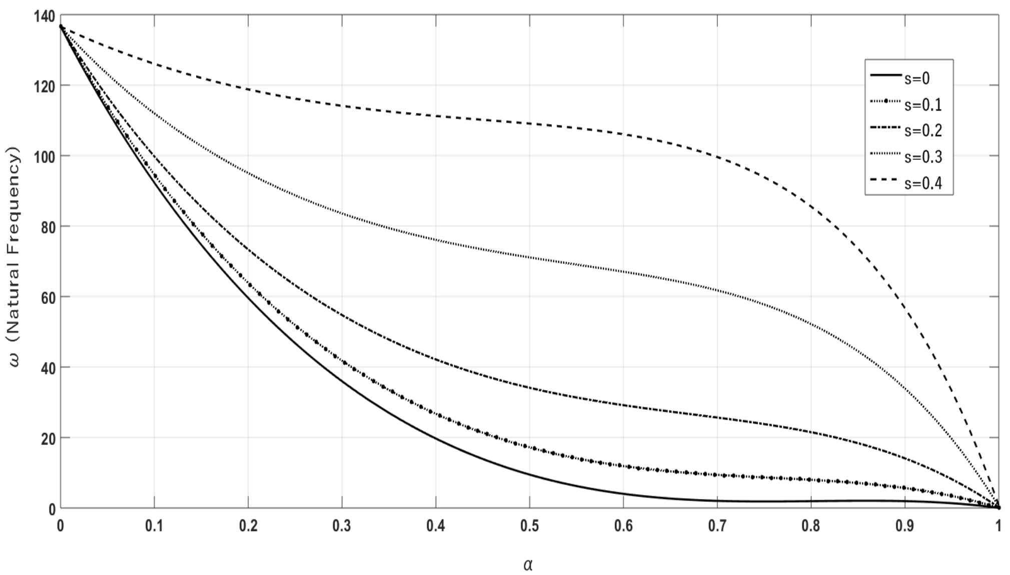


FIGURE 3. Natural frequency vs step location.

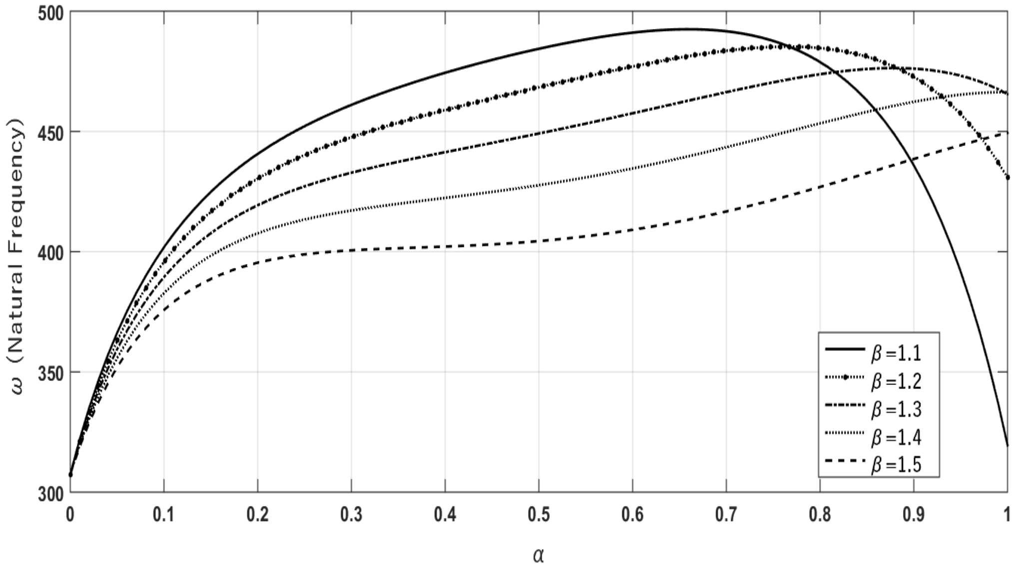


FIGURE 4. Natural frequency vs step location for different angles $\beta$.

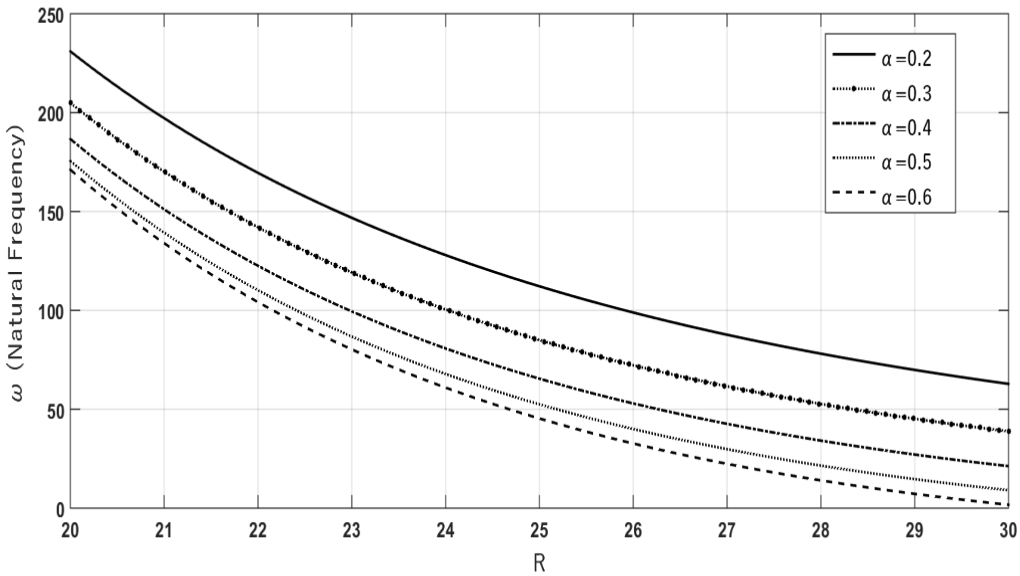


FIGURE 5. Natural frequency vs radius of the nano-arch.

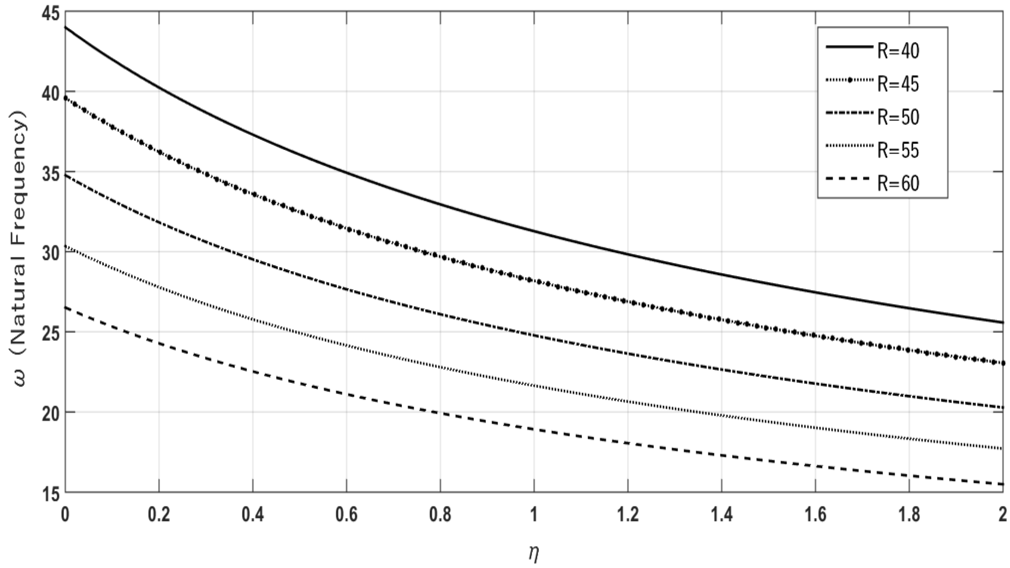


FIGURE 6. Natural frequency vs material constant of the nano-arch.

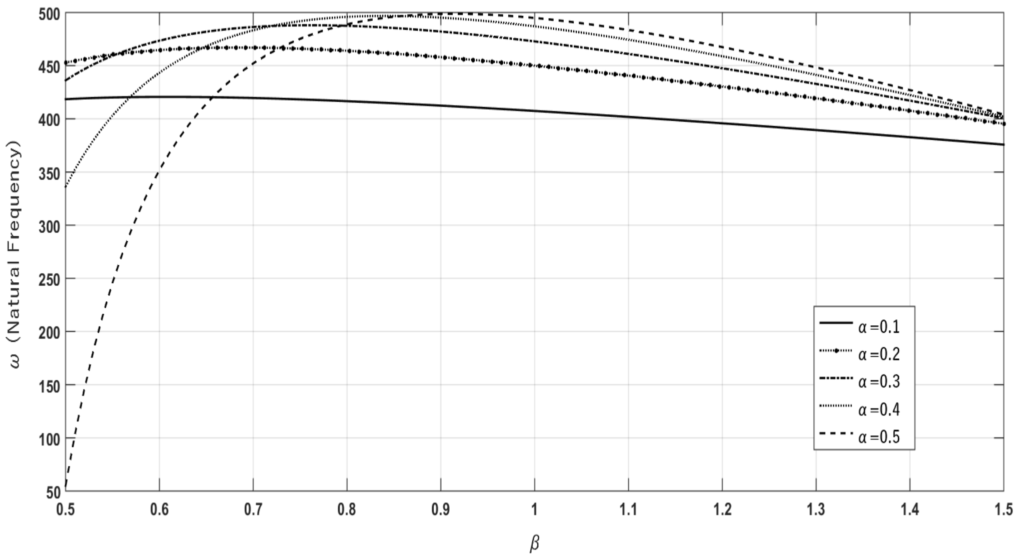


FIGURE 7. Natural frequency vs central angle of the nano-arch.

| Mode | $\eta$ | Present | Thai [27] |
|---|---|---|---|
| 1 | 0 | 9.75821445 | 9.2745 |
| | 1 | 7.05584295 | 8.8482 |
| | 2 | 5.80188130 | 8.4757 |
| | 3 | 5.04192108 | 8.1466 |
| | 4 | 4.51883759 | 7.8530 |

Table 1: Natural vibration of the nano-arches of the constant thickness.

Figure 4 reveals that in the case of smaller values of the angle $\alpha$ the lowest eigenfrequency is achieved in the case of the largest value of the central angle $\beta$. However, in the case when $\beta > 2\alpha$ this relationship is more complicated.

The relationship between the natural frequency and the radius $R$ of the nano-arch is shown in Figure 5 in the case $\beta = 1$ and $s = 0.7$ (here $\eta = 1\ nm$). Figure 5 reveals that the smaller is $\alpha$ the larger is the the natural frequency of the nano-arch. On the other hand, the larger is $\alpha$ the lower is the eigenfrequency.

The dependence of the eigenfrequency on the material parameter $\eta = (e_0 a)^2$ is demonstrated in Figure 6 for different values of the radius $R$. Generally speaking, the eigenfrequency decreases if the material parameter increases. Thus, in the case $\eta = 0$ the natural frequency has its maximal value.

The sensitivity of the eigenfrequency on the central angle $\beta$ of the nano-arch is shown in Figure 7 for radius $R = 30\ nm$. Different curves in Figure 7 correspond to nano-arches having the step in different cross sections (here $\eta = 1\ nm$). One can see from Figure 7 that in the case of larger values of the angle $\beta$ curves corresponding to different values of the step angle are quite close to each other. However, if $\beta$ is around the value 0.5 the discrepancies between these curves are large.

The results obtained in the current study are compared with those obtained by Thai [27] in the case when $\alpha = 0$, $\beta = 0.5°$. Table 1 shows that the results are in reasonable correspondence. One can see from Table 1 that for small values of $\eta$ the results are close to each other but for larger values of $\eta$ the discrepancy between corresponding results is larger.

## 7. Concluding remarks

Natural vibrations of curved nano-beams are treated in the current study. The nano-beams have piecewise constant thickness with stable cracks at the re-entrant corners of steps. A method for determination of natural frequencies based on the methods of the linear elastic fracture mechanics is developed. Calculations carried out showed that the cracks effect essentially the values of natural frequencies. It is shown that the lowest values of the natural frequency correspond to nano-arches without any defect. The results of the current study are compared with results obtained by Thai [27]. The results compare favourably in the case of small values of the parameter $\eta$.

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