ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 26, Number 1, June 2022
Available online at https://ojs.utlib.ee/index.php/ACUTM

Sasaki–Kenmotsu manifolds

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Abstract. In the present paper, we introduce a new class of structures on an even dimensional differentiable Riemannian manifold which combines, well known in literature, the Sasakian and Kenmotsu structures simultaneously. The structure will be called a Sasaki–Kenmotsu structure by us. Firstly, we discuss the normality of the Sasaki–Kenmotsu structure and give some basic properties. Secondly, we present some important results concerning with the curvatures of the Sasaki–Kenmotsu manifold. Finally, we show the existence of the Sasaki–Kenmotsu structure by giving some concrete examples.

1. Introduction

Yano introduced the notion of f-structure on a $(2n+s)$-dimensional manifold which satisfies $f^3 + f = 0$, where f is a $(1,1)$-tensor field of constant rank $2n$ [10]. An f-structure is a generalization of some structures defined on differentiable manifolds of different type. Almost complex $(s = 0)$ and almost contact $(s = 1)$ structures are well-known examples of f-structures. Later, some authors continued to study f-structures. Goldberg and Yano [7] introduced and studied globally framed metric f-structures on $(2n + s)$-dimensional differentiable manifolds. It is important to define and study new structures on differentiable manifolds for differential geometry. In this direction, Blair [2] defined K-structures which are special cases of S-structures and C-structures. We note that K-structures are the analogue of Kählerian structures in the almost complex geometry and also S-structures (resp., C-structures) of Sasakian structures in the almost contact geometry. In [1], Alegre, Fernandez and Prieto-Martin introduced a new class of metric f-manifolds which are called trans-S-manifolds because they play the same

Received October 24, 2021.
2020 *Mathematics Subject Classification.* Primary 53C53; secondary 53C15, 53C25.
Key words and phrases. f-structure, almost contact metric structure, Sasaki–Kenmotsu structure.
https://doi.org/10.12697/ACUTM.2022.26.06

role as trans-Sasakian manifolds in metric f-geometry. The class of trans-S-manifolds includes S-manifolds, C-manifolds and f-Kenmotsu manifolds as special cases. They prove some properties of trans-S-manifolds and show that several examples constructed by other authors are included in this class.

An f-structure with $s = 2$ has arisen in the study of hypersurfaces in almost contact spaces [4]. The structure has been studied first by Goldberg and Yano [6]. Inspired by this structure, Debnath and Konar [5] constructed a new kind of structure on an even dimensional Riemannian manifold with dimension $2n+2$ ($s = 2$), named as almost pseudo contact structure with two associated vector fields. They presented some basic properties of this structure and gave an example to show the existence of such structures.

Motivated by their research, in this paper, we study almost pseudo contact structures or more precisely we call them almost bi-contact structures to indicate the presence of two global vector fields, as a class of f-structures on a $(2n+2)$-dimensional Riemannian manifold. The structure which combines the Sasaki and Kenmotsu structures is a structure of new kind on an even dimensional differentiable Riemannian manifold and at the same time it is different from the trans-Sasakian structure and not a class of the trans-S-manifolds introduced by Alegre et al. [1].

The paper is organized as follows. In section 2, we review basic definitions and results that are needed to state and prove our results. In section 3, we study the normality of an almost bi-contact metric structure. In section 4, we consider a particular type of this class which we call Sasaki–Kenmotsu structure and we give some basic properties. Then, we construct an interesting class of examples to prove the existence of this type on $(2n+2)$-dimensional Euclidean space. The last section is devoted to establishing some basic results for Riemannian curvature tensor of a Sasaki–Kenmotsu manifold.

2. Review of definitions and needed results

Let (M, g) be a Riemannian manifold. The Lie algebra of all C^∞ vector fields on M will be denoted by $\mathfrak{X}(M)$. We denote by R and S the Riemannian curvature tensor and the Ricci tensor, respectively, defined for all $X, Y, Z \in \mathfrak{X}(M)$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1}$$

$$\begin{aligned} S(X,Y) &= \sum_{i=1}^{n} R(e_i, X, Y, e_i) \\ &= \sum_{i=1}^{n} g\big(R(e_i, X)Y, e_i\big), \end{aligned} \tag{2}$$

where $\{e_i\}_{i=1,n}$ is a local orthonormal basis.

2.1. Almost Hermitian manifolds. An almost complex manifold M is a differentiable manifold equipped with a $(1,1)$ tensor field J which satisfies $J^2 = -I$, where I is the identity. Such a manifold is even-dimensional. M^{2n} is an almost Hermitian manifold provided it is almost complex and has a Riemannian metric g for which

$$g(JX, JY) = g(X, Y)$$

for all $X, Y \in \mathfrak{X}(M^{2n})$. To describe the geometry of an almost Hermitian manifold M^{2n}, it is useful to consider two special tensors. The first, called the Nijenhuis tensor, is a $(1,2)$ tensor field N_J defined by

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]. \tag{3}$$

An almost complex structure J is integrable if its Nijenhuis tensor N_J vanishes. In this case, the almost Hermitian manifold M^{2n} is a Hermitian manifold. The second is a 2-form Ω, called the Kähler form, and it is defined by

$$\Omega(X, Y) = g(X, JY)$$

for all $X, Y \in \mathfrak{X}(M^{2n})$. The almost Hermitian manifold M^{2n} is an almost Kähler manifold if Ω is closed, i.e., $\mathrm{d}\Omega = 0$. If both $\mathrm{d}\Omega = 0$ and $N_J = 0$ are satisfied, then M^{2n} is called a Kähler manifold. Recall that $\mathrm{d}\Omega = 0$ and $N_J = 0$ are equivalent to

$$\nabla J = 0,$$

where ∇ denotes the Levi-Civita connection corresponding to g. For more background on almost complex manifolds, we refer to [11].

2.2. Almost contact metric manifolds. An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if on M^{2n+1} there exist a $(1,1)$-tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\begin{cases} \eta(\xi) = 1, \\ \varphi^2(X) = -X + \eta(X)\xi, \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \end{cases} \tag{4}$$

for any $X, Y \in \mathfrak{X}(M^{2n+1})$. On an almost contact metric manifold we immediately have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

The fundamental 2-form ϕ is defined by $\phi(X, Y) = g(X, \varphi Y)$. It is known that the almost contact structure (φ, ξ, η) is said to be normal if and only if

$$N^{(1)}(X, Y) = N_\varphi(X, Y) + 2d\eta(X, Y)\xi = 0 \tag{5}$$

for all $X, Y \in \mathfrak{X}(M^{2n+1})$, where N_φ denotes the Nijenhuis torsion of φ given by

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{6}$$

Some special cases are worthy of attention. A contact metric structure is an almost contact metric structure with $\mathrm{d}\eta = \phi$ and a Sasakian manifold is a normal contact metric manifold if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{7}$$

where ∇ denotes the Levi-Civita connection of g.

An almost contact metric structure is said to be almost cosymplectic if both η and ϕ are closed. In addition, if the structure is normal, then it is said to be cosymplectic, equivalently $(\nabla_X \varphi)Y = 0$. An almost contact metric manifold is an almost Kenmotsu manifold if $\mathrm{d}\eta = 0$ and $\mathrm{d}\phi = 2\eta \wedge \phi$. Moreover, if the structure (φ, ξ, η) is normal, the manifold is said to be Kenmotsu. Kenmotsu manifolds can also be characterized by

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{8}$$

where ∇ denotes the Levi-Civita connection of g. In [9], Oubina proves that (φ, ξ, η, g) is a trans-Sasakian structure of type (α, β) if and only if it is normal and

$$\mathrm{d}\eta = \alpha\phi, \qquad \mathrm{d}\phi = 2\beta\eta \wedge \phi, \tag{9}$$

where $\alpha = \frac{1}{2n}\delta\phi(\xi)$, $\beta = \frac{1}{2n} div\xi$ and δ is the codifferential of g. It is well known that the trans-Sasakian structure may be described as an almost contact metric structure satisfying

$$(\nabla_X \varphi)Y = \alpha\big(g(X, Y)\xi - \eta(Y)X\big) + \beta\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big). \tag{10}$$

For more background on almost contact metric manifolds, we refer to [3, 8, 9, 11].

2.3. Almost bi-contact metric manifolds. A $(2n+2)$-dimensional differentiable manifold M of class C^∞ is said to have a $(\varphi, \xi, \psi, \eta, \omega)$-structure or almost bi-contact structure if it admits a field φ of endomorphisms of the tangent spaces, two global vector fields ξ and ψ, and two 1-forms η and ω satisfying

$$\eta(\xi) = \omega(\psi) = 1, \qquad \varphi^2 = -I + \eta \otimes \xi + \omega \otimes \psi, \tag{11}$$

$$\eta(\psi) = \omega(\xi) = \varphi\xi = \varphi\psi = 0 \qquad \text{and} \qquad \eta \circ \varphi = \omega \circ \varphi = 0. \tag{12}$$

Here the endomorphism φ has rank $2n$.

If the manifold M^{2n+2} with an almost bi-contact structure $(\varphi, \xi, \psi, \eta, \omega)$ admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) - \omega(X)\omega(Y) \tag{13}$$

for all $X, Y \in \mathfrak{X}(M)$, then $(\varphi, \xi, \psi, \eta, \omega, g)$ is said to be an almost bi-contact metric structure and $(M^{2n+2}, \varphi, \xi, \psi, \eta, \omega, g)$ is called an almost bi-contact metric manifold. An immediate consequences is that

$$\eta(X) = g(X, \xi) \qquad \text{and} \qquad \omega(X) = g(X, \psi).$$

The 2-form ϕ on M^{2n+2} defined by

$$\phi(X,Y)=g(X,\varphi Y) \tag{14}$$

is called the fundamental 2-form of the almost bi-contact metric structure. If ∇ is the Riemannian connection of g, then it is easy to prove that

$$(\nabla_X\eta)Y=g(\nabla_X\xi,Y) \qquad \text{and} \qquad (\nabla_X\omega)Y=g(\nabla_X\psi,Y), \tag{15}$$

and

$$(\nabla_X\phi)(Y,Z)=g\big(Y,(\nabla_X\varphi)Z\big)=-g\big((\nabla_X\varphi)Y,Z\big). \tag{16}$$

The exterior derivatives of η, ω and ϕ are given by

$$2\mathrm{d}\eta(X,Y)=(\nabla_X\eta)Y-(\nabla_Y\eta)X, \tag{17}$$

$$2\mathrm{d}\omega(X,Y)=(\nabla_X\omega)Y-(\nabla_Y\omega)X, \tag{18}$$

$$3\mathrm{d}\phi(X,Y,Z)=\sigma(\nabla_X\phi)(Y,Z), \tag{19}$$

where σ denotes the cyclic sum over $X,Y,Z\in\mathfrak{X}(M)$.

3. Normality of almost bi-contact metric structure

Theorem 1. *For any almost bi-contact metric structure $(\varphi,\xi,\psi,\eta,\omega)$ on a Riemannian manifold (M^{2n+2},g) there exist an almost hermitian structure J on (M^{2n+2},g) given by*

$$J=\varphi-\xi\wedge\psi, \tag{20}$$

where the operator $\wedge$ is defined by

$$(X\wedge Y)Z=g(Y,Z)X-g(X,Z)Y, \tag{21}$$

for every tangent vectors fields X,Y and Z on M^{2n+2}.

Proof. Let $(M^{2n+2},\varphi,\xi,\psi,\eta,\omega,g)$ be an almost bi-contact metric manifold. For all $X\in\mathfrak{X}(M)$, we have

$$JX=\varphi X+\eta(X)\psi-\omega(X)\xi.$$

It follows that $J\xi=\psi$ and $J\psi=-\xi$. So, using formulas (11) and (12) we get

$$\begin{aligned} J^2X &= J\big(\varphi X+\eta(X)\psi-\omega(X)\xi\big)\\ &= \varphi^2X+\eta(X)\xi+\omega(X)\psi\\ &= -X. \end{aligned}$$

Moreover, we have

$$\begin{aligned} g(JX,JY) &= g\big(\varphi X+\eta(X)\psi-\omega(X)\xi,\varphi Y+\eta(Y)\psi-\omega(Y)\xi\big)\\ &= g(X,Y) \end{aligned}$$

for every tangent vectors fields X and Y on M^{2n+2} which completes the proof. $\square$

Using formulas (3) and (20), straightforward and long calculations give

$$\begin{aligned} N_J(X,Y) &= N_\varphi^{(1)}(X,Y) + N_\eta^{(2)}(X,Y)\psi + N_\omega^{(2)}(X,Y)\xi \\ &+ \Big(N_{\eta,\xi}^{(4)} \wedge \omega + \eta \wedge N_{\eta,\psi}^{(4)}\Big)(X,Y)\psi \\ &+ \Big(N_{\omega,\xi}^{(4)} \wedge \eta + \omega \wedge N_{\omega,\psi}^{(4)}\Big)(X,Y)\xi \\ &+ \Big(N_{\varphi,\xi}^{(3)} \wedge \omega + \eta \wedge N_{\varphi,\psi}^{(3)}\Big)(X,Y) \\ &+ (\eta\wedge\omega)(X,Y)[\xi,\psi], \end{aligned} \tag{22}$$

where $N_\varphi^{(1)}, N^{(2)}, N^{(3)}$ and $N^{(4)}$ are the following tensor fields on M^{2n+2} defined respectively by

$$\begin{aligned} N_\varphi^{(1)}(X,Y) &= \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X,\varphi Y] \\ &+ 2\mathrm{d}\eta(X,Y)\xi + 2\mathrm{d}\omega(X,Y)\psi, \end{aligned} \tag{23}$$

$$N_\eta^{(2)}(X,Y) = \big(L_{\varphi X}\eta\big)(Y) - \big(L_{\varphi Y}\eta\big)(X),$$
$$N_\omega^{(2)}(X,Y) = \big(L_{\varphi X}\omega\big)(Y) - \big(L_{\varphi Y}\omega\big)(X),$$
$$N_{\varphi,\xi}^{(3)}(X) = -\big(L_\xi\varphi\big)(X),$$
$$N_{\varphi,\psi}^{(3)}(X) = -\big(L_\psi\varphi\big)(X),$$
$$N_{\eta,\xi}^{(4)}(X) = \big(L_\xi\eta\big)(X),$$
$$N_{\eta,\psi}^{(4)}(X) = \big(L_\psi\eta\big)(X)$$

and

$$2\big(N_{\omega,\xi}^{(i)} \wedge \eta\big)(X,Y) = N_{\omega,\xi}^{(i)}(X)\eta(Y) - N_{\omega,\xi}^{(i)}(Y)\eta(X)$$

for $i \in \{3,4\}$, where L_X denotes the Lie derivative with respect to the vector field X.

Proposition 1. *For an almost bi-contact manifold $(M^{2n+2}, \varphi, \xi, \psi, \eta, \omega, g)$, the vanishing of the tensor field $N_\varphi^{(1)}$ implies the vanishing of the tensor fields $N^{(2)}, N^{(3)}$, $N^{(4)}$ and $[\xi,\psi]$.*

Proof. Suppose that $N_\varphi^{(1)} = 0$. Replacing $(X,Y) = (\xi,\psi)$ in the formula (23), we obtain $[\xi,\psi] = 0$. Setting $Y = \xi$ (resp., $Y = \psi$) and applying η (resp., ω), we get $\mathrm{d}\eta(X,\xi) = 0$ (resp., $\mathrm{d}\omega(X,\psi) = 0$), which easily gives $N^{(4)} = 0$.

Next, we notice that replacing X by φX in $N_\varphi^{(1)}(X,\xi)$ (resp., $N_\varphi^{(1)}(X,\psi)$), gives

$$0 = N_\varphi^{(1)}(\varphi X, \xi) = [\xi, \varphi X] - \varphi[\xi, X] = (L_\xi\varphi)X = N_{\varphi,\xi}^{(3)}(X),$$

$\big($resp., $0 = N_\varphi^{(1)}(\varphi X, \psi) = [\psi, \varphi X] - \varphi[\psi, X] = (L_\psi\varphi)X = N_{\varphi,\psi}^{(3)}(X).$ $\big)$

Finally, applying η (resp., ω) to $N_\varphi^{(1)}(\varphi X, Y)$ gives

$$\begin{aligned} 0 &= \eta\big(\big(N_\varphi^{(1)}(\varphi X, Y)\big) \\ &= \eta\big([\varphi^2 X, \varphi Y]\big) + \varphi X \eta(Y) - \eta\big([\varphi X, Y]\big) \end{aligned}$$

which simplifies to $N_\eta^{(2)} = 0$ (resp., $N_\omega^{(2)} = 0$). □

The almost bi-contact structure $(\varphi, \xi, \psi, \eta, \omega)$ is normal if and only if the tensor $N_\varphi^{(1)}$ vanishes. So, the normality condition transforms into

$$[\varphi, \varphi](X, Y) + 2\mathrm{d}\eta(X, Y)\xi + 2\mathrm{d}\omega(X, Y)\psi = 0, \tag{24}$$

where

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

Next, we will establish a formula for the covariant derivative of φ.

Proposition 2. *For an almost bi-contact structure $(\varphi, \xi, \psi, \eta, \omega, g)$, the covariant derivative of φ is given by*

$$\begin{aligned} 2g\big((\nabla_X \varphi)Y, Z\big) &= 3\mathrm{d}\phi(X, \varphi Y, \varphi Z) - 3\mathrm{d}\phi(X, Y, Z) + g\big(N_\varphi^{(1)}(Y, Z), \varphi X\big) \\ &+ N_\eta^{(2)}(Y, Z)\eta(X) + N_\omega^{(2)}(Y, Z)\omega(X) \\ &+ 2\mathrm{d}\eta(\varphi Y, X)\eta(Z) - 2\mathrm{d}\eta(\varphi Z, X)\eta(Y) \\ &+ 2\mathrm{d}\omega(\varphi Y, X)\omega(Z) - 2\mathrm{d}\omega(\varphi Z, X)\omega(Y). \end{aligned} \tag{25}$$

Proof. Using Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &+ g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X) \end{aligned}$$

and the formulas

$$2(\mathrm{d}\eta)(X, Y) = X\big(\eta(Y)\big) - Y\big(\eta(X)\big) - \eta\big([X, Y]\big)$$

$$2(\mathrm{d}\omega)(X, Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X, Y]\big)$$

$$\begin{aligned} 3(\mathrm{d}\phi)(X, Y, Z) &= X\big(\phi(Y, Z)\big) + Y\big(\phi(Z, X)\big) + Z\big(\phi(X, Y)\big) \\ &- \phi\big([X, Y], Z\big) - \phi\big([Y, Z], X\big) - \phi\big([Z, X], Y\big) \end{aligned}$$

with

$$(\nabla_X \varphi)Y = \nabla_X \varphi Y + \varphi \nabla_X Y,$$

the result is easily obtained. We omit standard calculations. □

Depending on Proposition 2, one can define several classes of almost bi-contact metric structures by discussing the different values for $\mathrm{d}\phi$, $\mathrm{d}\eta$ and $\mathrm{d}\omega$. While this is an area of possible future research, we study here an interesting structure which combines the Sasakian and Kenmotsu structures simultaneously. We refer to this structure as Sasaki–Kenmotsu structure.

4. Sasaki–Kenmotsu structure

Definition 1. An almost bi-contact metric structure $(\varphi, \xi, \psi, \eta, \omega, g)$ on M^{2n+2} is said to be a Sasaki–Kenmotsu structure if $(\varphi, \xi, \psi, \eta, \omega)$ is normal, ω is closed and $\mathrm{d}\eta = \phi - 2\eta \wedge \omega$.

Remark 1. Another class of Sasaki–Kenmotsu structures can be obtained if $(\varphi, \xi, \psi, \eta, \omega)$ is normal, η is closed and $\mathrm{d}\omega = \phi - 2\omega \wedge \eta$.

Remark 2. On a Sasaki–Kenmotsu manifold, differentiating the formula $\mathrm{d}\eta = \phi - 2\eta \wedge \omega$ and using $\mathrm{d}\omega = 0$ we get

$$\mathrm{d}\phi = 2\mathrm{d}\eta \wedge \omega,$$

which easily gives

$$\mathrm{d}\phi = 2\omega \wedge \phi.$$

Let $\mathcal{D}_\eta := \{X \in \mathfrak{X}(M) \ / \ \eta(X) = 0\}$ and $\mathcal{D}_\omega := \{X \in \mathfrak{X}(M) \ / \ \omega(X) = 0\}$ be two distributions of rank $2n+1$ transversal to the characteristic vector fields ξ and ψ, respectively. So, we notice that we have $\mathrm{d}\omega = 0$ and $\mathrm{d}\phi = 2\omega \wedge \phi$ on $\mathcal{D}_\eta$, and therefore $(\varphi, \psi, \omega, g)$ is a Kenmotsu structure on $\mathcal{D}_\eta$. Having $\mathrm{d}\eta = \phi$ on $\mathcal{D}_\omega$ implies that $(\varphi, \psi, \omega, g)$ is a Sasaki structure on $\mathcal{D}_\omega$. That is why we call these structures Sasaki–Kenmotsu structures.

Theorem 2. *If the structure $(\varphi, \xi, \psi, \eta, \omega, g)$ is a Sasaki–Kenmotsu structure, then*

$$(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\xi - \eta(Y)\varphi^2 X + g(\varphi X, Y)\psi - \omega(Y)\varphi X. \tag{26}$$

Proof. The proof follows easily from Proposition 2 with conditions in Definition 1 and Remark 2. □

Example 1. Let $(x^\alpha, y^\alpha, z, t)$ denote the Cartesian coordinates in $M = \mathbb{R}^{2m+2}$, $m \geq 1$. Latin indices take on values from 1 to $2m+2$, Greek indices will run from 1 to m and $\alpha' = \alpha + m$ for all $\alpha \in \{1, ..., m\}$.

Let (e_i) be the frame of vector fields on M defined by

$$e_\alpha = \frac{1}{t}\frac{\partial}{\partial x^\alpha}, \ e_{\alpha'} = \frac{1}{t}(\frac{\partial}{\partial y^\alpha} + 2x^\alpha \frac{\partial}{\partial z}), \ e_{2m+1} = \frac{1}{t^2}\frac{\partial}{\partial z}, \ e_{2m+2} = t\frac{\partial}{\partial t},$$

and let (θ^i) be the dual frame of differential 1-forms

$$\theta^\alpha = tdx^\alpha, \ \theta^{\alpha'} = tdy^\alpha, \ \theta^{2m+1} = t^2(-2\sum_\lambda x^\lambda dy^\lambda + dz), \ \theta^{2m+2} = \frac{1}{t}dt.$$

For the non-zero Lie brackets of (e_i), we have

$$[e_\alpha, e_{\beta'}] = 2\delta_{\alpha\beta}e_{2m+1}, \qquad [e_\alpha, e_{2m+2}] = e_\alpha,$$
$$[e_{\alpha'}, e_{2m+2}] = e_{\alpha'}, \qquad [e_{2m+1}, e_{2m+2}] = 2e_{2m+1}.$$

Define an almost bi-contact structure $(\varphi, \xi, \psi, \eta, \omega)$ on M by assuming

$$\varphi e_\alpha = e_{\alpha'}, \quad \varphi e_{\alpha'} = -e_\alpha \ , \ \varphi e_{2m+1} = \varphi e_{2m+2} = 0,$$

$$\xi = e_{2m+1}, \quad \psi = e_{2m+2}, \quad \eta = \theta^{2m+1}, \quad \omega = \theta^{2m+2}.$$

For the normality (i.e., vanishing of the tensor $N_\varphi^{(1)}$), it can be checked that

$$N_\varphi^{(1)}(e_i, e_j) = [\varphi, \varphi](e_i, e_j) + 2\mathrm{d}\eta(e_i, e_j)\xi + 2\mathrm{d}\omega(e_i, e_j)\psi = 0$$

for any i, j. Then $(\varphi, \xi, \psi, \eta, \omega)$ is normal.

Let g be a Riemannian metric on M for which (e_i) is an orthonormal frame, so that $g = \sum \theta^i \otimes \theta^i$. It is obvious that $(\varphi, \xi, \psi, \eta, \omega, g)$ is a bi-contact metric structure on M. For the fundamental form $\phi(X, Y) = g(X, \varphi Y)$, we have

$$\phi = -2\sum_\lambda \theta^\lambda \wedge \theta^{\lambda'} = -2t^2 \sum_\lambda dx^\lambda \wedge dy^\lambda,$$

from which it follows that

$$\phi = \mathrm{d}\theta^{2m+1} + 2\theta^{2m+1} \wedge \theta^{2m+2} = \mathrm{d}\eta + 2\eta \wedge \omega.$$

Hence $(\varphi, \xi, \psi, \eta, \omega)$ is normal (i.e., $N_\varphi^{(1)} = 0$) and $\mathrm{d}\omega = \mathrm{d}\theta^{2m+2} = 0$. As a result, $(\varphi, \xi, \psi, \eta, \omega, g)$ becomes a Sasaki–Kenmotsu structure on M.

To verify the result in Theorem 2, we give the Levi-Civita connection corresponding to g by

$$\begin{aligned}
&\nabla_{e_\alpha} e_\beta = \nabla_{e_{\alpha'}} e_{\beta'} = -\delta_{\alpha\beta}\psi,\\
&\nabla_{e_\alpha} e_{\beta'} = -\nabla_{e_{\alpha'}} e_\beta = \delta_{\alpha\beta}\xi,\\
&\nabla_{e_\alpha} \xi = \nabla_\xi e_\alpha = -\nabla_{e_{\alpha'}}\psi = -e_{\alpha'},\\
&\nabla_{e_\alpha} \psi = \nabla_{e_{\alpha'}}\xi = \nabla_\xi e_{\alpha'} = e_\alpha,\\
&\nabla_\xi \xi = -2\psi,\\
&\nabla_\xi \psi = 2\xi.
\end{aligned}$$

The non-zero components of $\nabla\varphi$ are the following:

$$\begin{aligned}
&(\nabla_{e_\alpha}\varphi)e_\beta = (\nabla_{e_{\alpha'}}\varphi)e_{\beta'} = \delta_{\alpha\beta}\xi,\\
&(\nabla_{e_\alpha}\varphi)e_{\beta'} = -(\nabla_{e_{\alpha'}}\varphi)e_\beta = \delta_{\alpha\beta}\psi,\\
&(\nabla_{e_\alpha}\varphi)\xi = (\nabla_{e_{\alpha'}}\varphi)\psi = e_\alpha,\\
&(\nabla_{e_\alpha}\varphi)\psi = -(\nabla_{e_{\alpha'}}\varphi)\xi = e_{\alpha'}.
\end{aligned}$$

One can easily check that for all $i \in \{1, ..., 2m+2\}$

$$(\nabla_{e_i}\varphi)e_j = g(e_i, e_j)\xi - \eta(e_j)e_i + g(\varphi e_i, e_j)\psi - \omega(e_j)\varphi e_i - \omega(e_i)\big(\xi \wedge \psi\big)e_j.$$

Based on formula (22), a Sasaki–Kenmotsu structure is normal (i.e., $N_\varphi^{(1)} = 0$), which implies the integrability of the almost complex structure J, i.e., $N_J = 0$.

On the other hand, using the formula (20) we get

$$\begin{aligned}
\Omega(X, Y) &= g(X, JY)\\
&= (\phi - 2\eta \wedge \omega)(X, Y)
\end{aligned}$$

$$= \mathrm{d}\eta(X,Y), \tag{27}$$

which implies that $\mathrm{d}\Omega = 0$. Thus, (J, g) becomes a Kählerian structure. Using this fact, it follows that $\nabla J = 0$.

Lemma 1. *Let $(M^{2n+2}, \varphi, \xi, \psi, \eta, \omega, g)$ be a Sasaki–Kenmotsu manifold. We have*

1) $\nabla_\xi \xi = -2\psi$ *and* $\nabla_\xi \psi = 2\xi$,
2) $\nabla_\psi \xi = \nabla_\psi \psi = 0$,
3) $\omega(\nabla_X \xi) = -2\eta(X)$,
4) $\eta(\nabla_X \psi) = 2\eta(X)$.

Proof. For the first formula, for all $X, Y \in \mathfrak{X}(M)$ we have

$$\Omega(X,Y) = \mathrm{d}\eta(X,Y) \Leftrightarrow 2g(X, JY) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X). \tag{28}$$

Taking $Y = \xi$ and using $g(\nabla_X \xi, \xi) = 0$, we have $\nabla_\xi \xi = -2\psi$. With the help of $\nabla J = 0$, we get $\nabla_\xi \psi = 2\xi$.

For the second, we have

$$\mathrm{d}\omega(X,Y) = 0 \Leftrightarrow g(\nabla_X \psi, Y) - g(\nabla_Y \psi, X) = 0. \tag{29}$$

Taking $Y = \psi$ and using $g(\nabla_X \psi, \psi) = 0$, we obtain $\nabla_\psi \psi = 0$ and $\nabla_\psi \xi = 0$.

For the third, we have

$$\begin{aligned} \omega(\nabla_X \xi) &= g(\nabla_X \xi, \psi) \\ &= 2\mathrm{d}\eta(X, \psi) + g(\nabla_\psi \xi, X) \\ &= 2g(X, J\psi) \\ &= -2\eta(X). \end{aligned}$$

Finally

$$\eta(\nabla_X \psi) = g(\nabla_\xi \psi, X) = 2\eta(X).$$

□

Now, we are going to give an explicit expression for $\nabla_X \xi$ and $\nabla_X \psi$ for any $X \in \mathfrak{X}(M)$.

Proposition 3. *For a Sasaki–Kenmotsu manifold, we have*

$$\nabla_X \xi = -\varphi X - 2\eta(X)\psi \qquad \textit{and} \qquad \nabla_X \psi = -\varphi^2 X + 2\eta(X)\xi. \tag{30}$$

Proof. In formula (26), setting $Y = \xi$ (resp., $Y = \psi$) gives

$$\varphi \nabla_X \xi = -\varphi^2 X \quad (\text{resp., } \varphi \nabla_X \psi = -\varphi X),$$

from which using formula (11) we get

$$\nabla_X \xi = -\varphi X + \omega(\nabla_X \xi)\psi \quad (\text{resp., } \varphi \nabla_X \psi = -\varphi^2 X + \eta(\nabla_X \psi)\xi).$$

Using formulas 3) and 4) in Lemma 1 we obtain

$$\nabla_X \xi = -\varphi X - 2\eta(X)\psi, \qquad (\textit{resp.}, \ \nabla_X \psi = -\varphi^2 X + 2\eta(X)\xi.$$

□

5. Curvature tensor

The equations (30) give important information about the curvature properties of Sasaki–Kenmotsu manifold. Now, we give the following proposition without proof, because it can easily be obtained by standard calculations.

Proposition 4. *Let $(M^{2n+2}, \varphi, \xi, \psi, \eta, \omega, g)$ be a Sasaki–Kenmotsu manifold. Then, we have*

$$R(X,Y)\xi = -(X\wedge Y)\xi + \varphi(X\wedge Y)\psi + 6(\eta\wedge\omega)(X,Y)\psi + 2g(X,\varphi Y)\psi, \quad (31)$$

$$R(X,Y)\psi = -(X\wedge Y)\psi - \varphi(X\wedge Y)\xi - 6(\eta\wedge\omega)(X,Y)\xi + 2g(X,\varphi Y)\xi, \quad (32)$$

$$R(X,\xi)Y = -(X\wedge\xi)Y - (\varphi X\wedge\psi)Y + 3\omega(X)(\xi\wedge\psi)Y - \omega(X)\varphi Y, \quad (33)$$

$$R(X,\psi)Y = (X\wedge\psi)Y - (\varphi X\wedge\xi)Y + 3\eta(X)(\xi\wedge\psi)Y + 2\eta(X)\varphi Y, \quad (34)$$

$$S(X,\xi) = -2(n+2)\eta(X), \quad (35)$$

$$S(X,\psi) = 2(n+2)\omega(X), \quad (36)$$

where R denotes the Riemannian curvature tensor and S is the Ricci curvature defined in (1) and (2), respectively and $(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ for all vectors fields X, Y and Z on M^{2n+2}.

Let X be the unit vector field orthogonal to ξ and ψ. Then, by using (31) (resp. (32)), we obtain $R(X,\xi)\xi = -X$ (resp., $R(X,\psi)\psi = -X$) which gives $g(R(X,\xi)\xi, X) = -1$ (resp., $g(R(X,\psi)\psi, X) = -1$). Thus we have

Proposition 5. *On a Sasaki–Kenmotsu manifold, the sectional curvature of all plane sections containing ξ or ψ is -1.*

Acknowledgements

The authors would like to thank the anonymous Referee for their helpful suggestions and their valuable comments which helped to improve the manuscript.

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