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Current Hom-Lie algebras

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Abstract. In this paper, we study Hom-Lie structures on tensor products. In particular, we consider current Hom-Lie algebras and discuss their representations. We determine faithful representations of minimal dimension of current Heisenberg Hom-Lie algebras. Moreover derivations, including generalized derivations and centroids, are studied. Furthermore, cohomology and extensions of current Hom-Lie algebras are also considered.

Introduction

Current algebra or Current Lie algebras were introduced first in Physics by Murray Gell-Mann to describe weak and electromagnetic currents of the strongly interacting particles, hadrons, leading to the Adler–Weisberger formula and other important physical results. Important examples include Affine Lie algebra, Chiral model, Virasoro algebra, Vertex operator algebra and Kac–Moody algebra. The concept of a Hom-Lie algebra was initially introduced by Hartwig, Larsson, and Silvestrov in [7]. It was motivated by quantum deformations of algebras of vector fields like Witt and Virasoro algebras. Hom-Lie structures were discussed in [4, 9], their derivations, representations, cohomology and deformations were studied first in [13, 2, 20]. In this paper we extend current Lie algebras theory introduced in [24, 25, 26] to Hom-Lie context, see also [1, 19]. A current Lie algebra is a Lie algebra of the form $L \otimes A$, where L is a Lie algebra, A is a commutative associative algebra, and the multiplication in $L \otimes A$ being defined by the formula $[x \otimes a, y \otimes b] = [x, y] \otimes (ab)$, for any $x, y \in L, a, b \in A$. More generally, Lie structures on tensor products were studied by Zusmanovich in

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[24], while Hom-Lie structures on a current Lie algebra $L \otimes A$ were considered by Makhlouf and Zusmanovich in [15]. The second aim of this paper is to discuss Hom-Lie structures on tensor products $L \otimes A$, where L and A are vector spaces such that either L or A is finite dimensional and endowed respectively with bilinear maps $[\cdot,\cdot]\colon L \times L \to L$ and $\mu\colon A \times A \to A$.

The paper is organized as follows. In Section 1, we review definitions and properties of Hom-Lie algebras and Hom-associative algebras. Moreover various relevant examples and low dimensional classification are given. In Section 2, we characterize Hom-Lie structures on tensor products $L \otimes A$, where either vector space L or vector space A is finite dimensional. We consider current Hom-Lie algebras $(L\otimes A, [\cdot,\cdot]_{L\otimes A}, \gamma)$, where $[\cdot,\cdot]_{L\otimes A}\colon L\otimes A \times L\otimes A \to L \otimes A$ is a bilinear map and $\gamma\colon L \otimes A \to L \otimes A$ is a linear map, for which we provide a classification of 4-dimensional current Hom-Lie structure algebras $L \otimes A$. Section 3 is dedicated to representation theory of current Hom-Lie algebras, and semidirect products and faithful representations of minimal dimension for current Heisenberg Hom-Lie algebras are considered there. In Section 4, we discuss derivations, including generalized derivations, and centroids of current Hom-Lie algebras. Moreover, explicit computations are provided. In Section 5, we study the second cohomology group of current Hom-Lie algebras with respect to trivial representation and determine explicitly the second cohomology group $H^2(\tilde{L}(\mathcal{G}))$ of Hom-Loop algebra and $H^2(\widehat{L}(\mathfrak{h}_1)_p)$ of Hom-truncated Heisenberg algebra. Finally, we study central extensions of current Hom-Lie algebras and establish their classification for Hom-Loop algebra and Hom-truncated Heisenberg algebra.

Throughout this paper, all the vector spaces are over the complex field $\mathbb{C}$ and all vector spaces are at least one-dimensional. Many of the results included in this paper are still valid if one considers any field.

1. Hom-Lie and Hom-associative algebras

In this section we summarize the relevant definitions and provide some examples of Hom-Lie and Hom-associative algebras.

1.1. Hom-Lie algebras.

Definition 1 ([2, 12, 7])**.** A Hom-Lie algebra is a triple $(\mathcal{G},\ [\cdot,\cdot],\ \alpha)$ consisting of a vector space $\mathcal{G}$, a bilinear map $[\cdot,\cdot] : \mathcal{G}\times\mathcal{G} \to \mathcal{G}$ and a linear map $\alpha : \mathcal{G} \to \mathcal{G}$ satisfying

$$[x,y] = -[y,x]\,, \text{(skew-symmetry)}$$
$$[\alpha(x),[y,z]] + [\alpha(z),[x,y]] + [\alpha(y),[z,x]] = 0, \text{(Hom-Jacobi identity)}$$

for all elements x, y, z in $\mathcal{G}$. A Hom-Lie algebra is called multiplicative if α is an algebra morphism, i.e. for any $x,\ y \in \mathcal{G}$ we have $\alpha([x,y]) = [\alpha(x), \alpha(y)]$, and it is called regular if α is an algebra automorphism. We recover Lie algebras when the linear map is the identity map.

Example 1 (Jackson $\boldsymbol{sl_2}(\mathbb{C})$, [3])**.** Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional vector space $sl_2(\mathbb{C})$ over $\mathbb{C}$. The following bracket $[\cdot,\cdot]$ and linear map α on $sl_2(\mathbb{C})$ define a Hom-Lie algebra over $\mathbb{C}$:
$[x_1, x_2] = -2ax_2$, $[x_1, x_3] = 2x_3$, $[x_2, x_3] = -\frac{1+a}{2}x_1$,
$\alpha(x_1) = ax_1$, $\alpha(x_2) = a^2x_2$, $\alpha(x_3) = ax_3$, where a is a parameter in $\mathbb{C}$.

Example 2 ([2, 12, 4])**.** Any non-abelian 2-dimensional complex multiplicative Hom-Lie algebra is isomorphic to one of the following isomorphism classes defined with respect to a basis $\{e_1, e_2\}$ by the bracket and a linear map represented by a matrix with respect to the basis:

(a) L_1: $[e_1, e_2] = -[e_2, e_1] = e_1$ and α_1 is represented by the matrix $\begin{pmatrix} 0 & \lambda \\ 0 & \mu \end{pmatrix}$.

(b) L_2 : $[e_1, e_2] = -[e_2, e_1] = e_1$ and α_2 is represented by the matrix $\begin{pmatrix} \gamma & \eta \\ 0 & 1 \end{pmatrix}$, with $\gamma \neq 0$.

Proposition 1 ([5],[22])**.** *Let $(\mathcal{G}, [\cdot,\cdot]')$ be a Lie algebra and α be a Lie algebra endomorphism. Then $(\mathcal{G}, \alpha \circ [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra.*
Moreover, let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a regular multiplicative Hom-Lie algebra. Then $(\mathcal{G}, \alpha^{-1} \circ [\cdot,\cdot])$ is a Lie algebra.

Example 3 (Heisenberg Hom-Lie algebras, [16])**.** Let $(\mathfrak{h}_m, [\cdot,\cdot])$ be a $(2m+1)$-dimensional Heisenberg Lie algebra and $\{x_1, \cdots, x_m, y_1, \cdots, y_m, z\}$ be a basis. The bracket is defined by $[x_i, y_j] = \delta_{ij}z$ for $i, j = 1, \cdots, m$, where δ_{ij} is the Kronecker symbol, other brackets are either zero or given by skew-symmetry.

Let α be a Lie algebra morphism with respect to the previous bracket. The morphisms are defined with respect to the basis $\{x_1, \cdots, x_m, y_1, \cdots, y_m, z\}$ by the following matrix :

$$\begin{pmatrix} X_{mm} & T_{mm} & 0_{m1} \\ Z_{mm} & Y_{mm} & 0_{m1} \\ L_{m1} & M_{m1} & \lambda \end{pmatrix}, \text{ where } \begin{pmatrix} X_{mm} & T_{mm} \\ Z_{mm} & Y_{mm} \end{pmatrix} \text{ is } \lambda\text{-symplectic.}$$

Acccording to the previous proposition, the bracket $[x_i, y_j]_\alpha = \delta_{ij}\alpha(z)$, defines a Hom-Lie algebra. .

Definition 2 ([3])**.** Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra. Let V be an arbitrary vector space, $\beta \in \mathcal{G}l(V)$ be an arbitrary linear self-map on V and $[\cdot,\cdot]_V : \mathcal{G} \times V \to V$, $(g, v) \mapsto [g, v]_V$ be a bilinear map.

The triple $(V, [\cdot,\cdot]_V, \rho)$ is called a representation of the Hom-Lie algebra $\mathcal{G}$ or a $\mathcal{G}$-module V if the bilinear map $[\cdot,\cdot]_V$ satisfies, for $x, y \in \mathcal{G}$ and $v \in V$,

$$[[x, y], \rho(v)]_V = [\alpha(x), [y, v]_V]_V - [\alpha(y), [x, v]_V]_V . \quad (1)$$

When $[\cdot,\cdot]_V$ is the zero-map, we say that the $\mathcal{G}$-module V is trivial.

Example 4. We construct a representation of the Hom-Lie algebra L_1 defined in Example 2. Let V_1 be a 2-dimensional vector space and let $\{v_1, v_2\}$ be its basis. Define $\rho \in End(V_1)$ by $\rho(v_1) = 0$ and $\rho(v_2) = \eta\, v_2$, and a bilinear map $[\cdot,\cdot]_{V_1} : L_1 \times V_1 \to V_1$ by

$$[e_1, v_1]_{V_1} = tv_1, \quad [e_1, v_2]_{V_1} = 0, \quad [e_2, v_1]_{V_1} = -\frac{\lambda}{\eta}\, t\, v_1, \quad [e_2, v_2]_{V_1} = 0.$$

Then $(V_1, [\cdot,\cdot]_{V_1}, \rho)$ is a representation of L_1.

1.2. Hom-associative algebras. In this section, we summarize some basics about Hom-associative algebras. For more details, see [12, 13, 11, 2].

Definition 3. A Hom-associative algebra is a triple $(A,\, \mu,\, \beta)$, in which A is a vector space, $\beta : A \to A$ a linear map and $\mu : A \times A \to A$ a bilinear map, with notation $\mu(a, a') = aa'$, satisfying, for all $a, a', a'' \in A$: $\beta(a)(a'\, a'') = (a\, a')\ \beta(a'')$, called the Hom-associativity condition.
A Hom-associative algebra is called multiplicative if for all $a, b \in A$ $\beta(ab) = \beta(a)\beta(b)$. A Hom-associative algebra is said to be unital if there exists a unit element 1 such that $\beta(1) = 1$ satisfying $\beta(a) = 1\, a = a\, 1$.

Example 5 (Laurent polynomials Hom-associative algebra)**.** Consider the Laurent polynomials algebra $A = \mathbb{K}[t, t^{-1}]$. Let β_i be an algebra endomorphism of A which is uniquely determined by the polynomial $\beta_i(f)(t) = f((qt)^i)$. Define μ by $\mu(f, g)(t) = f(\beta_i(t))g(\beta_i(t))$ for any f, g in A. Then $A_i = (A, \mu, \beta_i)$ is a unital commutative Hom-associative algebra.

Proposition 2 ([14])**.** *Any 2-dimensional complex commutative multiplicative Hom-associative algebra with basis $\{f_1, f_2\}$ is isomorphic to one of the following isomorphism classes, where the linear map β is given by its matrix with respect to the basis:*

A_i	μ_i	β_i
A_1	$f_1f_1 = -f_1,\ f_1f_2 = f_2\ ,\ f_2f_2 = f_1.$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$
A_2	$f_1f_1 = f_1,\ f_1f_2 = 0\ ,\ f_2f_2 = f_2.$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$
A_3	$f_1f_1 = f_1,\ f_1f_2 = 0\ ,\ f_2f_2 = 0.$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$
A_4	$f_1f_1 = f_1,\ f_1f_2 = f_2\ ,\ f_2f_2 = 0.$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$
A_5	$f_1f_1 = f_1,\ f_1f_2 = 0\ ,\ f_2f_2 = 0.$	$\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}.$
A_6	$f_1f_1 = f_2,\ f_1f_2 = 0\ ,\ f_2f_2 = 0.$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$
A_7	$f_1f_1 = 0,\ f_1f_2 = a\, f_1\ ,\ f_2f_2 = b\, f_1.$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$

Definition 4. Let (A, μ, β) be a Hom-associative algebra, M be a vector space and $\varphi : M \to M$ be a linear map.

(i) A left A-module structure on (M, φ) consists of a bilinear map $\mu_M : A \times M \to M$; $(a, m) \mapsto a \bullet m$ satisfying the conditions:

$$\varphi(a \bullet m) = \beta(a) \bullet \varphi(m), \quad \beta(a) \bullet (a' \bullet m) = (a\, a') \bullet \varphi(m), \tag{2}$$

for all $a, a' \in A$ and $m \in M$.

(ii) A right A-module structure on (M, φ) consists of a bilinear map $\mu_M : M \times A \to M$; $(m, a) \mapsto m \bullet a$ satisfying the conditions:

$$\varphi(m \bullet a) = \varphi(m) \bullet \beta(a), \quad \varphi(m) \bullet (aa') = (m \bullet a) \bullet \beta(a'), \tag{3}$$

for all $a, a' \in A$ and $m \in M$.

(iii) A two sided A-module structure on (M, φ) or an A-bimodule consists on a left A-module structure and a right A-module structure on (M, φ) satisfying the compatibility condition: $\beta(a) \bullet (m \bullet a') = (a \bullet m) \bullet \beta(a')$, for all $a, a' \in A$ and $m \in M$.

If A is unital we assume that $1 \bullet m = m \bullet 1 = \varphi(m)$ for all $m \in M$.

Throughout the article, we mean by a representation (M, μ_M, φ) of a Hom-associative algebra (A, μ, β) an A-bimodule structure on (M, φ).

Now, we construct left modules and representations of the Hom-associative algebra A_1 defined in Example 2.

Example 6. Let W_1 be a 2-dimensional vector space and $\{w_1, w_2\}$ be its basis. Define $\varphi_1 \in End(W_1)$ by $\varphi_1(w_1) = -w_1$ and $\varphi_1(w_2) = w_2$. Define a bilinear map $[\cdot, \cdot]_{W_1} : A_1 \times W_1 \to W_1$ by $[f_1, w_1]_{W_1} = w_1$, $[f_1, w_2]_{W_1} = -w_2$, $[f_2, w_1]_{W_1} = s\, w_2$, $[f_2, w_2]_{W_1} = -\frac{1}{s}\, w_1$, where s is a parameter. Then $(W_1, [\cdot, \cdot]_{W_1}, \varphi_1)$ is a left A_1-module.

Example 7. Let W_1' be a 2-dimensional vector space and $\{w_1, w_2\}$ be its basis. Define $\varphi_1 \in End(W_1')$ by $\varphi_1(w_1) = -w_1$ and $\varphi_1(w_2) = w_2$. Define a bilinear map $[\cdot, \cdot]_{W_1'} : A_1 \times W_1' \to W_1'$ by
$[f_1, w_1]_{W_1'} = dw_1$, $[f_1, w_2]_{W_1'} = w_2$, $[f_2, w_1]_{W_1'} = sw_2$, $[f_2, w_2]_{W_1'} = \frac{1}{s}\, w_1$, and a bilinear map $[\cdot, \cdot]_{W_1'} : W_1' \times A_1 \to W_1'$ by
$[w_1, f_1]_{W_1'} = \frac{1}{d}\, w_1$, $[w_1, f_2]_{W_1'} = -w_1$, $[w_2, f_1]_{W_1'} = -\frac{1}{d}\, w_2$, $[w_2, f_2]_{W_1'} = w_2$, where d, s are parameters.
Then (W_1', φ_1) is a two-sided A_1-module or a representation of A_1.

2. Hom-Lie structures on tensor products $\mathcal{G} \otimes A$

In this section, we aim to characterize tensor products that provide a Hom-Lie algebra structure and discuss current Hom-Lie algebras. Moreover, we give some examples of current Hom-Lie algebras and a classification of four dimensional current Hom-Lie algebras, where the Hom-Lie algebra and the Hom-associative algebra are 2-dimensional. First, we recall the following relevant result of linear algebra.

Proposition 3 ([25], Lemma 1.1)**.** *Let U, W be two vector spaces where either U and W or both U and W are finite-dimensional. Let $S, S' \in Hom(U, \cdot)$, $T, T' \in Hom(W, \cdot)$. Then*

$$Ker(S \otimes T) \cap Ker(S' \otimes T') \simeq (KerS \cap KerS') \otimes W + KerS \otimes kerT' + KerS' \cap kerT + U \otimes (KerT \cap KerT').$$

One has the following corollary, see [25].

Corollary 1. *Let $\mathcal{G}$ and A be two vector spaces such that at least one of $\mathcal{G}$ and A is finite-dimensional. Let S, S' and T, T' be linear operators defined on the spaces of n-linear maps $\mathcal{G}^n \to \mathcal{G}$ and $A^n \to A$, respectively. Let $\alpha_i : \mathcal{G}^n \to \mathcal{G}$ and $\beta_i : A^n \to A$ be n-linear maps.*
If $\sum_{i\in I} S(\alpha_i) \otimes T(\beta_i) = 0$ and $\sum_{i\in I} S'(\alpha_i) \otimes T'(\beta_i) = 0$, then the indexing set is partitioned into the four subsets $I = I_1 \cup I_2 \cup I_3 \cup I_4$ such that:

(i) $S(\alpha_i) = 0$ and $S'(\alpha_i) = 0$ for any $i \in I_1$;

(ii) $S(\alpha_i) = 0$ and $T'(\beta_i) = 0$ for any $i \in I_2$;

(iii) $S'(\alpha_i) = 0$ and $T(\beta_i) = 0$ for any $i \in I_3$;

(iv) $T(\beta_i) = 0$ and $T'(\beta_i) = 0$ for any $i \in I_4$.

Let $\mathcal{G}$ and A be two vector spaces such that at least one of $\mathcal{G}$ and A is finite-dimensional. Let $[\cdot,\cdot]_{\mathcal{G}} \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and $\mu \colon A \times A \to A$ be bilinear maps such that $[\cdot,\cdot]_{\mathcal{G}}$ is not symmetric. Define a bilinear map $[\cdot,\cdot] : \mathcal{G}\otimes A \times \mathcal{G}\otimes A \to \mathcal{G}\otimes A$ by $[x\otimes a, y\otimes b] = [x, y]_{\mathcal{G}} \otimes \mu(a, b)$. An arbitrary linear map $\psi \colon \mathcal{G}\otimes A \to \mathcal{G}\otimes A$ can be written in the form $\psi = \sum_{i\in I} \alpha_i \otimes \beta_i$, where $\alpha_i \colon \mathcal{G} \to \mathcal{G}$, $\beta_i \colon A \to A$ are (finite) families of linearly independent linear maps indexed by a set I.

Theorem 1. *With the above notations, $(\mathcal{G} \otimes A, [\cdot,\cdot], \psi)$ is a Hom-Lie algebra if and only if $[\cdot,\cdot]_{\mathcal{G}}$ is skew-symmetric, μ is symmetric and there exists a decomposition of the set of indices $I = I_1 \cup I_2 \cup I_3 \cup I_4$ such that one of the following condition is satisfied:*

(i) $[[x, z]_{\mathcal{G}}, \alpha_i(x)]_{\mathcal{G}} = 0$, *for any* $i \in I_1$;
(ii) $\beta_i(a)(bc) = 0$, *for any* $i \in I_2$;
(iii) $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \alpha_i)$ *is a Hom-Lie algebra and* (A, μ, β_i) *is a Hom-associative algebra, for any* $i \in I_3$;
(iv) $[\alpha_i(x), [y, z]_{\mathcal{G}}]_{\mathcal{G}} = [\alpha_i(y), [x, z]_{\mathcal{G}}]_{\mathcal{G}}$ *and* $\beta_i(a)(bc) + \beta_i(b)(ac) + \beta_i(c)(ab) = 0$ *for any* $i \in I_4$.

Proof. For any $x \otimes a, y \otimes b \in \mathcal{G} \otimes A$, we should have $[x \otimes a, y \otimes b] = -[y \otimes b, x \otimes a]$, that is $[x, y]_{\mathcal{G}} \otimes ab = -[y, x]_{\mathcal{G}} \otimes ba$. This implies $[x, y]_{\mathcal{G}} = -\lambda[y, x]_{\mathcal{G}}$ and $ba = \lambda ab$. Then $-[y, x]_{\mathcal{G}} \otimes ba = \lambda[x, y]_{\mathcal{G}} \otimes \lambda ab = \lambda^2[x, y]_{\mathcal{G}} \otimes ab$. Hence $\lambda^2 = 1$. Since $[\cdot, \cdot]_{\mathcal{G}}$ is not symmetric, we have $\lambda = 1$. Therefore, $[\cdot, \cdot]_{\mathcal{G}}$ is skew-symmetric and μ is symmetric.

The Hom-Jacobi identity with respect to ψ may be written

$$\sum_{i \in I} [\alpha_i(x), [y, z]_{\mathcal{G}}]_{\mathcal{G}} \otimes \beta_i(a)(bc) + [\alpha_i(y), [x, z]_{\mathcal{G}}]_{\mathcal{G}} \otimes \beta_i(b)(ca) + [\alpha_i(z), [x, y]_{\mathcal{G}}]_{\mathcal{G}} \otimes \beta_i(c)(ab) = 0. \tag{4}$$

Cyclically permuting x, y, z, in the last equality and summing up the obtained 3 equalities, we get

$$\sum_{i \in I} \Big([\alpha_i(x), [y, z]_{\mathcal{G}}]_{\mathcal{G}} + [\alpha_i(y), [z, x]_{\mathcal{G}}]_{\mathcal{G}} + [\alpha_i(z), [x, y]_{\mathcal{G}}]_{\mathcal{G}} \Big) \otimes \Big(\beta_i(a)(bc) + \beta_i(b)(ac) + \beta_i(c)(ba)\Big) = 0.$$

Skew-symmetrizing the equality (4) with respect to x, y, leads to

$$\sum_{i \in I} \Big([\alpha_i(x), [y, z]_{\mathcal{G}}]_{\mathcal{G}} + [\alpha_i(y), [z, x]_{\mathcal{G}}]_{\mathcal{G}} \Big) \otimes \Big(\beta_i(a)(bc) - \beta_i(b)(ac)\Big) = 0. \tag{5}$$

By applying Corollary 1 derived from Proposition 3 (see [25, Lemma 1.1]) to the last two equalities, we complete the proof. □

Now, we consider the subclass of Hom-Lie algebras provided by Type (*iii*) of Theorem 1, which corresponds to so called current Hom-Lie algebras.

Definition 5. A current Hom-Lie algebra is a tensor product of the form $(\mathcal{G} \otimes A, [\cdot, \cdot]_{\mathcal{G}} \otimes \mu, \alpha \otimes \beta)$, where $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \alpha)$ is a Hom-Lie algebra and (A, μ, β) is a Hom-associative commutative algebra. The current Hom-Lie algebra is denoted by $(\mathcal{G} \otimes A, [\cdot, \cdot]_{\mathcal{G} \otimes A}, \gamma)$ instead of $(\mathcal{G} \otimes A, [\cdot, \cdot]_{\mathcal{G}} \otimes \mu, \alpha \otimes \beta)$.

Example 8 (Loop Hom-Lie algebras)**.** For any Hom-Lie algebra $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \alpha)$, set $\widetilde{\mathcal{G}} = \mathcal{G} \otimes \mathbb{C}[t, t^{-1}]$, where $\mathbb{C}[t, t^{-1}]$ denote Laurent polynomials. We define a bracket $[\cdot, \cdot]$ on $\widetilde{\mathcal{G}}$ by

$$[x \otimes t^n, y \otimes t^m] = [x, y]_{\mathcal{G}} \otimes (qt)^{n+m}, \forall x, y \in \mathcal{G},\ \forall n, m \in \mathbb{Z},$$

and an endomorphism $\gamma \colon \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$ by $\gamma = \alpha \otimes \beta$ where $\beta = \beta_1$ (see Example 2). Then $(\widetilde{\mathcal{G}}, [\cdot, \cdot], \gamma)$ is a multiplicative Hom-Lie algebra, which we call a Loop Hom-Lie algebra.

Example 9 (Truncated current Hom-Lie algebras)**.** Let $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \alpha)$ be a Hom-Lie algebra over the complex field $\mathbb{C}$, and fix a positive integer p. Define an endomorphism $\beta\colon \mathbb{C}[t]/t^{p+1}\mathbb{C}[t] \to \mathbb{C}[t]/t^{p+1}\mathbb{C}[t]$ by $\beta(f)(t) = f(qt)$. The tensor product $\widehat{\mathcal{G}_p} = \mathcal{G} \otimes \mathbb{C}[t]/t^{p+1}\mathbb{C}[t]$ with the bracket

$$[x \otimes f, y \otimes g] = [x, y]_{\mathcal{G}} \otimes f(qt)g(qt), \forall x, y \in \mathcal{G},\ \forall f, g \in \mathbb{C}[t]/t^{p+1}\mathbb{C}[t]$$

and the linear map $\gamma = \alpha \otimes \beta$ is a Hom-Lie algebra, which we call Truncated current Hom-Lie algebra.

We end this section by a remark about the classification of 4-dimensional current Hom-Lie algebras.

Remark 1. Every current Hom-Lie algebra where both the Hom-Lie algebra and the Hom-associative algebra are 2-dimensional is isomorphic to one of the following non-isomorphic current Hom-Lie algebras $\mathcal{G} \otimes A = (\mathcal{G}_p \otimes A_q, [\cdot,\cdot]_{\mathcal{G}_p} \otimes \mu_q, \alpha_p \otimes \beta_q)$, where $\big(\mathcal{G}_p, [\cdot,\cdot]_{\mathcal{G}_p}, \alpha_p\big)$, $p = 1, 2$, is a Hom-Lie algebra given in Example 2 and (A_q, μ_q, β_q), $q = 1, \cdots, 7$, is a Hom-associative algebra given in Example 2.

3. Representations of current Hom-Lie algebras

Let $(\mathcal{G} \otimes A, [\cdot,\cdot]_{\mathcal{G}\otimes A}, \alpha \otimes \beta)$ be a current Hom-Lie algebra, V and W be two vector spaces, $[\cdot,\cdot]_V\colon \mathcal{G} \times V \to V$ and $\bullet\colon A \times W \to W$ be two bilinear maps. Define a bilinear map $[\cdot,\cdot]_{V\otimes W}\colon \mathcal{G} \otimes A \times V \otimes W \to V \otimes W$ by

$$[x \otimes a, v \otimes w]_{V\otimes W} = [x, v]_V \otimes a \bullet w,$$

for all $x \in \mathcal{G}, a \in A, v \in V, w \in W$.
Let $\psi = \sum_{i\in I} \alpha_{iV} \otimes \beta_{iW}$ be an endomorphism of $V \otimes W$. Assume that $(V \otimes W, [\cdot,\cdot]_{V\otimes W}, \psi)$ is a representation of the current Hom-Lie algebra $(\mathcal{G} \otimes A, [\cdot,\cdot]_{\mathcal{G}\otimes A}, \alpha \otimes \beta)$. That is, we have

$$\begin{aligned}&\sum_{i\in I} [[x, y], \alpha_{iV}(v)]_V \otimes (ab) \bullet \beta_{iW}(w) \\ &= [\alpha(x), [y, v]] \otimes \beta(a) \bullet (b \bullet w) - [\alpha(y), [x, v]] \otimes \beta(b) \bullet (a \bullet w).\end{aligned} \tag{6}$$

Skew-symmetrizing the previous equality with respect to x, y leads to

$$\big([\alpha(x), [y, v]] + [\alpha(y), [x, v]]\big) \otimes (\beta(a) \bullet (b \bullet w) - \beta(b) \bullet (a \bullet w)) = 0. \tag{7}$$

We have the following result.

Theorem 2. *The triple $(V \otimes W, [\cdot,\cdot]_{V\otimes W}, \psi_{V\otimes W})$ is a representation of a current Hom-Lie algebra $(\mathcal{G} \otimes A, [\cdot,\cdot]_{\mathcal{G}\otimes A}, \alpha \otimes \beta)$ if and only if one of the following cases holds.*

(1) *There is a subset J of I and a sequence of complex numbers $(\lambda_j)_{j\in J}$ such that*

$$\beta(a) \bullet (b \bullet w) = \beta(b) \bullet (a \bullet w) = \sum_{j\in J} \lambda_j\, (ab) \bullet \beta_{jW}(w)$$

for all $a, b \in A, w \in W$ *and*

$$[[x,y], \alpha_{jV}(v)]_V = \lambda_j \left([\alpha(x), [y,v]_V]_V - [\alpha(y), [x,v]_V]_V\right)$$

for all $j \in J$, $x, y \in \mathcal{G}, v \in V$. *Hence* $(W, \bullet, \sum_{i\in J} \lambda_i \beta_{iW})$ *is a representation of* (A, μ, β) *and* $(V, [\cdot,\cdot]_V, \frac{1}{\lambda_j}\alpha_{jV})$ *is a representation of* $\mathcal{G}$.

(2) *There is a subset* J *of* I *and a complex sequence* $(\lambda_j)_{j\in J}$ *such that*

$$[\alpha(x), [y,v]_V]_V = -\,[\alpha(y), [x,v]_V]_V = \sum_{i\in J} \lambda_i \; [[x,y], \alpha_{iV}(v)]_V$$

$$\text{and } (ab) \bullet \beta_{jW}(w) = \lambda_j \; (\beta(a) \bullet (b \bullet w) + \beta(b) \bullet (a \bullet w))$$

for all $j \in J$, $x, y \in \mathcal{G}, v \in V, w \in W$.

Proof. One uses [25, Lemma 1.1], the proof is similar to Theorem 1. □

Corollary 2. *Let* $(V, [\cdot,\cdot]_V, \alpha_V)$ *be a representation of the Hom-Lie algebra* $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \alpha)$ *and* $(W, \bullet, \beta_W)$ *be a representation of the Hom-associative algebra* (A, μ, β). *Then* $(V\otimes W, [\cdot,\cdot]_{V\otimes W}, \alpha_V \otimes \beta_W)$ *is a representation of the current Hom-Lie algebra* $(\mathcal{G}\otimes A, [\cdot,\cdot]_{\mathcal{G}\otimes A}, \alpha\otimes\beta)$.

Example 10. Let L_1 be the Hom-Lie algebra defined in Example 2 and A_1 be the Hom-associative algebra defined in Example 2. Let V_1 be the representation of L_1 given in Example 4 and let W_1 be the representation of A_1 defined in Example 6.

Define a bilinear map $[\cdot,\cdot]_{V_1\otimes W_1} \colon (L_1\otimes A_1)\times(V_1\otimes W_1) \to V_1\otimes W_1$ by

$$\begin{aligned}
&[e_1\otimes f_1, v_1\otimes w_1]_{V_1\otimes W_1} = t\, v_1\otimes w_1, && [e_1\otimes f_1, v_1\otimes w_2]_{V_1\otimes W_1} = -t\, v_1\otimes w_2,\\
&[e_1\otimes f_1, v_2\otimes w_1]_{V_1\otimes W_1} = 0, && [e_1\otimes f_1, v_2\otimes w_2]_{V_1\otimes W_1} = 0,\\
&[e_1\otimes f_2, v_1\otimes w_1]_{V_1\otimes W_1} = t\, s\, v_1\otimes w_2, && [e_1\otimes f_2, v_1\otimes w_2]_{V_1\otimes W_1} = \frac{t}{s}\, v_1\otimes w_1,\\
&[e_1\otimes f_2, v_2\otimes w_1]_{V_1\otimes W_1} = 0, && [e_1\otimes f_2, v_2\otimes w_2]_{V_1\otimes W_1} = 0,\\
&[e_2\otimes f_1, v_1\otimes w_1]_{V_1\otimes W_1} = -\frac{\lambda}{\eta} t\, v_1\otimes w_1, && [e_2\otimes f_1, v_1\otimes w_2]_{V_1\otimes W_1} = \frac{\lambda}{\eta} t\, v_1\otimes w_2,\\
&[e_2\otimes f_1, v_2\otimes w_1]_{V_1\otimes W_1} = 0, && [e_2\otimes f_1, v_2\otimes w_2]_{V_1\otimes W_1} = 0,\\
&[e_2\otimes f_2, v_1\otimes w_1]_{V_1\otimes W_1} = -\frac{\lambda}{\eta} t\, s\, v_1\otimes w_2, && [e_2\otimes f_2, v_1\otimes w_2]_{V_1\otimes W_1} = \frac{\lambda}{\eta}\frac{t}{s}\, v_1\otimes w_1,\\
&[e_2\otimes f_2, v_2\otimes w_1]_{V_1\otimes W_1} = 0, && [e_2\otimes f_2, v_2\otimes w_2]_{V_1\otimes W_1} = 0,
\end{aligned}$$

and define a bilinear map $[\cdot,\cdot]_{V_1\otimes W_1} \colon (V_1\otimes W_1)\times(L_1\otimes A_1) \to V_1\otimes W_1$ by

$$\begin{aligned}
&[v_1\otimes w_1, e_1\otimes f_1]_{V_1\otimes W_1} = \frac{1}{d} t\, v_1\otimes w_1, && [v_1\otimes w_2, e_1\otimes f_1]_{V_1\otimes W_1} = -\frac{1}{d}\, t\, v_1\otimes w_2,\\
&[v_2\otimes w_1, e_1\otimes f_1]_{V_1\otimes W_1} = 0, && [v_2\otimes w_2, e_1\otimes f_1]_{V_1\otimes W_1} = 0,\\
&[v_1\otimes w_1, e_1\otimes f_2]_{V_1\otimes W_1} = -t\, v_1\otimes w_1, && [v_1\otimes w_2, e_1\otimes f_2]_{V_1\otimes W_1} = t\, v_1\otimes w_1,\\
&[v_2\otimes w_1, e_1\otimes f_2]_{V_1\otimes W_1} = 0, && [v_2\otimes w_2, e_1\otimes f_2]_{V_1\otimes W_1} = 0,
\end{aligned}$$

$$[v_1\otimes w_1, e_2\otimes f_1]_{V_1\otimes W_1} = -\frac{\lambda}{\eta} t\, v_1\otimes w_1, \qquad [v_1\otimes w_2, e_2\otimes f_1]_{V_1\otimes W_1} = \frac{\lambda}{\eta} t\, v_1\otimes w_2,$$
$$[v_2\otimes w_1, e_2\otimes f_1]_{V_1\otimes W_1} = 0, \qquad [v_2\otimes w_2, e_2\otimes f_1]_{V_1\otimes W_1} = 0,$$
$$[v_1\otimes w_1, e_2\otimes f_2]_{V_1\otimes W_1} = \frac{\lambda}{\eta} t, v_1\otimes w_1, \qquad [v_1\otimes w_2, e_2\otimes f_2]_{V_1\otimes W_1} = -\frac{\lambda}{\eta} t\, v_1\otimes w_2,$$
$$[v_2\otimes w_1, e_2\otimes f_2]_{V_1\otimes W_1} = 0, \qquad [v_2\otimes w_2, e_2\otimes f_2]_{V_1\otimes W_1} = 0.$$

Define $\alpha_{V_1}\otimes\beta_{W_1} = \rho\otimes\varphi_1 \in End(V_1\otimes W_1)$ by

$$\alpha_{V_1}\otimes\beta_{W_1}(v_1\otimes w_1) = 0, \qquad \alpha_{V_1}\otimes\beta_{W_1}(v_1\otimes w_2) = 0,$$
$$\alpha_{V_1}\otimes\beta_{W_1}(v_2\otimes w_1) = -\eta\, v_2\otimes w_1, \qquad \alpha_{V_1}\otimes\beta_{W_1}(v_2\otimes w_2) = \eta\, v_2\otimes w_2.$$

Then, using Corollary 2, $(V_1\otimes W_1, [\cdot,\cdot]_{V_1\otimes W_1}, \alpha_{V_1}\otimes\beta_{W_1})$ is a representation of $L_1\otimes A_1$.

3.1. Semidirect Product. Given a representation $(V, [\cdot,\cdot]_V, \beta)$ of a Hom-Lie algebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$. Define a skew-symmetric bilinear bracket $[\cdot,\cdot]_{\mathcal{G}\times V}\colon (\mathcal{G}\oplus V)^2 \to \mathcal{G}\oplus V$ by $[(x,v),(y,w)] = ([x,y], [x,v]_V - [y,w]_V)$, and a linear map $\alpha+\beta\colon \mathcal{G}\oplus V \to \mathcal{G}\oplus V$ by $(\alpha+\beta)(x,v) = (\alpha(x), \beta(v))$.

Proposition 4 ([20])**.** *With the above notations, $(\mathcal{G}\oplus V, [\cdot,\cdot]_{\mathcal{G}\times V}, \alpha+\beta)$ is a Hom-Lie algebra, which we call the semidirect product of the Hom-Lie algebra $\mathcal{G}$ and V.*

One may use Example 10 to construct a semidirect product on the Hom-Lie algebra L_1 defined in Example 4 and A_1 the Hom-associative algebra defined in Example 2.

3.2. Faithful representations of current Heisenberg Hom-Lie algebras. The faithful representations of Lie algebras and superalgebras are studied in [8, 21]. In this section we extend the study of faithful representations of minimal dimension of current Heisenberg Lie algebras, see [10], to current Heisenberg Hom-Lie algebras.

Definition 6. A representation $(V, [\cdot,\cdot]_V, \beta)$ of a multiplicative Hom-Lie algebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$ is said to be faithful if β is a bijective map satisfying $\beta([x,v]_V) = [\alpha(x), \beta(v)]_V$ for all $x\in\mathcal{G}$, $v\in V$ and the map $\rho\colon \mathcal{G}\to End(V)$, defined as $\rho(x)(v) = [x,v]_V$, is injective.

Let $\mathcal{G}$ be a Hom-Lie algebra and set

$$\mu(\mathcal{G}) = \min\{\dim V \mid V \text{ is a faithful } \mathcal{G}\text{-module}\}.$$

Let $\mathfrak{h}_m$ be the $(2m+1)$-dimensional Heisenberg Lie algebra defined in Example 3 with respect to a basis $\{x_1,\cdots,x_m,y_1,\cdots,y_m,z\}$ such that the only non-zero brackets are $[x_i,y_i]' = z$ for all $i\in\{1,\cdots,m\}$ and let $\mathbb{C}[t]$ be the polynomials algebra in one variable. Let $p = \sum_{k=0}^{d-1} a_k t^k + t^d$ be a nonzero monic polynomial and let (p) be the principal ideal generated by p. Let $\mathfrak{h}_{m,p} = \mathfrak{h}_m \otimes \mathbb{C}[t]/(p)$ be the current Lie algebra associated to $\mathfrak{h}_m$ and $\mathbb{C}[t]/(p)$. Let α be an algebra isomorphism of the Heisenberg Lie algebra $(\mathfrak{h}_m, [\cdot,\cdot]')$, defined in Example 3 and let $\beta\colon \mathbb{C}[t]/(p) \to \mathbb{C}[t]/(p)$ be an isomorphism defined by $\beta(t^k) = (qt)^k$ for all $k \in \{0,\cdots,d-1\}$. Define a bracket $[\cdot,\cdot]$ on $\mathfrak{h}_m$ by

$$[x_i \otimes t^k, y_j \otimes t^l] = \delta_{ij} q^{k+l} \alpha(z) \otimes t^{k+l},$$
$$[x_i \otimes t^k, x_j \otimes t^l] = [y_i \otimes t^k, y_j \otimes t^l] = [x_i \otimes t^k, z \otimes t^l] = [y_j \otimes t^k, z \otimes t^l] = 0,$$

for all $i, j \in \{1,\cdots,m\}$; $k, l \in \{0, 1 \cdots, d-1\}$. With the above notations, $(\mathfrak{h}_{m,p}, [\cdot,\cdot], \alpha \otimes \beta)$ is a Hom-Lie algebra, which we call the current Heisenberg Hom-Lie algebra.

Proposition 5. *Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a regular multiplicative Hom-Lie algebra. Define the bilinear bracket $[\cdot,\cdot]'\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ by $[x,y]' = [\alpha^{-1}(x), \alpha^{-1}(y)]$ for all $x, y \in \mathcal{G}$. Let $(V, [\cdot,\cdot]_V, \beta)$ be a representation of the Hom-Lie algebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$. We assume that β is bijective and satisfies $\beta([x,v]_V) = [\alpha(x), \beta(v)]_V$ for all $x \in \mathcal{G}$, $v \in V$. Define the bilinear bracket $[\cdot,\cdot]'_V\colon \mathcal{G} \times V \to V$ by $[x,v]'_V = [\alpha^{-1}(x), \beta^{-1}(v)]_V$ for all $x \in \mathcal{G}$, $v \in V$. Then $(V, [\cdot,\cdot]'_V)$ is a representation of the Lie algebra $(\mathcal{G}, [\cdot,\cdot]')$.*

Proof. By Proposition 1, $(\mathcal{G}, [\cdot,\cdot]')$ is a Lie algebra. Set $x = \alpha(a)$ and $v = \beta(u)$. Then

$$\beta^{-1}([x,v]_V) = \beta^{-1}([\alpha(a), \beta(u)]_V) = \beta^{-1}\beta([a,u]_V) = [\alpha^{-1}(x), \beta^{-1}(v)]_V.$$

Hence

$$\begin{aligned}
[[x,y]', v]'_V &= [[\alpha^{-1}(x), \alpha^{-1}(y)], v]'_V \\
&= [[\alpha^{-2}(x), \alpha^{-2}(y)], \beta^{-1}(v)]_V \\
&= [[\alpha(\alpha^{-2}(x)), [\alpha^{-2}(y), \beta^{-2}(v)]]_V - [[\alpha(\alpha^{-2}(y)), [\alpha^{-2}(x), \beta^{-2}(v)]]_V \\
&= [\alpha^{-1}(x), \beta^{-1}[\alpha^{-1}(y), \beta^{-1}(v)]]_V - [\alpha^{-1}(y), \beta^{-1}[\alpha^{-1}(x), \beta^{-1}(v)]]_V \\
&= [\alpha^{-1}(x), \beta^{-1}[y,v]'_V]_V - [\alpha^{-1}(y), \beta^{-1}[x,v]'_V]_V \\
&= [x, [y,v]'_V]'_V - [y, [x,v]'_V]'_V.
\end{aligned}$$

Thus, $(V, [.,.]'_V)$ is a representation of the Lie algebra $(G, [.,.]')$. $\square$

Let $(V, [\cdot,\cdot]_V, \beta)$ be a faithful representation of the current Heisenberg Hom-Lie algebra $\mathfrak{h}_{m,p}$. Then $(V, [\cdot,\cdot]'_V)$ is a representation of Heisenberg Lie algebra $(\mathfrak{h}_{m,p}, [\cdot,\cdot]')$ (Proposition 5). So, by [10],

$$\dim V \geq m \deg p + [2\sqrt{\deg p}]. \tag{8}$$

Proposition 6. *Let $(V, [\cdot,\cdot]'_V)$ be a representation of a Lie algebra $(\mathcal{G}, [\cdot,\cdot]')$. Let α be a Lie algebra isomorphism on $\mathcal{G}$ and α_V be an endomorphism of*

V satisfying $\alpha_V([x,v]'_V) = [\alpha(x), \alpha_V(v)]'_V$ for all $x \in \mathcal{G}$, $v \in V$. Then $(V, \alpha_V \circ [\cdot,\cdot]'_V, \alpha_V)$ is a representation of the Hom-Lie algebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$.

Proof. By Proposition 1, $(\mathcal{G}, [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra.
Set $[\cdot,\cdot]_V = \alpha_V \circ [\cdot,\cdot]'_V$. Then we have

$$\alpha_V([x,v]) = \alpha_V \circ \alpha_V([x,v]'_V) = \alpha_V([\alpha(x), \alpha_V(v)]'_V) = [\alpha(x), \alpha_V(v)]_V$$

and

$$\begin{aligned}
&[[x,y], \alpha_V(v)] = [[\alpha(x), \alpha(y)]', \alpha_V(v)] = [\alpha([x,y]'), \alpha_V(v)]\\
&= \alpha_V \circ [\alpha([x,y]'), \alpha_V(v)]'_V = \alpha_V \circ \alpha_V([[x,y]', v]'_V)\\
&= \alpha_V \circ \alpha_V([x,[y,v]'_V]'_V - [y,[x,v]'_V]'_V)\\
&= \alpha_V([\alpha(x), \alpha_V([y,v]')]'_V - [\alpha(y), \alpha_V([x,v]')]'_V)\\
&= [\alpha(x), \alpha_V([y,v]')]_V - [\alpha(y), \alpha_V([x,v]')]_V\\
&= [\alpha(x), [y,v]]_V - [\alpha(y), [x,v]]_V.
\end{aligned}$$

Hence $(V, \alpha_V \circ [.,.]'_V, \alpha_V)$ is a representation ofthe Hom-Liealgebra $(\mathcal{G}, [.,.], \alpha)$. ∎

Proposition 7 ([10]). *Let a, b two integers such that $ab \geq d$ and $a + b = \lceil 2\sqrt{d} \rceil$. Here $\lceil 2\sqrt{d} \rceil$ is the closest integer that is greater than or equal to $2\sqrt{d}$. Consider matrices $P \in \mathcal{M}_{d,d}$, $A \in \mathcal{M}_{a,d}$ and $B \in \mathcal{M}_{d,b}$, where*

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & \ddots & & 0 & -a_1 \\ & \ddots & 0 & \vdots & \\ & & & 1 & -a_{d-1} \end{pmatrix};$$

$$A_{ij} = \begin{cases} 1 & \text{if } j = d-(a-i)b; \\ 0 & \text{otherwise}; \end{cases} \qquad B_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise}. \end{cases}$$

Define a map $\rho_{A,B} \colon \mathfrak{h}_{m,p} \to End(\mathbb{C}^{md+\lceil 2\sqrt{d} \rceil})$ by

$$\rho_{A,B}\left(\sum_{i=1}^{m} x_i \otimes q_{1i}(t) + \sum_{i=1}^{m} y_i \otimes q_{2i}(t) + z \otimes q_3(t)\right)$$
$$= \begin{pmatrix} 0_{aa} & Aq_{11}(P) & \dots & Aq_{1m}(P) & Aq_3(P)B \\ & & & & q_{21}(P)B \\ & & 0 & & \vdots \\ & & & & q_{2m}(P)B \\ & & & & 0_{b,b} \end{pmatrix}.$$

With the above notations, $(\rho_{A,B}, \mathbb{C}^{md+\lceil 2\sqrt{d} \rceil})$ is a faithful representation of the current Heisenberg Lie algebra $(\mathfrak{h}_{m,p}, [\cdot,\cdot]')$.

By Proposition 6 and Proposition 7, we obtain the following result.

Proposition 8. *Let* $V = \mathbb{C}^{md+\lceil 2\sqrt{d}\rceil}$ *and* α_V *be an endomorphism of* V *satisfying* $\alpha_V \circ \rho_{A,B}(u \otimes f) = \rho_{A,B}\left(\alpha \otimes \beta(u \otimes f)\right) \circ \alpha_V$.
Then, $(V, \alpha_V \circ [\cdot,\cdot]'_V, \alpha_V)$ *is a faithful representation of the current Heisenberg Hom-Lie algebra* $(\mathfrak{h}_{m,p}, [\cdot,\cdot], \alpha \otimes \beta)$.

Let $(V, [\cdot,\cdot]_V, \beta)$ be a faithful representation of the current Heisenberg Hom-Lie algebra $\mathfrak{h}_{m,p}$. Then, by the previous proposition,

$$\mu(\mathfrak{h}_{m,p}) \leq m \deg p + \lceil 2\sqrt{\deg p}\rceil. \tag{9}$$

By (8) and (9) we obtain the following result.

Theorem 3. *The equality*

$$\mu(\mathfrak{h}_{m,p}) = m \deg p + \lceil 2\sqrt{\deg p}\rceil$$

holds, where $\lceil 2\sqrt{\deg p}\rceil$ *is the closest integer that is greater than or equal to* $2\sqrt{\deg p}$.

Example 11. Let $m = 1$ and $p = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + t^5$. Then

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\alpha = \begin{pmatrix} \nu & 0 & 0 \\ 0 & \frac{\lambda}{\nu} & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, $V = \mathbb{C}^9$ and α_V be an endomorphism of V satisfying

$$\begin{aligned} \alpha_V \circ \rho_{A,B}(x_1 \otimes t^k) &= \rho_{A,B}\left(\alpha \otimes \beta(x_1 \otimes t^k)\right) = \nu q^k \rho_{A,B}(x_1 \otimes t^k) \circ \alpha_V; \\ \alpha_V \circ \rho_{A,B}(y_1 \otimes t^k) &= \frac{\lambda}{\nu} q^k \rho_{A,B}(y_1 \otimes t^k) \circ \alpha_V; \\ \alpha_V \circ \rho_{A,B}(z \otimes t^k) &= \lambda q^k \rho_{A,B}(z \otimes t^k) \circ \alpha_V. \end{aligned}$$

Using a computer algebra system, we obtain $\alpha_V = \begin{pmatrix} 0 & \cdots & 0 & x_{1,8} & x_{1,9} \\ 0 & \cdots & 0 & x_{2,8} & x_{2,9} \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$.

4. Derivations and centroids of current Hom-Lie algebras

The purpose of this section is to study γ^r-derivations and the γ^r-centroid of current Hom-Lie algebras viewed as a $(1,1,0)$-derivation and a $(1,1,1)$-derivation of current Hom-Lie algebras.
Let $(\mathcal{G} \otimes A, [\cdot,\cdot]_{\mathcal{G}\otimes A}, \gamma)$ be a current Hom-Lie algebra and V be a $\mathcal{G}$-module.

4.1. (λ',μ',γ')**-derivations of Hom-Lie algebras.** In this subsection we extend (λ',μ',γ')-derivation theory of Lie algebras introduced in [17] to Hom-Lie context.

Definition 7. Let λ',μ',γ' be elements of $\mathbb{K}$ (for example $\mathbb{K}=\mathbb{C}$). A linear map $d\colon \mathcal{G}\to V$ is a (λ',μ',γ')-α^k-derivation of $\mathcal{G}$ on V if for all $x,y\in\mathcal{G}$ we have

$$\lambda'\,d([x,y]) = -\mu'\,[\alpha^k(y),d(x)]_V + \gamma'[\alpha^k(x),d(y)]_V.$$

We denote the set of all (λ',μ',γ')-α^k-derivations by

$$Der_{\alpha^k}^{(\lambda',\mu',\gamma')}(\mathcal{G},V) = \bigoplus_{k\geq 0} Der_{\alpha^k,}^{(\lambda',\mu',\gamma')}(\mathcal{G},V).$$

In particular, with the adjoint representation $(V=\mathcal{G})$, we set

$$Der^{(\lambda',\mu',\gamma')}(\mathcal{G}) = \bigoplus_{k\geq 0} Der_{\alpha^k}^{(\lambda',\mu',\gamma')}(\mathcal{G}).$$

Proposition 9. *For any $\lambda',\mu',\gamma'\in\mathbb{K}$, there exists $\delta\in\mathbb{K}$ such that the subspace $Der^{(\lambda',\mu',\gamma')}(\mathcal{G})$ is equal to one of the following subspaces:*
(a) $Der_{\alpha^k}^{(\delta,0,0)}(\mathcal{G})$, (b) $Der_{\alpha^k}^{(\delta,1,-1)}(\mathcal{G})$, (c) $Der_{\alpha^k}^{(\delta,1,0)}(\mathcal{G})$, (d) $Der_{\alpha^k}^{(\delta,1,1)}(\mathcal{G})$.

Definition 8. The set $\Gamma(\mathcal{G}) = \bigoplus_{k\geq 0} Der_{\alpha^k}^{(1,1,0)}(\mathcal{G})$ is the centroid of $\mathcal{G}$.

Definition 9. An element $d\in Der_{\alpha^k}^{(0,1,0)}(\mathcal{G})\cap Der_{\alpha^k}^{(1,0,0)}(\mathcal{G})$ is called an α^k-central derivation. We denote the set of all α^k-central derivations by

$$C(\mathcal{G}) = \bigoplus_{k\geq 0} C_{\alpha^k}(\mathcal{G}) = \bigoplus_{k\geq 0} Der_{\alpha^k}^{(0,1,0)}(\mathcal{G})\cap Der_{\alpha^k}^{(1,0,0)}(\mathcal{G}).$$

4.2. (λ',μ',γ')**-derivations of Hom-associative algebras.** In this subsection, we extend to Hom-associative algebras the concept of (λ',μ',γ')-derivation of associative algebras introduced in [18]. Let (A,μ,β) be a Hom-associative algebra. We denote by $\mathcal{S}^1(A)$ the set of all linear maps $g: A\to A$ which are symmetric in the sense that $g(ab)=g(ba)$ for all $a,b\in A$.

Definition 10. Let λ',μ',γ' be elements of $\mathbb{K}$. A linear map $g\in\mathcal{S}^1(A)$ is a (λ',μ',γ')-β^k-derivation of A if, for all $a,b\in A$, we have

$$\lambda' g(ab) = \mu' g(a)\beta^r(b) + \gamma'\beta^r(a)g(b).$$

We denote the set of all (λ',μ',γ')-β^k-derivations of A by

$$Der^{(\lambda',\mu',\gamma')}(A) = \bigoplus_{k\geq 0} Der_{\beta^k}^{(\lambda',\mu',\gamma')}(A).$$

If β is an isomorphism, we have

$$Der_{\beta^r}^{(\lambda',1,0)}(A) = \{g\in End(A)\mid \exists u\in A;\ g(a)=u\beta^{r-1}(a)\}.$$

Proposition 10. *We have the isomorphism*

$$Der_{\beta^r}^{\lambda',1,0}(A) \cong A.$$

In the sequel we will consider multiplicative Hom-associative algebras $(\mathcal{A}, \mu, \beta)$ which are finite dimensional, unital and are the direct sum of generalized eigenspaces of β: $\mathcal{A} = \ker \beta \oplus E(1,\beta) \oplus E(\lambda_2,\beta) \oplus \cdots \oplus E(\lambda_s,\beta)$, where $E(\lambda,\beta)$ is the eigenspace associated to an eigenvalue λ of the linear map β.

4.3. $(\lambda', \mu', \gamma')$**-derivation of current-Hom-Lie algebras.** Let $\Phi : \mathcal{G} \otimes \mathcal{A} \longrightarrow \mathcal{G} \otimes \mathcal{A}$ be a $(\lambda', \mu', \gamma')$-$\gamma^r$-derivation of the current multiplicative Hom-Lie algebra $(\mathcal{G} \otimes \mathcal{A}, [\cdot,\cdot], \gamma)$. Then

$$\lambda'\Phi\left([x \otimes a, y \otimes b]\right) = \mu'\left[\Phi(x \otimes a), \gamma^r(y \otimes b)\right] + \gamma'\left[\gamma^r(x \otimes a), \Phi(y \otimes b)\right], \quad (10)$$

and Φ can be written in the form $\Phi = \sum_{i \in I} f_i \otimes g_i$ and $\gamma^r = \alpha^r \otimes \beta^r$, where I is a finite set of indices, and f_i and g_i are linear maps $f_i : \mathcal{G} \to \mathcal{G}$, $g_i : \mathcal{A} \to \mathcal{A}$, respectively. From this and (10) we obtain

$$\sum_{i \in I} \lambda' f_i([x,y]_{\mathcal{G}}) \otimes g_i(ab) - (\mu'[f_i(x), \alpha^r(y)]_{\mathcal{G}} \otimes g_i(a)\beta^r(b) \qquad (11)$$
$$+\gamma'[\alpha^r(x), f_i(y)]_{\mathcal{G}} \otimes \beta^r(a) g_i(b)) = 0.$$

Proposition 11. *We have*

$$Der_{\gamma^r}^{\delta',1,0}(\mathcal{G} \otimes \mathcal{A}) = C_{\alpha^r}(\mathcal{G}) \otimes End(\mathcal{A}) + \sum_{i=1}^{s}\sum_{j=1}^{s} Der_{\alpha^r}^{\delta'\frac{\lambda_i}{\lambda_j},1,0}(\mathcal{G}) \otimes Der_{\beta^r}^{\frac{\lambda_j}{\lambda_i},1,0}(\mathcal{A})$$
$$+ Der_{\alpha^r}^{0,1,0}(\mathcal{G}) \otimes Der_{\beta^r}^{1,0,0}(\mathcal{A}) + Der_{\alpha^r}^{1,0,0}(\mathcal{G}) \otimes Der_{\beta^r}^{0,1,0}(\mathcal{A}).$$

Proof. We have $(\lambda', \mu', \gamma') = (\delta', 1, 0)$. Let $(e_1^k, \cdots, e_{s_k}^k)$ be an ordered basis of $E(\lambda_k, \beta)$. Taking $a = e_q^k$ and $b = 1$ in (11), then using $a1 = \beta(a)$ and $g_i(a)1 = \beta(g_i(a))$, we obtain $\delta'\lambda_q f_i\left([x,y]_{\mathcal{G}}\right) = \lambda_k [f_i(x), \alpha^r(y)]_{\mathcal{G}}$. Replacing $[f_i(x), \alpha^r(y)]_{\mathcal{G}}$ by $\frac{\delta'\lambda_q}{\lambda_k} f_i\left([x,y]_{\mathcal{G}}\right)$ in (11), we obtain

$$\sum_{i \in I} f_i\left([x,y]_{\mathcal{G}}\right) \otimes \left(g_i(ab) - \frac{\lambda_q}{\lambda_k} g_i(a)\beta^r(b)\right) = 0.$$

Hence, there is a partition $I = I_1 \cup I_2 \cup I_3 \cup I_4$ such that

(a) $f_i\left([x,y]_{\mathcal{G}}\right) = [f_i(x), \alpha^r(y)]_{\mathcal{G}} = 0$ for any $i \in I_1$,
(b) $\delta'\frac{\lambda_q}{\lambda_k} f_i\left([x,y]_{\mathcal{G}}\right) = [f_i(x), \alpha^r(y)]_{\mathcal{G}}$ and $\frac{\lambda_k}{\lambda_q} g_i(ab) = g_i(a)\beta^r(b)$ for any $i \in I_2$,
(c) $f_i\left([x,y]_{\mathcal{G}}\right) = 0$ and $g_i(a)\beta^r(b) = 0$ for any $i \in I_3$,
(d) $[f_i(x), \alpha^r(y)]_{\mathcal{G}} = 0$ and $g_i(ab) = 0$ for any $i \in I_4$.

□

Proposition 12. *If β is invertible, then*

$$Der^{\delta',1,1}_{\gamma^r}(\mathcal{G}\otimes\mathcal{A})=C_{\alpha^r}(\mathcal{G})\otimes End(\mathcal{A})+\Big(Der^{1,0,0}_{\alpha^r}(\mathcal{G})\cap Der^{0,1,1}_{\alpha^r}(\mathcal{G})\Big)\otimes Der^{0,1,-1}_{\beta^r}(\mathcal{A})$$
$$+\sum_{i=1}^{s} Der^{\frac{\delta'}{2\lambda_i},1,0}_{\alpha^r}(\mathcal{G})\otimes Der^{2\lambda_i,1,1}_{\beta^r}(\mathcal{A})+\sum_{1\le i,j\le s} Der^{\delta'\frac{\lambda_i}{\lambda_j},1,1}_{\alpha^r}(\mathcal{G})\otimes Der^{\frac{\lambda_j}{\lambda_i},1,0}_{\beta^r}(\mathcal{A})$$
$$+\sum_{1\le i,j\le s} Der^{\delta'\frac{\lambda_i}{\lambda_j},1,0}_{\alpha^r}(\mathcal{G})\otimes Der^{\frac{\lambda_j}{\lambda_i},1,1}_{\beta^r}(\mathcal{A}).$$

Proof. Suppose $(\lambda',\mu',\gamma')=(\delta',1,1)$. Skew-symmetrizing the equality (11) with respect to x, y, we get

$$\sum_{i\in I}\left([f_i(x),\alpha^r(y)]_{\mathcal{G}}-[\alpha^r(x),f_i(y)]_{\mathcal{G}}\right)\otimes\left(g_i(a)\beta^r(b)-\beta^r(a)g_i(b)\right)=0.$$

Hence, the index set can be partitioned as $I=I_1\cup I_2$ in such a way that $[f_i(x),\alpha^r(y)]_{\mathcal{G}}=[\alpha^r(x),f_i(y)]_{\mathcal{G}}$ for any $i\in I_1$, and $g_i(a)\beta^r(b)-\beta^r(a)g_i(b)=0$ for any $i\in I_2$. Then (11) can be rewritten as

$$\sum_{i\in I_1}\delta' f_i\left([x,y]_{\mathcal{G}}\right)\otimes g_i(ab)-[f_i(x),\alpha^r(y)]_{\mathcal{G}}\otimes\left(g_i(a)\beta^r(b)+\beta^r(a)g_i(b)\right)=0 \tag{12}$$

and

$$\sum_{i\in I_1}\delta' f_i\left([x,y]_{\mathcal{G}}\right)\otimes g_i(ab)-\left([f_i(x),\alpha^r(y)]_{\mathcal{G}}+[\alpha^r(x),f_i(y)]_{\mathcal{G}}\right)\otimes g_i(a)\beta^r(b)=0. \tag{13}$$

Let $\{e^k_1,\cdots,e^k_{s_k}\}$ be an ordered basis of $E(\lambda_k,\beta)$ and $\beta(g_i(1))=\lambda_k g_i(1)$. Denote by $I_{11}=\{i\in I_1\mid g_i(1)\neq 0\}$ and $I_{12}=\{i\in I_1\mid g_i(1)=0\}$.

Taking $a=b=1$ in (12), then using $\beta(1)=1$ and $1g_i(1)=\beta(g_i(1))=\lambda_k g_i(1)$, we obtain $\delta' f_i\left([x,y]_{\mathcal{G}}\right)=2\lambda_k[f_i(x),\alpha^r(y)]_{\mathcal{G}}$. Plugging this in (12), we get $\sum_{i\in I_{11}}[f_i(x),\alpha^r(y)]_{\mathcal{G}}\otimes(2\lambda_k g_i(ab)-g_i(a)\beta^r(b)-\beta^r(a)g_i(b))=0$. Hence, there is a partition $I_{11}=J_{11}\cup J_{12}$ such that

$$f_i\left([x,y]_{\mathcal{G}}\right)=[f_i(x),\alpha^r(y)]_{\mathcal{G}}=0 \qquad \text{for any } i\in J_{11},$$

and

$$\delta' f_i\left([x,y]_{\mathcal{G}}\right)=2\lambda_k[f_i(x),\alpha^r(y)]_{\mathcal{G}},\ 2\lambda_k g_i(ab)=g_i(a)\beta^r(b)+\beta^r(a)g_i(b)\ \forall i\in J_{12}.$$

Taking $a=e^k_j$ and $b=1$ in (12), then using $\beta(1)=1$ and $1g_i(a)=\beta(g_i(a))=\lambda_j g_i(a)$, we obtain $\delta'\lambda_k f_i\left([x,y]_{\mathcal{G}}\right)=\lambda_j[f_i(x),\alpha^r(y)]_{\mathcal{G}}$. Plugging this in (12), we get

$$\sum_{i\in I_{12}}[f_i(x),\alpha^r(y)]_{\mathcal{G}}\otimes\left(\frac{\lambda_j}{\lambda_k}g_i(ab)-g_i(a)\beta^r(b)-\beta^r(a)g_i(b)\right)=0.$$

Hence, there is a partition $I_{12} = J_{21} \cup J_{22}$ such that

$$f_i\left([x,y]_{\mathcal{G}}\right) = [f_i(x), \alpha^r(y)]_{\mathcal{G}} = 0 \qquad \text{for any } i \in J_{21},$$

and

$$\delta' \lambda_k f_i\left([x,y]_{\mathcal{G}}\right) = \lambda_j [f_i(x), \alpha^r(y)]_{\mathcal{G}}, \ \frac{\lambda_j}{\lambda_k} g_i(ab) = g_i(a)\beta^r(b) + \beta^r(a) g_i(b) \ \forall i \in J_{22}.$$

Taking $a = e_j^k$ and $b = 1$ in (13), we obtain $\delta' \lambda_k f_i\left([x,y]_{\mathcal{G}}\right) = \lambda_j \left([f_i(x), \alpha^r(y)]_{\mathcal{G}} + [\alpha^r(x), f_i(y)]_{\mathcal{G}}\right)$. Plugging this in (13), we get $\sum_{i \in I_2} f_i\left([x,y]_{\mathcal{G}}\right) \otimes \left(\frac{\lambda_j}{\lambda_k} g_i(ab) - g_i(a)\beta^r(b)\right) = 0$. Hence we may assume that the indexing set is partitioned into two subsets $I_2 = I_{21} \cup I_{22}$ such that $f_i\left([x,y]_{\mathcal{G}}\right) = [f_i(x), \alpha^r(y)]_{\mathcal{G}} + [\alpha^r(x), f_i(y)]_{\mathcal{G}} = 0$, for all $i \in I_{21}$, and for all $i \in I_{22}$ we have $\delta' \frac{\lambda_k}{\lambda_j} f_i\left([x,y]_{\mathcal{G}}\right) = [f_i(x), \alpha^r(y)]_{\mathcal{G}} + [\alpha^r(x), f_i(y)]_{\mathcal{G}} = 0$, and $\frac{\lambda_j}{\lambda_k} g_i(ab) = g_i(a)\beta^r(b)$.

□

4.4. Centroids of current Hom-Lie algebras. Using Proposition 11 and the fact that β is an isomorphism, we get the following result.

Proposition 13. *One has*

$$\Gamma_{\gamma^r}(\mathcal{G} \otimes \mathcal{A}) = C_{\alpha^r}(\mathcal{G}) \otimes End(\mathcal{A}) + \sum_{i=1}^{s} \sum_{j=1}^{s} Der_{\alpha^r}^{\frac{\lambda_i}{\lambda_j},1,0}(\mathcal{G}) \otimes Der_{\beta^r}^{\frac{\lambda_j}{\lambda_i},1,0}(\mathcal{A}).$$

Corollary 3. *Suppose $\mathcal{G}$ is finite dimensional, simple and $\beta = id_{\mathcal{A}}$. Then*

$$\Gamma_{\gamma^0}(\mathcal{G} \otimes \mathcal{A}) \cap C^1_{\gamma}(\mathcal{G} \otimes \mathcal{A}) \cong \mathcal{A}.$$

Theorem 4. *If $\mathcal{G}$ is a perfect Hom-Lie algebra, then*

$$\Gamma_{\gamma^r}(\mathcal{G} \otimes \mathcal{A}) = \sum_{i=1}^{s} \sum_{j=1}^{s} Der_{\alpha^r}^{\frac{\lambda_i}{\lambda_j},1,0}(\mathcal{G}) \otimes Der_{\beta^r}^{\frac{\lambda_j}{\lambda_i},1,0}(\mathcal{A}).$$

Theorem 5. *Suppose $\mathcal{G}$ is finite dimensional and perfect. Then*

$$\Gamma_{\gamma^r}(\mathcal{G} \otimes \mathbb{C}[t]) \cong \Gamma_{\alpha^r}(\mathcal{G}) \otimes \mathbb{C}[t].$$

4.5. Derivations of current Hom-Lie algebras. Letting $\delta' = 1$ in Proposition 12, we obtain the following result.

Theorem 6. *Any derivation in $Der_{\gamma^r}(\mathcal{G} \otimes \mathcal{A})$ is a linear combination of γ^r-derivations $f \otimes g$ of the five following types:*

(*i*) $f([x,y]_{\mathcal{G}}) = [f(x), \alpha^r(y)]_{\mathcal{G}} = 0$;

(*ii*) $f([x,y]_{\mathcal{G}}) = \frac{1}{2\lambda_i}[f(x), \alpha^r(y)]_{\mathcal{G}}, \qquad 2\lambda_i g(ab) = g(a)\beta^r(b) + \beta^r(a)g(b)$;

(*iii*) $f([x,y]_{\mathcal{G}}) = 0, \ [f(x), \alpha^r(y)]_{\mathcal{G}} + [\alpha^r(x), f(y)]_{\mathcal{G}} = 0, \ g(a)\beta^r(b) = \beta^r(a)g(b)$;

(*iv*) $\frac{\lambda_i}{\lambda_j} f([x,y]_{\mathcal{G}}) = [f(x), \alpha^r(y)]_{\mathcal{G}} + [\alpha^r(x), f(y)]_{\mathcal{G}}, \quad \frac{\lambda_j}{\lambda_i} g(ab) = g(a)\beta^r(b)$, $\forall i, j \in \{1, \cdots, s\}$;

(v) $\frac{\lambda_j}{\lambda_i} f([x,y]_{\mathcal{G}}) = [f(x), \alpha^r(y)]_{\mathcal{G}}, \quad \frac{\lambda_i}{\lambda_j} g(ab) = g(a)\beta^r(b) + \beta^r(a)g(b)\ \forall i,j \in \{1,\cdots,s\}$;

for all $x, y \in \mathcal{G},\ a, b \in \mathcal{A}$.

4.6. Derivations of current Hom-Lie algebras of small dimensions. In the following, we describe the γ^r-derivations of four-dimensional complex current Hom-Lie algebras corresponding to the classification provided in Remark 1. Let $\{e_1, e_2\}$ be a basis of L_i and $\{f_1, f_2\}$ be a basis of A_j, for $i \in \{1,2\}$ and $j \in \{1,\ldots,7\}$. We consider the following basis for $L_i \otimes A_j$, with $i \in \{1,2\}$ and $j \in \{1,\ldots,7\}$, $\{u_1 = e_1 \otimes f_1,\ u_2 = e_1 \otimes f_2,\ u_3 = e_2 \otimes f_1,\ u_4 = e_2 \otimes f_2\}$. In the following table, we set $D_{i,j} := \dim(Der_{\gamma^r_{ij}}(L_i \otimes A_j))$.

$L_1 \otimes A_j$	$Der_{\gamma^r_{ij}}(L_i \otimes A_j)$	$D_{i,j}$
$L_1 \otimes A_1$	$d(u_1) = d(u_2) = 0,\ d(u_3) = d_{44}(-1)^{r+1}\left(\frac{\lambda}{\mu}u_1 + u_3\right)$ $d(u_4) = d_{44}\left(\frac{\lambda}{\mu}u_2 + u_4\right)$	1
$L_1 \otimes A_2$	$d(u_1) = d(u_2) = 0,\ d(u_3) = d_{13}\left(u_1 + \frac{\mu}{\lambda}u_3\right),\ d(u_4) = d_{24}u_2$	2
$L_1 \otimes A_3$	$d(u_1) = d_{11}\left(u_1 - (1-\mu^r)u_2\right),\ d(u_2) = 0$ $d(u_3) = d_{13}u_1 + \frac{\lambda}{\mu}(1-\mu^r)d_{11}u_2 + (d_{11} + \frac{\mu}{\lambda}d_{13})u_3$ $d(u_4) = d_{24}u_2$	3
$L_1 \otimes A_4$	$d(u_1) = d(u_2) = 0,\ d(u_3) = d_{13}u_1 + d_{23}u_2 + d_{13}\frac{\mu}{\lambda}u_3 + d_{23}\frac{\mu}{\lambda}u_4$ $d(u_4) = d_{14}u_1 + d_{24}u_2 + d_{14}\frac{\mu}{\lambda}u_3 + d_{24}\frac{\mu}{\lambda}u_4$	4
$L_1 \otimes A_5$	$d(u_1) = d_{11}u_1 + d_{21}u_2 - (d_{11} + d_{21})u_3$ $d(u_2) = d_{12}u_1 + d_{22}u_2 + d_{32}u_3$ $d(u_3) = d_{13}u_1 + d_{23}u_2 + d_{33}u_3$ $d(u_4) = -\frac{\lambda}{\mu}d_{12}u_1 + d_{24}u_2 - \frac{\lambda}{\mu}d_{32}u_3 + \left(d_{22} + \frac{\mu}{\lambda}d_{24}\right)u_4$	9
$L_1 \otimes A_6$	$d(u_1) = d_{11}u_1 + d_{21}u_2,\ d(u_2) = \mu^r d_{11}u_2$ $d(u_3) = d_{13}u_1 + d_{23}u_2 + \left(d_{11} + \frac{\mu}{\lambda}d_{13}\right)u_3 + + \left(d_{21} + \frac{\mu}{\lambda}d_{23}\right)u_4$ $d(u_4) = d_{14}u_1 + d_{24}u_2 + \frac{\mu}{\lambda}d_{14}u_3 + \left(\mu^r d_{11} + \frac{\mu}{\lambda}d_{24}\right)u_4$	6
$r \in \{0,1\}$ $L_1 \otimes A_7$	$d(u_1) = d_{11}(u_1 - u_3),\ d(u_2) = d_{12}u_1 + d_{32}u_3$ $d(u_3) = \frac{\lambda}{\mu}(d_{44} - d_{11})u_1 + \left(\frac{\lambda}{\mu}d_{11} + d_{44}\right)u_3$ $d(u_4) = d_{14}u_1 + d_{24}u_2 + d_{34}u_3 + d_{44}u_4$	8
$L_2 \otimes A_1$ $(\lambda = 1)$	$d(u_1) = d(u_2) = 0,\ d(u_3) = (-1)^r d_{24}u_1$ $d(u_4) = d_{24}u_2$	1
$L_2 \otimes A_2$ $(\lambda = 1)$	$d(u_1) = d(u_2) = 0$ $d(u_3) = d_{13}u_1,\ d(u_4) = d_{24}u_2$	2
$L_2 \otimes A_3$	$d(u_1) = \frac{\lambda-1}{\mu}d_{13}u_1,\ d(u_2) = 0,\ d(u_3) = d_{13}u_1,\ d(u_4) = d_{24}u_2$	2
$L_2 \otimes A_4$	$d(u_1) = \frac{\lambda-1}{\mu}d_{13}u_1,\ d(u_2) = \frac{\lambda-1}{\lambda^r-1}\frac{1}{\mu}(\lambda^r d_{24} - d_{13})u_2$ $d(u_3) = d_{13}u_1,\ d(u_4) = d_{24}u_2 + \frac{\lambda-1}{\lambda^r-1}\frac{1}{\mu}(d_{24} - d_{13})u_4$	4
$L_2 \otimes A_5$	$d(u_1) = d_{11}u_1,\ d(u_2) = d_{22}u_2,\ d(u_3) = d_{33}u_3$ $d(u_4) = d_{24}u_2 + \left(d_{22} - \frac{\lambda-1}{\mu}d_{24}\right)u_4$	4

$L_2 \otimes A_6$	$d(u_1) = d_{11}u_1 + d_{21}u_2$ $d(u_2) = \Big((\lambda^r + 1)d_{11} - \lambda^r(\frac{\lambda-1}{\mu})d_{13}\Big)\, u_2$ $d(u_3) = d_{13}u_1 + d_{23}u_2 + \Big(d_{11} + \frac{1-\lambda}{\mu}d_{13}\Big)\, u_3 + \Big(d_{21} + \frac{1-\lambda}{\mu}d_{23}\Big)\, u_4$ $d(u_4) = d_{24}u_2 + \Big((\lambda^r + 1)d_{11} + \frac{1-\lambda}{\mu}\lambda^r d_{13} + \frac{1-\lambda}{\mu}\lambda^r d_{24}\Big)\, u_4$	5
$L_2 \otimes A_7$	$d(u_1) = d_{11}\,(u_1 - u_3)$ $d(u_2) = d_{12}u_1 + (1+\mu)d_{11}u_2 + d_{32}u_3 - \lambda d_{11}u_4$ $d(u_3) = d_{13}u_1 - \frac{1+\mu}{\lambda}d_{11}u_3$ $d(u_4) = d_{14}u_1 + \Big(\frac{(1+\lambda)(1+\mu)\mu}{\lambda^2}d_{11} + \frac{d_{13}}{\lambda}\Big)\, u_2 + d_{34}u_3 - \frac{1+\mu+\mu\lambda}{\lambda}d_{11}u_4$	6

4.7. Extensions by derivations. Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra and d be a derivation of this Hom-Lie algebra. Define a skew-symmetric bilinear map $[\cdot,\cdot]_d$ on the direct sum $\mathcal{G} \oplus \mathbb{C}d$ by $[x + \lambda' d, y + \mu' d] = [x, y]_{\mathcal{G}} + \lambda' d(y) - \mu' d(x)$.
Define $\alpha_d \in End(\mathcal{G} \oplus \mathbb{C}d)$ by $\alpha_d(x + \lambda' d) = \alpha(x) + \lambda' d$.

Theorem 7 ([20]). *With the above notations, $(\mathcal{G} \oplus \mathbb{C}d, [\cdot,\cdot]_d, \alpha_d)$ is a Hom-Lie algebra.*

Example 12. Define a linear map $d\colon L(\mathcal{G}) \to L(\mathcal{G})$ by $d(x \otimes t^n) = nx \otimes t^n$. Then $\big(\big(\mathcal{G} \otimes \mathbb{C}[t, t^{-1}]\big) \oplus (\mathbb{C}\alpha \otimes t\frac{d}{dt}\cdot), [\cdot,\cdot]_d, \gamma_d\big)$ is a Hom-Lie algebra.

5. Scalar second cohomology group

The general Chevalley–Eilenberg cohomology theory of Hom-Lie algebras was initiated in [13] and established in [2, 20]. We deal here only with scalar cohomology. A scalar k-cochain is an alternating k-linear map from $(\mathcal{G} \otimes A)^k$ to $\mathbb{C}$. The vector space of scalar k-cochains is denoted by $C^k(\mathcal{G} \otimes A, \mathbb{C})$ and by definition $C^0(\mathcal{G} \otimes A, \mathbb{C}) = \mathbb{C}$. In this section, we study the second cohomology group of current Hom-Lie algebras with coefficients in a trivial representation.
The coboundary operator $\delta^k\colon C^k(\mathcal{G} \otimes A, \mathbb{C}) \to C^{k+1}(\mathcal{G} \otimes A, \mathbb{C})$ is given by

$$\begin{aligned}
&\delta^k(f \otimes g)(x_0 \otimes a_0, \ldots, x_k \otimes a_k)\\
&= \sum_{0 \le s < t \le k} (-1)^t \times f\left(\alpha(x_0), \cdots, \alpha(x_{s-1}), [x_s, x_t], \alpha(x_{s+1}), \cdots, \widehat{x_t}, \cdots, \alpha(x_k)\right)\\
&\qquad\qquad \otimes g\left(\beta(a_0), \cdots, \beta(a_{s-1}), a_s a_t, \beta(a_{s+1}), \cdots, \widehat{a_t}, \cdots, \beta(a_k)\right).
\end{aligned}$$

Denote by $Z^k(\mathcal{G} \otimes A)$ and $B^k(\mathcal{G} \otimes A)$ the corresponding space of k-cocycles and k-coboundaries, respectively. We denote the resulting cohomology by $H^k(\mathcal{G} \otimes A)$.

In the following, we give a result similar to [26, Theorem 1.1], in the case of current Hom-Lie algebras.

Theorem 8. *Let $\mathcal{G} \otimes A$ be a current Hom-Lie algebra such that either $\mathcal{G}$ or A is finite dimensional. We denote by $\mathcal{S}^2(A)$ the set of all bilinear maps $g : A \times A \to A$ which are symmetric in the sense that $g(a, b) = g(b, a)$ for all $a, b \in A$ and $\mathcal{C}^2(A)$ the set of all bilinear maps $g : A \times A \to A$ which are skew-symmetric in the sense that $g(a, b) = -g(b, a)$ for all $a, b \in A$.*
Then each cocycle in $Z^2(\mathcal{G} \otimes A)$ is a linear combination of cocycles of the 8 following types:

(1) $f([x,z],\alpha(y))=0,\ f\in C^2(\mathcal{G})$ *and* $g\in S^2(A)$;
(2) $f\in Z^2(\mathcal{G},\mathbb{C}),\ g(ac,\beta(b))=g(bc,\beta(a))$ *and* $g\in S^2(A)$;
(3) $f([x,z],\alpha(y))=f(\alpha(x),[y,z]),\ g(ab,\beta(c))+g(ac,\beta(b))+g(\beta(a),bc)=0$ *and* $f\otimes g\in C^2(\mathcal{G})\otimes S^2(A)$;
(4) $g(ab,\beta(c))=0$ *and* $f\otimes g\in C^2(\mathcal{G})\otimes S^2(A)$;
(5) $f([x,z],\alpha(y))=0$ *and* $f\otimes g\in S^2(\mathcal{G})\otimes C^2(A)$;
(6) $-f([x,y],\alpha(z))+f([x,z],\alpha(y))-f(\alpha(x),[y,z])=0$, $g(ab,\beta(c))=g(bc,\beta(a))$ *and* $f\otimes g\in S^2(\mathcal{G})\otimes C^2(A)$;
(7) $f([x,z],\alpha(y))+f(\alpha(x),[y,z])=0$, $g(ab,\beta(c))+g(ac,\beta(b))+g(bc,\beta(a))=0$ *and* $f\otimes g\in S^2(\mathcal{G})\otimes C^2(A)$;
(8) $g(ab,\beta(c))=0$ *and* $f\otimes g\in S^2(\mathcal{G})\otimes C^2(A)$.

Now, we will describe the second cohomology group of Loop Hom-Lie algebra $\tilde{L}(\mathcal{G})$, where the Hom-Lie algebra $\mathcal{G}$ is multiplicative simple (for the definition of Loop Hom-Lie algebra, see Example 8).
First we give a relationship between simple multiplicative Hom-Lie algebras and Lie algebras, as well as some relevant properties.

Lemma 1 ([5])**.** *Define the bracket* $[\cdot,\cdot]'\colon \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ *by* $[x,y]'=[\alpha^{-1}(x),\alpha^{-1}(y)]$ *for all* $x,\,y\in\mathcal{G}$. *The induced Lie algebra* $(\mathcal{G},[x,y]')$ *of the multiplicative simple Hom-Lie algebra* $(\mathcal{G},[\cdot,\cdot],\alpha)$ *is semisimple and can be decomposed into a direct sum of isomorphic simple ideals:* $\mathcal{G}=\mathcal{G}_1\oplus\alpha(\mathcal{G}_1)\oplus\cdots\oplus\alpha^r(\mathcal{G}_1)$.

Lemma 2. *For all* $i,\,j\in\{0,\cdots,r\}$, *the ideals* $\alpha^i(\mathcal{G}_1)$ *and* $\alpha^j(\mathcal{G}_1)$ *of the Lie algebra* $(\mathcal{G},[\cdot,\cdot]')$ *are isomorphic.*

The previous lemmas lead us to see the Lie case.

Lemma 3 ([6])**.** *A finite-dimensional simple Lie algebra* $\mathcal{G}$ *has only trivial* 2-*cocycle.*

Lemma 4 ([6])**.** *Every symmetric associative bilinear form on a simple Lie algebra is proportional to the Cartan-Killing form:* $K(x,y)=tr\,(ad_x\circ ad_y)$, *for all* $x,\,y\in\mathcal{G}$.

Lemma 5. *Every* 2-*cocycle on the induced Lie algebra* $(\mathcal{G},[\cdot,\cdot]')$ *is a linear combination of the* 2-*cocycles* Φ_i *given by* $\Phi_i(x,y)=\begin{cases}\Phi(x,y), & \text{if } x,\,y\in\alpha^i(\mathcal{G}_1)\\ 0, & \text{otherwise.}\end{cases}$

Lemma 6. *A skew-symmetric bilinear map* Φ *is a* 2-*cocycle on the multiplicative simple Hom-Lie algebra* $(\mathcal{G},[\cdot,\cdot],\alpha)$ *if and only if it is a* 2-*cocycle on the induced Lie algebra* $(\mathcal{G},[\cdot,\cdot]')$.

Now, we state the main result of this section.

Theorem 9. *Let* $(\mathcal{G},[\cdot,\cdot],\alpha)$ *be a finite-dimensional simple Hom-Lie algebra. Then the space* $H^2(\tilde{L}(\mathcal{G}))$ *is generated by the maps* $\Phi_i\colon\tilde{L}(\mathcal{G})\times\tilde{L}(\mathcal{G})\to\mathbb{C}$ *defined by:*

$$\Phi_i(x\otimes t^n,y\otimes t^m)=\begin{cases} n\ \delta_{n+m,0}\ K(x,y), & \text{if } x,y\in\alpha^i(\mathcal{G}_1)\\ 0, & \text{otherwise.}\end{cases}$$

Hence, $\dim H^2(\tilde{L}(\mathcal{G}))=r+1$.

Proof. Let $\Phi = f \otimes g$ be a 2-cocycle on the simple Hom-Lie algebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$.
Φ is a 2-cocycle of type 1: Let $x, y \in \mathcal{G}$. Since $\mathcal{G}$ is simple and α is an isomorphisme, we can write $x = [a, b]$ and $y = \alpha(c)$. Then $f(x, y) = f([a, b], \alpha(c)) = 0$.
Φ is a 2-cocycle of type 2: By Lemma 3, Lemma 2 and Lemma 5, we obtain that the 2-cocycle f is trivial.
Taking $a = t^n$, $b = t^m$, $c = t^s$, we get

$$\begin{aligned} g(ab, \beta(c)) = g(bc, \beta(a)) &\implies g((qt)^n(qt)^m, (qt)^s) = g((qt)^m(qt)^s, (qt)^n) \\ &\implies g(t^{n+m}, t^s) = g(t^{m+s}, t^n). \end{aligned} \tag{14}$$

Taking $m + s = 0$ in (14), we get $g(t^{n+m}, t^{-m}) = g(1, t^n)$. Let $h(t^n) = q^{-n}g(1, t^n)$.
Then, $g(t^n, t^m) = g(1, t^{n+m}) = \varrho h(t^{n+m})$, $\varrho = q^{-(n+m)}$. Thus g is trivial.
Since f and g are trivial, one can deduce that $\Phi = f \otimes g$ is trivial.
Φ is a 2-cocycle of type 3: We have $g(ab, \beta(c)) + g(ac, \beta(b)) + g(\beta(a), bc) = 0$. Then

$$g(t^{n+m}, t^s) + g(t^{n+s}, t^m) + g(t^n, t^{m+s}) = 0. \tag{15}$$

Taking $s = 0$ in (15), we obtain $g(t^n, t^m) = \frac{1}{2}g\left(t^{n+m}, 1\right)$. Then, using (15) and that g is symmetric, one can deduce $g\left(t^{n+m+s}, 1\right) = 0$. Thus $g = 0$.
Φ is a 2-cocycle of type 4: By $g(ab, \beta(c)) = 0$, we obtain $g = 0$.
Φ is a 2-cocycle of type 5: Similarly to type 1, we obtain $f = 0$.
Φ is a 2-cocycle of type 6: We have $g(ab, \beta(c)) = g(bc, \beta(a))$. Then

$$g(t^{n+m}, t^s) = g(t^{m+s}, t^n). \tag{16}$$

Taking $m = 0$ in (16), we obtain $g(t^n, t^s) = g(t^s, t^n)$. Since g is skew-symmetric, one can deduce $g = 0$.
Φ is a 2-cocycle of type 7: Let $x' = \alpha(x)$, $y' = \alpha(y)$ and $z' = \alpha(z)$. We have

$$\begin{aligned} f\left(x', [y', z']'\right) &= f\left(\alpha(x), [\alpha^{-1}(y'), \alpha^{-1}(z')]\right) = f\left(\alpha(x), [\alpha^{-1}(y'), \alpha^{-1}(z')]\right) \\ &= f\left([x, \alpha^{-1}(y')], z'\right) = f\left([\alpha^{-1}(x'), \alpha^{-1}(y')], z'\right) = f\left([x', y']', z'\right). \end{aligned}$$

Then the symmetric bilinear form f is associative in the induced Lie algebra $(\mathcal{G}, [\cdot,\cdot]')$.
Define a symmetric associative bilinear form f_i by

$$f_i(x, y) = \begin{cases} f(x, y), & \text{if } x,\, y \in \alpha^i(\mathcal{G}_1) \\ 0, & \text{otherwise.} \end{cases}$$

Since $\alpha^i(\mathcal{G}_1)$ is a simple ideal of the Lie algebra $(\mathcal{G}, [\cdot,\cdot]')$, by Lemma 4, one can deduce $f_i(x, y) = \lambda_i K(x, y)$ for all $x, y \in \alpha^i(\mathcal{G}_1)$ and $f(x, y) = \sum_{i=0}^{m} \lambda_i f_i(x, y)$.
By $g(ab, \beta(c)) + g(ac, \beta(b)) + g(bc, \beta(a)) = 0$, we obtain

$$g(t^{n+m}, t^s) + g(t^{n+s}, t^m) + g(t^{m+s}, t^n) = 0. \tag{17}$$

Take $s = 0$ in (17). Since g is skew-symmetric, we obtain $g(t^{n+m}, 1) = 0$.
Take $n + s = 0$ in (17). Using $g(t^m, 1) = 0$ and that g is skew-symmetric, we obtain $g(t^{n+m}, t^{-n}) = g(t^n, t^{m-n})$. Fix $k \in \mathbb{Z}$ and let $n + m + s = k$. Then

$$g(t^{n+m}, t^{k-n-m}) + g(t^{k-m}, t^m) + g(t^{k-n}, t^n) = 0.$$

Therefore, $g(t^{n+m+k},t^{-n-m}) = g(t^{m+k},t^{-m}) + g(t^{n+k},t^{-n})$.
Let $U_n = g(t^{n+k},t^{-n})$. Then $U_{n+m} = U_n + U_m$. Hence, $U_m = m\ U_1$. Therefore, $g(t^{m+k},t^{-m}) = m\ U_1$. Thus $g(t^m,t^n) = \delta_{n+m,k}\ m\ U_1^{(k)}$ and $g(t^n,t^m) = \delta_{n+m,k}\ n\ U_1^{(k)} = -\delta_{n+m,k}\ (m-k)\ U_1^{(k)}$. Since g is skew-symmetric, one can deduce $k=0$ or $U_1^{(k)}=0$, which gives $g(t^n,t^m) = \delta_{n+m,0}\ n\ U_1^{(0)}$.
Φ is a 2-cocycle of type 8: By $g(ab,\beta(c)) = 0$, one can deduce $g=0$. □

Example 13. The induced Lie algebra of $(sl_2(\mathbb{C}),[\cdot,\cdot],\alpha)$ (see Example 1) is given by $[x_1,x_2]' = -\frac{2}{a^2}x_2$, $[x_1,x_3]' = [\alpha^{-1}(x_1),\alpha^{-1}(x_3)] = \frac{2}{a^2}x_3$, $[x_2,x_3]' = -\frac{1+a}{2a^3}x_1$. By $\dim(sl_2(\mathbb{C})) = (r+1)\dim(\mathcal{G}_1)$ and $[\alpha^i(\mathcal{G}_1),\alpha^j(\mathcal{G}_1)]' = \delta_{i,j}[\alpha^i(\mathcal{G}_1),\alpha^i(\mathcal{G}_1)]$, we obtain $r=0$. Hence $\dim H^2(\tilde{L}(sl_2(\mathbb{C}))) = 1$ and each non trivial 2-cocycle of $\tilde{L}(sl_2(\mathbb{C}))$ $(a \notin \{-1,0\})$ is proportional to the linear map $\Phi\colon \tilde{L}(sl_2(\mathbb{C})) \times \tilde{L}(sl_2(\mathbb{C})) \to \mathbb{C}$ defined by $\Phi(x\otimes t^n, y\otimes t^m) = \delta_{m+n,0}\ n\ K(x,y)$. Furthermore,

$$\begin{aligned}
\Phi(x_1\otimes t^n, x_1\otimes t^m) &= \delta_{m+n,0}\ n\ K(x_1,x_1) = \delta_{m+n,0}\ n\ tr\,(ad(x_1)\circ ad(x_1))\\
&= \delta_{m+n,0}\ n\ \frac{8}{a^4};\\
\Phi(x_1\otimes t^n, x_2\otimes t^m) &= \delta_{m+n,0}\ n\ K(x_1,x_2) = \delta_{m+n,0}\ n\ tr\,(ad(x_1)\circ ad(x_2)) = 0;\\
\Phi(x_1\otimes t^n, x_3\otimes t^m) &= \delta_{m+n,0}\ n\ K(x_1,x_3) = \delta_{m+n,0}\ n\ tr\,(ad(x_1)\circ ad(x_3)) = 0;\\
\Phi(x_2\otimes t^n, x_2\otimes t^m) &= \delta_{m+n,0}\ n\ K(x_2,x_2) = \delta_{m+n,0}\ n\ tr\,(ad(x_2)\circ ad(x_2)) = 0;\\
\Phi(x_2\otimes t^n, x_3\otimes t^m) &= \delta_{m+n,0}\ n\ K(x_2,x_3) = \delta_{m+n,0}\ n\ tr\,(ad(x_2)\circ ad(x_3))\\
&= \delta_{m+n,0}\ n\ 2\frac{1+a}{a^5};\\
\Phi(x_3\otimes t^n, x_3\otimes t^m) &= \delta_{m+n,0}\ n\ K(x_3,x_3) = \delta_{m+n,0}\ n\ tr\,(ad(x_3)\circ ad(x_3))\\
&= \delta_{m+n,0}\ n(-2)\frac{1+a}{a^5}.
\end{aligned}$$

Using Theorem 8, one obtains the second cohomology group of the truncated Hom-Lie algebra $\widehat{L}_p$ (see Example 9 for the definition of $\widehat{L}_p$):

Theorem 10. *Each non-trivial cocycle in $Z^2(\widehat{L}_p)$ can be represented as the sum of decomposable cocycles $f\otimes g$ where $f\colon L\times L\to\mathbb{C}$ and $g\colon \mathbb{C}[t]/t^{p+1}\mathbb{C}[t]\times\mathbb{C}[t]/t^{p+1}\mathbb{C}[t]\to\mathbb{C}$ are of one of the following 3 types:*

(1) $f(x_1,z) = f(x_2,z) = f(z,z) = 0$, $f\in S^2(\mathfrak{h}_1)$, $g\in C^2(\mathbb{C}[t]/t^{p+1}\mathbb{C}[t])$;
(2) $f(z,z) = 0$, $f\in S^2(\mathfrak{h}_1)$, $g\in C^2(\mathbb{C}[t]/t^{p+1}\mathbb{C}[t])$, $g(t^n,t^m) = g(1,1)$;
(3) $f(x_1,z) = f(x_2,z) = f(z,z) = 0$, $f\in C^2(\mathfrak{h}_1)$, $g\in S^2(\mathbb{C}[t]/t^{p+1}\mathbb{C}[t])$.

6. Extensions of current Hom-Lie algebras

The aim of this section is to provide a method to construct Hom-Lie algebras by extensions of current Hom-Lie algebras.

Definition 11 ([23])**.** An extension of a Hom-Lie algebra $(\mathcal{G},[\cdot,\cdot],\alpha)$ by a representation $(V,[\cdot,\cdot]_V,\beta)$ is an exact sequence

$$0 \longrightarrow (V,\beta) \xrightarrow{i} (K,\gamma) \xrightarrow{\pi} (\mathcal{G},\alpha) \longrightarrow 0$$

satisfying $\gamma\ o\ i = i\ o\ \beta$ and $\alpha\ o\ \pi = \pi\ o\ \gamma$. This extension is said to be central if $[K, i(V)]_K = 0$. In particular, if $K = \mathcal{G} \times V$, $i(v) = v$, for all $v \in V$ and $\pi(x) = x$, for all $x \in \mathcal{G}$, then we have $\gamma(x, v) = (\alpha(x), \beta(v))$ and we denote

$$0 \longrightarrow (V, \beta) \longrightarrow (K, \gamma) \longrightarrow (\mathcal{G}, \alpha) \longrightarrow 0.$$

For convenience, we denote $K = \mathcal{G} \times V = \mathcal{G} \oplus V$ and $C^{k,l} = Hom(\mathcal{G}^k V^l, V)$ where $\mathcal{G}^k V^l$ is the subspace of $C^{k+l}(K, K)$ consisting of products of k elements from $\mathcal{G}$ and l elements from V. Let $d = \mu + \lambda + f$ where $\mu \in C^2(\mathcal{G}, \mathcal{G})$, $\lambda \in C^{1,1}$ and $f \in C^{2,0}$. Let $\gamma' = (\alpha', \beta') \in End(\mathcal{G} \oplus V)$. Now we shall determine the 2-cochains d satisfying (K, d, γ') is a Hom-Lie algebra. Let $d = \mu + \lambda + f$, where $\mu \in C^2(\mathcal{G}, \mathcal{G})$, $\lambda \in C^{1,1}$ and $f \in C^{2,0}$. We have

$$\circlearrowleft_{x,y,z} d(\gamma(x + a), d(y + b, z + c))$$

$$= \Big(\circlearrowleft_{x,y,z} \mu(\alpha'(x), \mu(y, z)), \lambda\Big(\alpha'(x), \lambda(y, c)\Big) - \lambda\Big(\alpha'(y), \lambda(x, c)\Big) - \lambda\Big(\mu(x, y), \beta'(c)\Big)$$

$$+ \lambda\Big(\alpha'(z), \lambda(x, b)\Big) - \lambda\left(\alpha'(x), \lambda(z, b)\right) - \lambda\Big(\mu(z, x), \beta'(b)\Big) + \delta^2(f)(x, y, z)\Big)$$

where

$$\delta^2(f)(x, y, z) = \Big(\lambda\left(\alpha'(x), f(y, z)\right) + \lambda\left(\alpha'(y), f(z, x)\right) + \lambda\left(\alpha'(z), f(x, y)\right)$$

$$+ f\left(\alpha'(x), \mu(y, z)\right) + f\left(\alpha'(y), \mu(z, x)\right) + f\left(\alpha'(z), \mu(x, y)\right)\Big).$$

Then $\circlearrowleft_{x,y,z} d(\gamma(x + a), d(y + b, z + c) = 0 \implies \circlearrowleft_{x,y,z} \mu(\alpha'(x), \mu(y, z)) = 0$.
Hence $(\mathcal{G}, \mu, \alpha')$ is a Hom-Lie algebra. We assume that (V, λ, β') is a representation of $(\mathcal{G}, \mu, \alpha')$.

Theorem 11. *If (V, λ, β) is a representation of $(\mathcal{G}, \mu, \alpha)$. Then (K, d, γ) is a Hom-Lie algebra if and only if f is a 2-cocycle on V.*

Theorem 12. *A cohomology class $[f] \in H^2(\mathcal{G}, V)$ defines an extension of the Hom-Lie algebra $\mathcal{G}$ which is unique up to equivalence.*

6.1. Classification of central extensions of Hom-Loop algebras. By Theorem 8, Theorem 9 and Theorem 12, we obtain the following result.

Proposition 14. *Any central extension of a Hom-Loop algebra is equivalent to the extension defined by the skew-symmetric map $d\colon \tilde{L}(\mathcal{G}) \times \tilde{L}(\mathcal{G}) \to \tilde{L}(\mathcal{G})$ given by $d(x \otimes t^n, y \otimes t^m) = q^{n+m}[x, y] \otimes t^{n+m} + \delta_{n+m,0}\ n\ K(x, y)$ and the endomorphism $\gamma\colon \tilde{L}(\mathcal{G}) \to \tilde{L}(\mathcal{G})$ given by $\gamma\left(x \otimes t^n\right) = q^n \alpha(x) \otimes t^n$.*

Example 14. Any central extension of a Hom-Loop algebra $\tilde{L}(sl_2(\mathbb{C}))$ is equivalent to the extension given by

$$d(x_1 \otimes t^n, x_1 \otimes t^m) = \delta_{m+n,0}\ n\ c;\ d(x_1 \otimes t^n, x_2 \otimes t^m) = -2aq^{n+m} x_2 \otimes t^{n+m};$$

$$d(x_1 \otimes t^n, x_3 \otimes t^m) = 2q^{n+m} x_3 \otimes t^{n+m};\ d(x_2 \otimes t^n, x_2 \otimes t^m) = 0;$$

$$d(x_2 \otimes t^n, x_3 \otimes t^m) = -\frac{1+a}{2} q^{n+m} x_1 \otimes t^{n+m} + \frac{1+a}{4a} \delta_{m+n,0}\ n\ c;$$

$$d(x_3 \otimes t^n, x_3 \otimes t^m) = -\frac{1+a}{4a}\delta_{m+n,0}\ n\ c;$$
$$\gamma(x_1 \otimes t^n) = aq^n x_1 \otimes t^n,\ \gamma(x_2 \otimes t^n) = a^2q^n x_2 \otimes t^n,\ \gamma(x_3 \otimes t^n) = aq^n x_3 \otimes t^n;$$
$$\text{and } c = \frac{8}{a^4}.$$

6.2. Classification of central extensions of Hom-truncated Heisenberg algebras. In Theorem 10, we denote $f(x,x) = f_{11}$, $f(y,x) = f(x,y) = f_{12}$, $f(x,z) = f(z,x) = f_{13}$, $f(y,y) = f_{22}$, $f(z,y) = f(y,z) = f_{23}$, $f(z,z) = f_{23}$ and $g(t^m,t^n) = g(t^n,t^m) = \gamma_{nm}$ for all $n,m \in \{0,\cdots,p\}$.

Theorem 13. *Any central extension of a Hom-truncated Heisenberg algebra $\widehat{L}(\mathfrak{h}_1)_p$ is equivalent to one of following extensions:*

(1) $[x\otimes t^n, x\otimes t^m] = f_{11}\,\gamma_{nm}\,c_1$; $[x\otimes t^n, y\otimes t^m] = q^{n+m}z\otimes t^{n+m} + f_{12}\,\gamma_{nm}c_1$; $[y\otimes t^n, y\otimes t^m] = f_{22}\gamma_{nm}\,c_1$;

(2) $[x\otimes t^n, x\otimes t^m] = f_{11}\,\gamma_{nm}\,c_1$; $[x\otimes t^n, y\otimes t^m] = q^{n+m}z\otimes t^{n+m} + f_{12}\,\gamma_{nm}\,c_1$; $[x\otimes t^n, z\otimes t^m] = f_{13}\,\gamma_{nm}\,c_1$; $[y\otimes t^n, y\otimes t^m] = f_{22}\,\gamma_{nm}\,c_1$; $[y\otimes t^n, z\otimes t^m] = f_{23}\,\gamma_{nm}\,c_1$;

(3) $[x\otimes t^n, x\otimes t^m] = f_{11}\,\gamma_{nm}\,c_1$; $[x\otimes t^n, y\otimes t^m] = q^{n+m}z\otimes t^{n+m} + f_{12}\,\gamma_{nm}\,c_1$; $[y\otimes t^n, y\otimes t^m] = f_{22}\,\gamma_{nm}\,c_1$;

and $\gamma(x\otimes t^n) = \lambda_1 q^n x\otimes t^n$, $\gamma(y\otimes t^n) = \frac{\lambda}{\lambda_1}q^n y\otimes t^n$, $\gamma(z\otimes t^n) = \lambda q^n z\otimes t^n$.

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