ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 26, Number 1, June 2022
Available online at https://ojs.utlib.ee/index.php/ACUTM

Diophantine equations involving the bi-periodic Fibonacci and Lucas sequences

Lyes Ait-Amrane, Djilali Behloul, and Akila Djoumakh

Abstract. In this paper, we present new identities involving the bi-periodic Fibonacci and Lucas sequences. Then we solve completely some quadratic Diophantine equations involving the bi-periodic Fibonacci and Lucas sequences.

1. Introduction

The well known Fibonacci sequence $(F_n)_n$ and Lucas sequence $(L_n)_n$ are defined, respectively, by

$$\begin{cases} F_0 = 0, \quad F_1 = 1, \\ F_n = F_{n-1} + F_{n-2}, \end{cases} \quad \text{and} \quad \begin{cases} L_0 = 2, \quad L_1 = 1, \\ L_n = L_{n-1} + L_{n-2}, \end{cases} \quad n \geq 2.$$

These sequences and their different generalizations satisfy several properties, e.g., [4, 9, 10, 15, 17, 19]. One of the last generalizations of the sequences $(F_n)_n$ and $(L_n)_n$ is given by the bi-periodic Fibonacci sequence $(q_n)_n$, which is defined by Edson and Yayenie [8], and the bi-periodic Lucas sequence $(l_n)_n$, which is defined by Bilgici [3].

The bi-periodic Fibonacci sequence $(q_n)_n$ is defined by

$$q_0 = 0, \quad q_1 = 1, \quad q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even}, \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd}, \end{cases} \quad n \geq 2, \tag{1}$$

where a and b are nonzero real numbers. It is clear that if $a = b = 1$, we get the classical Fibonacci sequence. In [8], the authors extended Binet's

Received December 21, 2021.

2020 *Mathematics Subject Classification.* Primary 11D09; secondary 11D45, 11B37, 11B39.

Key words and phrases. Bi-periodic Fibonacci sequence, bi-periodic Lucas sequence, Diophantine equation, Pell's equation.

https://doi.org/10.12697/ACUTM.2022.26.09

formula to the bi-periodic Fibonacci sequence $(q_n)_n$ as follows

$$q_n = \frac{a^{\xi(n+1)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right), \quad n \geq 0, \tag{2}$$

where $\alpha = \frac{ab+\sqrt{\Delta}}{2}$ and $\beta = \frac{ab-\sqrt{\Delta}}{2}$ are the roots of the quadratic equation $x^2 - abx - ab = 0$, $\Delta = a^2b^2 + 4ab$ and $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd. Then, they deduce a number of mathematical properties. In particular, they generalize Cassini's, Catalan's and d'Ocagne's identities.

The bi-periodic Lucas sequence $(l_n)_n$ is defined by

$$l_0 = 2, \quad l_1 = a, \quad l_n = \begin{cases} bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even}, \\ al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd}, \end{cases} \quad n \geq 2. \tag{3}$$

If $a = b = 1$, we get the classical Lucas sequence. In [3], the author obtains some properties of the bi-periodic Lucas sequence $(l_n)_n$ and gives some relations between the sequences $(q_n)_n$ and $(l_n)_n$. In particular, he extended Binet's formula as follows

$$l_n = \frac{a^{\xi(n)}}{(ab)^{\left\lfloor \frac{n+1}{2} \right\rfloor}} \left(\alpha^n + \beta^n \right), \qquad n \geq 0. \tag{4}$$

The numbers α and β verify the following properties

$$\alpha^2 = ab(\alpha+1), \quad \beta^2 = ab(\beta+1), \quad \alpha+\beta = ab, \quad \alpha\beta = -ab, \quad \alpha-\beta = \sqrt{\Delta}.$$

From Binet's formulas (2) and (4), we get

$$q_{-n} = (-1)^{n+1} q_n, \qquad n \in \mathbb{Z}, \tag{5}$$

and

$$l_{-n} = (-1)^n l_n, \qquad n \in \mathbb{Z}.$$

Many authors have studied properties of the bi-periodic Fibonacci and Lucas sequences, e.g., [3, 8, 13, 20]. In this paper, we give new identities involving the sequences $(q_n)_n$ and $(l_n)_n$. Then we use these identities to define some Diophantine equations that we solve completely. Throughout this paper, a and b are assumed to be nonzero integers and $\Delta > 0$. By a simple induction, using (1) and (3), we can see that $a \mid q_{2n}$ and $a \mid l_{2n+1}$ for any $n \in \mathbb{Z}$. Thus, we put $t_n = \frac{q_{2n}}{a}$ and $s_n = \frac{l_{2n+1}}{a}$.

The study of Diophantine equations involving recurrence sequences has interested several authors, e.g., [1, 3, 6, 7, 11, 14, 16, 18]. Our purpose is to determine all integer solutions (x, y) of the following Diophantine equations

$$x^2 - \Delta y^2 = 4,$$

$$x^2 - \Delta y^2 = 4^k,$$

$$(ab)x^2 - \left(\frac{\Delta}{ab} \right) y^2 = -4,$$

$$x^2 + abs_nxy - aby^2 = q_{2n+1}^2,$$
$$x^2 - l_{2n}xy + y^2 = -(ab)t_n^2,$$
$$x^2 - \left(\frac{\Delta}{ab}\right) q_{2n+1}xy + \left(\frac{\Delta}{ab}\right) y^2 = -(ab)s_n^2,$$
$$x^2 - \Delta t_nxy - \Delta y^2 = -(ab)l_{2n}^2,$$
$$x^2 - \left(\frac{\Delta}{ab}\right) q_{2n+1}xy + \left(\frac{\Delta}{ab}\right) y^2 = s_n^2,$$
$$x^2 - (ab)s_nxy - (ab)y^2 = \left(\frac{\Delta}{ab}\right) q_{2n+1}^2,$$
$$x^2 - l_{2n}xy + y^2 = \left(\frac{\Delta}{ab}\right) t_n^2.$$

2. Some identities for $(q_n)_n$ and $(l_n)_n$

Let $n, m \in \mathbb{Z}$. We list here the identities that are necessary in our paper, they can be found in [3]:

$$q_{n-1} + q_{n+1} = l_n, \tag{6}$$
$$l_{n-1} + l_{n+1} = (ab + 4)q_n, \tag{7}$$

$$\Delta \left(\frac{1}{a^2}\right)^{\xi(m+1)\xi(n+1)} \left(\frac{1}{ab}\right)^{1-\xi(n+1)\xi(m+1)} q_mq_n + \left(\frac{b}{a}\right)^{\xi(n)\xi(m)} l_ml_n = 2l_{m+n}, \tag{8}$$

$$\left(\frac{b}{a}\right)^{\xi(n+1)\xi(m)} q_nl_m - \left(\frac{b}{a}\right)^{\xi(n)\xi(m+1)} q_ml_n = 2(-1)^m q_{n-m}, \tag{9}$$

$$\left(\frac{b}{a}\right)^{\xi(n)\xi(m)} l_nl_m - \Delta \left(\frac{1}{a^2}\right)^{\xi(n+1)\xi(m+1)} \left(\frac{1}{ab}\right)^{1-\xi(n+1)\xi(m+1)} q_nq_m = 2(-1)^m l_{n-m}, \tag{10}$$

$$\left(\frac{b}{a}\right)^{\xi(n+1)\xi(m)} q_nl_m + \left(\frac{b}{a}\right)^{\xi(n)\xi(m+1)} q_ml_n = 2q_{n+m}, \tag{11}$$
$$\left(\frac{b}{a}\right)^{\xi(n)} l_n^2 - \Delta \left(\frac{1}{a^2}\right)^{\xi(n+1)} \left(\frac{1}{ab}\right)^{\xi(n)} q_n^2 = 4(-1)^n. \tag{12}$$

We now give some new identities involving the bi-periodic Fibonacci and Lucas sequences.

Proposition 1. *Let $n, m, k \in \mathbb{Z}$. Then*

$$\left(-\frac{b}{a}\right)^{\xi(n+k)[1-\xi(n+m)]} q_{n+m}^2 + (-1)^{1+\xi(n+k)[1-\xi(n+m)]} \left(\frac{b}{a}\right)^{\xi(n+k)} l_{n-k} q_{n+m} q_{m+k} + \left(-\frac{b}{a}\right)^{\xi(n+k)\xi(n+m)} q_{m+k}^2 = \left(-\frac{b}{a}\right)^{\xi(n+m)[1-\xi(n+k)]} q_{n-k}^2.$$

Proof. According to the parity of $(n+m)$ and $(m+k)$, there are four cases.

(1) If $(n+m)$ and $(n+k)$ are even, we have to prove

$$q_{n+m}^2 - l_{n-k} q_{n+m} q_{m+k} + q_{m+k}^2 = q_{n-k}^2. \tag{13}$$

We consider the case n, m and k even. The case n, m and k odd is done with the same method. From identity (11), we have

$$\begin{cases} q_n l_m + q_m l_n = 2q_{n+m}, \\ q_k l_m + q_m l_k = 2q_{m+k}. \end{cases} \tag{14}$$

Multiplying the first identity of (14) by q_k and the second by q_n, and subtracting the results, we get

$$q_m (l_n q_k - l_k q_n) = 2(q_{n+m} q_k - q_{m+k} q_n).$$

Using identity (9), we obtain

$$q_m q_{n-k} = q_{m+k} q_n - q_{n+m} q_k. \tag{15}$$

Multiplying the first identity of (14) by l_k and the second by l_n, and subtracting the results, we get

$$l_m (q_n l_k - q_k l_n) = 2(q_{n+m} l_k - q_{m+k} l_n).$$

Using identity (9), we obtain

$$l_m q_{n-k} = q_{n+m} l_k - q_{m+k} l_n. \tag{16}$$

From (12), we have

$$(l_m q_{n-k})^2 - \left(\frac{\Delta}{a^2}\right)(q_m q_{n-k})^2 = 4q_{n-k}^2. \tag{17}$$

Replacing the left-hand sides of (15) and (16) in (17), expanding and using identities (10) and (12), we get the result.

(2) If $(n+m)$ is even and $(n+k)$ is odd, we have to prove

$$-\left(\frac{b}{a}\right) q_{n+m}^2 + \left(\frac{b}{a}\right) l_{n-k} q_{n+m} q_{m+k} + q_{m+k}^2 = q_{n-k}^2. \tag{18}$$

We consider the case n, m even and k odd. The case n, m odd and k even is done with the same method. From (11), we have

$$\begin{cases} q_n l_m + q_m l_n = 2q_{n+m}, \\ q_k l_m + \left(\frac{b}{a}\right) q_m l_k = 2q_{m+k}. \end{cases}$$

With the same process as before, we obtain

$$q_m q_{n-k} = q_{n+m} q_k - q_{m+k} q_n \tag{19}$$

and

$$l_m q_{n-k} = q_{m+k} l_n - \left(\frac{b}{a}\right) q_{n+m} l_k. \tag{20}$$

From (12), we have

$$(l_m q_{n-k})^2 - \left(\frac{\Delta}{a^2}\right) (q_m q_{n-k})^2 = 4q_{n-k}^2. \tag{21}$$

Replacing the left-hand sides of (19) and (20) in (21), expanding and using identities (10) and (12), we get the result.

(3) If $(n+m)$ is odd and $(n+k)$ is even, we have to prove

$$q_{n+m}^2 - l_{n-k} q_{n+m} q_{m+k} + q_{m+k}^2 = -\left(\frac{b}{a}\right) q_{n-k}^2. \tag{22}$$

We consider the case n, k odd and m even. The case n, k even and m odd is done with the same method. From Formula (11), we have

$$\begin{cases} q_n l_m + \left(\frac{b}{a}\right) q_m l_n = 2q_{n+m}, \\ q_k l_m + \left(\frac{b}{a}\right) q_m l_k = 2q_{m+k}. \end{cases} \tag{23}$$

With the same process as before, we obtain

$$q_m q_{n-k} = \left(\frac{a}{b}\right) (q_{n+m} q_k - q_{m+k} q_n) \tag{24}$$

and

$$l_m q_{n-k} = q_{m+k} l_n - q_{n+m} l_k. \tag{25}$$

From (12), we have

$$(l_m q_{n-k})^2 - \left(\frac{\Delta}{a^2}\right) (q_m q_{n-k})^2 = 4q_{n-k}^2. \tag{26}$$

Replacing the left-hand sides of (24) and (25) in (26), expanding and using identities (10) and (12), we get the result.

(4) If $(n+m)$ and $(n+k)$ are odd, we have to prove

$$q_{n+m}^2 - \left(\frac{b}{a}\right) l_{n-k} q_{n+m} q_{m+k} - \left(\frac{b}{a}\right) q_{m+k}^2 = q_{n-k}^2. \tag{27}$$

We consider the case n odd and m, k even. The case n even and m, k odd is done with the same method. From identity (11), we have

$$\begin{cases} q_n l_m + \left(\frac{b}{a}\right) q_m l_n = 2q_{n+m}, \\ q_k l_m + q_m l_k = 2q_{m+k}. \end{cases} \tag{28}$$

With the same process as before, we obtain

$$q_m q_{n-k} = q_{m+k} q_n - q_{n+m} q_k \tag{29}$$

and

$$l_m q_{n-k} = q_{n+m} l_k - \left(\frac{b}{a}\right) q_{m+k} l_n. \tag{30}$$

From (12), we have

$$(l_m q_{n-k})^2 - \left(\frac{\Delta}{a^2}\right) (q_m q_{n-k})^2 = 4q_{n-k}^2. \tag{31}$$

Replacing the left-hand sides of (29) and (30) in (31), expanding and using identities (10) and (12), we get the result.

□

Proposition 2. *Let $n, m, k \in \mathbb{Z}$. Then*

$$\begin{aligned} \left(\frac{b}{a}\right)^{\xi(n+m)[1-\xi(n+k)]} l_{n+m}^2 - \Delta \left(\frac{1}{a^2}\right)^{1-\xi(n+k)} \left(\frac{1}{ab}\right)^{\xi(n+k)} q_{n-k} l_{n+m} q_{m+k} \\ + (-1)^{1-\xi(n+k)} \left(\frac{\Delta}{ab}\right) \left(\frac{b}{a}\right)^{[1-\xi(n+k)][1-\xi(n+m)]} q_{m+k}^2 \\ = (-1)^{\xi(n+m)+\xi(n+k)} \left(\frac{b}{a}\right)^{[1-\xi(n+m)]\xi(n+k)} l_{n-k}^2. \end{aligned}$$

Proof. According to the parity of $(n + m)$ and $(m + k)$, there are four cases.

(1) If $(n + m)$ and $(n + k)$ are even, we have to prove

$$l_{n+m}^2 - \left(\frac{\Delta}{a^2}\right) l_{n+m} q_{n-k} q_{m+k} - \left(\frac{\Delta}{a^2}\right) q_{m+k}^2 = l_{n-k}^2. \tag{32}$$

We consider the case n, m and k even. The case n, m and k odd is done with the same method. From identities (8) and (11), we have

$$\begin{cases} \left(\frac{\Delta}{a^2}\right) q_n q_m + l_n l_m = 2l_{n+m}, \\ q_k l_m + q_m l_k = 2q_{m+k}. \end{cases} \tag{33}$$

Multiplying the first identity of (33) by q_k and the second by l_n, and subtracting the results, we obtain

$$q_m \left[\left(\frac{\Delta}{a^2}\right) q_n q_k - l_k l_n \right] = 2 \left(l_{n+m} q_k - q_{m+k} l_n \right).$$

Using identity (10), we get

$$q_m l_{n-k} = q_{m+k} l_n - l_{n+m} q_k. \tag{34}$$

Multiplying identity (8) by l_k and identity (11) by $\left(\frac{\Delta}{a^2}\right) q_n$, and subtracting the results, we obtain

$$l_m \left[l_n l_k - \left(\frac{\Delta}{a^2}\right) q_n q_k \right] = 2 \left[l_{n+m} l_k - \left(\frac{\Delta}{a^2}\right) q_{m+k} q_n \right].$$

Using identity (10) we get

$$l_m l_{n-k} = l_{n+m} l_k - \left(\frac{\Delta}{a^2}\right) q_{m+k} q_n. \tag{35}$$

From (12), we have

$$(l_m l_{n-k})^2 - \left(\frac{\Delta}{a^2}\right) (q_m l_{n-k})^2 = 4 l_{n-k}^2. \tag{36}$$

Replacing the left-hand sides of (34) and (35) in (36), expanding and using identities (9) and (12), we get the result.

(2) If $(n+m)$ is even and $(n+k)$ is odd, we have to prove

$$l_{n+m}^2 - \left(\frac{\Delta}{ab}\right) l_{n+m} q_{n-k} q_{m+k} + \left(\frac{\Delta}{ab}\right) q_{m+k}^2 = -\left(\frac{b}{a}\right) l_{n-k}^2. \tag{37}$$

We consider the case n, m even and k odd. The case n, m odd and k even is done with the same method. From identities (8) and (11), we have

$$\begin{cases} \left(\frac{\Delta}{a^2}\right) q_n q_m + l_n l_m = 2 l_{n+m}, \\ q_k l_m + \left(\frac{b}{a}\right) q_m l_k = 2 q_{m+k}. \end{cases}$$

With the same process as before, we obtain

$$q_m l_{n-k} = \left(-\frac{b}{a}\right) (l_{n+m} q_k - q_{m+k} l_n) \tag{38}$$

and

$$l_m l_{n-k} = l_{n+m} l_k - \left(\frac{\Delta}{ab}\right) q_{m+k} q_n. \tag{39}$$

From (12), we have

$$(l_m l_{n-k})^2 - \left(\frac{\Delta}{a^2}\right) (q_m l_{n-k})^2 = 4 l_{n-k}^2. \tag{40}$$

Replacing the left-hand sides of (38) and (39) in (40), expanding and using identities (9) and (12), we get the result.

(3) If $(n+m)$ is odd and $(n+k)$ is even, we have to prove

$$l_{n+m}^2 - \left(\frac{\Delta}{ab}\right) l_{n+m} q_{n-k} q_{m+k} - \left(\frac{\Delta}{b^2}\right) q_{m+k}^2 = -\left(\frac{a}{b}\right) l_{n-k}^2. \tag{41}$$

We consider the case n,k odd and m even. The case n,k even and m odd is done with the same method. From identities (8) and (11), we have

$$\begin{cases} \left(\frac{\Delta}{ab}\right) q_n q_m + l_n l_m = 2l_{n+m}, \\ q_k l_m + \left(\frac{b}{a}\right) q_m l_k = 2q_{m+k}. \end{cases}$$

With the same process as before, we obtain

$$q_m l_{n-k} = l_{n+m} q_k - q_{m+k} l_n \tag{42}$$

and

$$l_m l_{n-k} = \left(\frac{\Delta}{ab}\right) q_{m+k} q_n - \left(\frac{b}{a}\right) l_{n+m} l_k. \tag{43}$$

From (12), we have

$$(l_m l_{n-k})^2 - \left(\frac{\Delta}{a^2}\right) (q_m l_{n-k})^2 = 4l_{n-k}^2. \tag{44}$$

Replacing the left-hand sides of (42) and (43) in (44), expanding and using identities (9) and (12), we get the result.

(4) If $(n+m)$ and $(n+k)$ are odd, we have to prove

$$l_{n+m}^2 - \left(\frac{\Delta}{ab}\right) l_{n+m} q_{n-k} q_{m+k} + \left(\frac{\Delta}{ab}\right) q_{m+k}^2 = l_{n-k}^2. \tag{45}$$

We consider the case n odd and m,k even. The case n even and m,k odd is done with the same method. From identities (8) and (11), we have

$$\begin{cases} \left(\frac{\Delta}{ab}\right) q_n q_m + l_n l_m = 2l_{n+m}, \\ q_m l_k + q_k l_m = 2q_{m+k}. \end{cases}$$

With the same process as before, we obtain

$$q_m l_{n-k} = q_{m+k} l_n - l_{n+m} q_k. \tag{46}$$

and

$$l_m l_{n-k} = l_{n+m} l_k - \left(\frac{\Delta}{ab}\right) q_{m+k} q_n. \tag{47}$$

From (12), we have

$$(l_m l_{n-k})^2 - \left(\frac{\Delta}{a^2}\right) (q_m l_{n-k})^2 = 4l_{n-k}^2. \tag{48}$$

Replacing the left-hand sides of (46) and (47) in (48), expanding and using identities (9) and (12), we get the result.

□

Proposition 3. *Let $n,m,k \in \mathbb{Z}$. Then*

$$\left(\frac{-b}{a}\right)^{\xi(n+m)\xi(n+k)} l_{n+m}^2 - (-1)^{\xi(n+m)\xi(n+k)} \left(\frac{b}{a}\right)^{\xi(n+k)} l_{n-k} l_{n+m} l_{m+k}$$

$$+\left(\frac{-b}{a}\right)^{[1-\xi(n+m)]\xi(n+k)} l_{m+k}^2 = \left(\frac{\Delta}{ab}\right)\left(\frac{-b}{a}\right)^{[1-\xi(n+m)][1-\xi(n+k)]} q_{n-k}^2.$$

Proof. According to the parity of $(n+m)$ and $(m+k)$, there are four cases.

(1) If $(n+m)$ and $(n+k)$ are even, we have to prove

$$l_{n+m}^2 - l_{n-k}l_{n+m}l_{m+k} + l_{m+k}^2 = -\left(\frac{\Delta}{a^2}\right) q_{n-k}^2. \tag{49}$$

We consider the case n, m and k even. The case n, m and k odd is done with the same method. From Identity (8), we have

$$\begin{cases} \left(\frac{\Delta}{a^2}\right) q_n q_m + l_n l_m = 2l_{n+m}, \\ \left(\frac{\Delta}{a^2}\right) q_m q_k + l_m l_k = 2l_{m+k}. \end{cases} \tag{50}$$

Multiplying the first identity of (50) by l_k and the second by l_n, and subtracting the results, we obtain

$$\left(\frac{\Delta}{a^2}\right) q_m \left(q_n l_k - q_k l_n\right) = 2\left(l_{n+m}l_k - l_{m+k}l_n\right).$$

Using identity (9), we get

$$q_m q_{n-k} = \left(\frac{a^2}{\Delta}\right)\left(l_{n+m}l_k - l_{m+k}l_n\right). \tag{51}$$

Multiplying the first identity of (50) by q_k and the second by q_n, and subtracting the results, we obtain

$$l_m \left(l_n q_k - q_n l_k\right) = 2\left(l_{n+m}q_k - l_{m+k}q_n\right).$$

Using identity (9), we get

$$l_m q_{n-k} = l_{m+k}q_n - l_{n+m}q_k. \tag{52}$$

From (12), we have

$$\left(l_m q_{n-k}\right)^2 - \left(\frac{\Delta}{a^2}\right)\left(q_m q_{n-k}\right)^2 = 4q_{n-k}^2. \tag{53}$$

Replacing the left-hand sides of (51) and (52) in (53), expanding and using Identities (10) and (12), we get the result.

(2) If $(n+m)$ is even and $(n+k)$ is odd, we have to prove

$$l_{n+m}^2 - \left(\frac{b}{a}\right) l_{n-k}l_{n+m}l_{m+k} - \left(\frac{b}{a}\right) l_{m+k}^2 = \left(\frac{\Delta}{ab}\right) q_{n-k}^2. \tag{54}$$

We consider the case n, m even and k odd. The case n, m odd and k even is done with the same method. From identity (8), we have

$$\begin{cases} \left(\frac{\Delta}{a^2}\right) q_n q_m + l_n l_m = 2l_{n+m}, \\ \left(\frac{\Delta}{ab}\right) q_m q_k + l_m l_k = 2l_{m+k}. \end{cases}$$

With the same process as before, we obtain

$$q_m q_{n-k} = \left(\frac{ab}{\Delta}\right)(l_{m+k} l_n - l_{n+m} l_k) \tag{55}$$

and

$$l_m q_{n-k} = l_{n+m} q_k - \left(\frac{b}{a}\right) l_{m+k} q_n. \tag{56}$$

From (12), we have

$$(l_m q_{n-k})^2 - \left(\frac{\Delta}{a^2}\right)(q_m q_{n-k})^2 = 4q_{n-k}^2. \tag{57}$$

Replacing the left-hand sides of (55) and (56) in (57), expanding and using identities (10) and (12), we get the result.

(3) If $(n+m)$ is odd and $(n+k)$ is even, we have to prove

$$l_{n+m}^2 - l_{n-k} l_{n+m} l_{m+k} + l_{m+k}^2 = \left(\frac{\Delta}{ab}\right) q_{n-k}^2. \tag{58}$$

We consider the case n, k odd and m even. The case n, k even and m odd is done with the same method. From identity (8), we have

$$\begin{cases} \left(\frac{\Delta}{ab}\right) q_n q_m + l_n l_m = 2l_{n+m}, \\ \left(\frac{\Delta}{ab}\right) q_m q_k + l_m l_k = 2l_{m+k}. \end{cases}$$

With the same process as before, we obtain

$$q_m q_{n-k} = \left(\frac{ab}{\Delta}\right)(l_{m+k} l_n - l_{n+m} l_k) \tag{59}$$

and

$$l_m q_{n-k} = l_{n+m} q_k - l_{m+k} q_n. \tag{60}$$

From (12), we have

$$(l_m q_{n-k})^2 - \left(\frac{\Delta}{a^2}\right)(q_m q_{n-k})^2 = 4q_{n-k}^2. \tag{61}$$

Replacing the left-hand sides of (59) and (60) in (61), expanding and using identities (10) and (12), we get the result.

(4) If $(n+m)$ and $(n+k)$ are odd, we have to prove

$$\left(\frac{-b}{a}\right) l_{n+m}^2 + \left(\frac{b}{a}\right) l_{n-k} l_{n+m} l_{m+k} + l_{m+k}^2 = \left(\frac{\Delta}{ab}\right) q_{n-k}^2. \tag{62}$$

We consider the case n odd and m, k even. The case n even and m, k odd is done with the same method. From identity (8), we have

$$\begin{cases} \left(\frac{\Delta}{ab}\right) q_n q_m + l_n l_m = 2l_{n+m}, \\ \left(\frac{\Delta}{a^2}\right) q_m q_k + l_m l_k = 2l_{m+k}. \end{cases}$$

With the same process as before, we obtain

$$q_m q_{n-k} = \left(\frac{ab}{\Delta}\right)(l_{n+m}l_k - l_{m+k}l_n) \tag{63}$$

and

$$l_m q_{n-k} = l_{m+k}q_n - \left(\frac{b}{a}\right) l_{n+m}q_k. \tag{64}$$

From (12), we have

$$(l_m l_{n-k})^2 - \left(\frac{\Delta}{a^2}\right)(q_m l_{n-k})^2 = 4q_{n-k}^2. \tag{65}$$

Replacing the left-hand sides of (63) and (64) in (65), expanding and using identities (10) and (12), we get the result.

□

3. Solutions of some Diophantine equations

The identities (12), (18), (22), (37), (41), (45), (54) and (58) suggest to explore the solutions of the Diophantine equations listed before. To do this, we need the following proposition which is given in [5, Proposition 6.3.16. p. 355].

Proposition 4 (The Structure Theorem)**.** *If $D > 0$ is not a square and is congruent to 0 or 1 modulo 4, the Pell's equation $x^2 - Dy^2 = \pm 4$ has an infinity of solutions given in the following way. If (x_0, y_0) is a solution with the least strictly positive y_0 (and $x_0 > 0$, say), the* ***general solution*** *is given by*

$$\frac{x+\sqrt{D}y}{2} = \pm\left(\frac{x_0+\sqrt{D}y_0}{2}\right)^k,$$

for any $k \in \mathbb{Z}$.

Remark 1. It is well known (see [2]) that if D is a positive not square integer, then all integer solutions of $x^2 - Dy^2 = 1$ are given by

$$x + \sqrt{D}y = \pm\left(x_0 + \sqrt{D}y_0\right)^n, \quad n \in \mathbb{Z},$$

where (x_0, y_0) is the fundamental positive solution. Since $x_0 + \sqrt{D}y_0 = -\left(-x_0 + \sqrt{D}y_0\right)^{-1}$, so $x + \sqrt{D}y = \pm\left(-x_0 + \sqrt{D}y_0\right)^k$ with $k = -n \in \mathbb{Z}$.

The following lemma exists in an equivalent form in [12], but we prefer to give a simple proof involving bi-periodic numbers.

Lemma 1. *All integer solutions of the Diophantine equation*

$$x^2 - \Delta y^2 = 4 \tag{66}$$

are $(x, y) = \pm\left(l_{2k}, \frac{q_{2k}}{a}\right)$ *where* $k \in \mathbb{Z}$.

Proof. A fundamental solution of the equation (66) is $(x_0, y_0) = (ab+2, 1)$. Thus, according to the Structure Theorem, the solutions of equation (66) verify

$$\frac{x + \sqrt{\Delta} y}{2} = \pm \left(\frac{ab + 2 + \sqrt{\Delta}}{2} \right)^k, \quad k \in \mathbb{Z}.$$

Thus, we get for the " + " sign

$$\begin{cases} \frac{x+\sqrt{\Delta}y}{2} = \left(\frac{ab+2+\sqrt{\Delta}}{2}\right)^k = (\alpha+1)^k = \left(\frac{\alpha^2}{ab}\right)^k, & k \in \mathbb{Z}, \\ \frac{x-\sqrt{\Delta}y}{2} = \left(\frac{ab+2-\sqrt{\Delta}}{2}\right)^k = (\beta+1)^k = \left(\frac{\beta^2}{ab}\right)^k, & k \in \mathbb{Z}, \end{cases}$$

which is equivalent to

$$\begin{cases} x = \left(\frac{\alpha^2}{ab}\right)^k + \left(\frac{\beta^2}{ab}\right)^k = \frac{\alpha^{2k}+\beta^{2k}}{(ab)^k} = l_{2k}, & k \in \mathbb{Z}, \\ y = \frac{1}{\sqrt{\Delta}}\left[\left(\frac{\alpha^2}{ab}\right)^k - \left(\frac{\beta^2}{ab}\right)^k\right] = \frac{1}{(ab)^k}\left[\frac{\alpha^{2k}-\beta^{2k}}{\sqrt{\Delta}}\right] = t_{2k}, & k \in \mathbb{Z}. \end{cases}$$

Proceeding in the same way for the " $-$ " sign, we get that all solutions are $(x, y) = \pm(l_{2k}, \frac{q_{2k}}{a})$.

Conversely, from (12) we have

$$l_{2k}^2 - \Delta \left(\frac{q_{2k}}{a}\right)^2 = 4,$$

which means that $\pm\left(l_{2k}, \frac{q_{2k}}{a}\right)$ are solutions of (66). ☐

Lemma 2. *Assume that* $x^2 - \Delta y^2 = 4^k$ *with* $k \geq 2$. *If* ab *is odd, then* x *and* y *are even numbers.*

Proof. Assume that ab is odd and let $ab = 2c + 1$. Then $(ab)^2 + 4ab = (2c+1)^2 + 4(2c+1) = 4c(c+3) + 5 \equiv 5[8]$. Since $k \geq 2$, we have $x^2 - \left[(ab)^2 + 4ab\right] y^2 \equiv 0[8]$. Thus, $x^2 - \left[(ab)^2 + 4ab\right] y^2 \equiv x^2 - 5y^2[8]$. But $x^2 - 5y^2 \equiv 0[8]$ if and only if either $x^2, y^2 \equiv 0[8]$ or $x^2, y^2 \equiv 4[8]$. We conclude that either $x, y \equiv 0, 4[8]$ or $x, y \equiv 2, 6[8]$, i.e., we conclude that x and y are even. ☐

Remark 2. If ab is even, the conclusion of Lemma 2 is not true. Indeed, for $k = 4$ and $ab = 16$, the equation $x^2 - \Delta y^2 = 4^k$ becomes $x^2 - 2^6 \cdot 5y^2 = 2^8$. We deduce that $2^3 \mid x$. Let $x = 2^3 x_1$. Then, the equation $x^2 - 2^6 \cdot 5y^2 = 2^8$ becomes $x_1^2 - 5y^2 = 4$. It is clear that $(x_1, y) = (3, 1)$ is a solution and y is odd.

Corollary 1. *Let $k \geq 1$ be an integer. Then all integer solutions (x, y) of the equation $x^2 - \Delta y^2 = 4^k$ with ab odd are $\pm\left(2^{k-1}l_{2m}, 2^{k-1}\frac{q_{2m}}{a}\right)$ with $m \in \mathbb{Z}$.*

Proof. By induction using Lemmas 1 and 2, the details are left to the reader. □

To solve the Diophantine equation in the following lemma, it is necessary to study several cases.

Lemma 3. *All integer solutions of the Diophantine equation*

$$(ab)x^2 - \left(\frac{\Delta}{ab}\right) y^2 = -4 \tag{67}$$

are $(x, y) = \pm\left(\frac{l_{2k-1}}{a}, q_{2k-1}\right)$, where $k \in \mathbb{Z}$.

Proof. Let $t = -ab$. Then equation (67) becomes

$$tx^2 - (t-4)y^2 = 4. \tag{68}$$

It is clear that $\gcd(t, t-4) = 1, 2$ or 4. Thus, we have three cases.

(1) If $\gcd(t, t-4) = 1$, then $t \equiv 1[2]$. Since $x^2 \equiv x[2]$ and $y^2 \equiv y[2]$, we get from equation (68) that $x - y \equiv 0[2]$, i.e., x and y have the same parity. We make the following change of variables:

$$\left\{\begin{array}{lcl} u &=& \frac{-1}{2}\left[tx + (t-4)y\right] \\ v &=& \frac{1}{2}(x+y) \end{array}\right. \iff \left\{\begin{array}{lcl} x &=& \frac{-1}{2}\left[u + (t-4)v\right] \\ y &=& \frac{1}{2}(u+tv) \end{array}\right. .$$

Then, using the fact that $t(t-4) = \Delta$, the equation (68) becomes

$$u^2 - \Delta v^2 = 4. \tag{69}$$

Then, from Lemma 1, we have

$$\left\{\begin{array}{ll} u = l_{2k}, & k \in \mathbb{Z}, \\ v = \frac{q_{2k}}{a}, & k \in \mathbb{Z}. \end{array}\right.$$

From identities (6) and (7), we get

$$u = q_{2k-1} + q_{2k+1} \quad \text{and} \quad v = \frac{l_{2k-1} + l_{2k+1}}{a(4-t)}.$$

Thus,

$$x = -\frac{1}{2}\left[l_{2k} + (t-4)\frac{l_{2k-1} + l_{2k+1}}{a(4-t)}\right] = \frac{l_{2k-1}}{a}$$

and

$$y = \frac{1}{2}\left(q_{2k-1} + q_{2k+1} - bq_{2k}\right) = q_{2k-1}.$$

Proceeding in the same way for the "$-$" sign, we get that all solutions are $(x, y) = \pm\left(\frac{l_{2k-1}}{a}, q_{2k-1}\right)$.

(2) If $\gcd(t, t-4) = 2$, let $s = \frac{t}{2}$. Then, the equation (68) becomes

$$sx^2 - (s-2)y^2 = 2. \tag{70}$$

Since s is odd, we deduce that x and y have the same parity. Thus, we use the following change of variables:

$$\left\{\begin{array}{lcl} u &=& -\frac{1}{2}\left[sx + (s-2)y\right] \\ v &=& \frac{1}{2}(x+y) \end{array}\right. \iff \left\{\begin{array}{lcl} x &=& -u-(s-2)v \\ y &=& u+sv \end{array}\right. .$$

Equation (70) becomes

$$u^2 - s(s-2)v^2 = 1. \tag{71}$$

The fundamental solution of (71) is $(u_0, v_0) = (s-1, 1)$. Therefore, according to Remark 1, the solutions of equation (71) verify

$$u + \sqrt{s(s-2)}v = \pm\left(1 - s + \sqrt{s(s-2)}\right)^k, \quad k \in \mathbb{Z}.$$

Thus, we get for the "$+$" sign

$$\begin{cases} u + \sqrt{s(s-2)}v = (\alpha+1)^k = \left(\frac{\alpha^2}{ab}\right)^k, & k \in \mathbb{Z}, \\ u - \sqrt{s(s-2)}v = (\beta+1)^k = \left(\frac{\beta^2}{ab}\right)^k, & k \in \mathbb{Z}, \end{cases}$$

which is equivalent to

$$\begin{cases} u = \frac{1}{2}\left[\left(\frac{\alpha^2}{ab}\right)^k + \left(\frac{\beta^2}{ab}\right)^k\right], & k \in \mathbb{Z}, \\ v = \frac{1}{\sqrt{\Delta}}\left[\left(\frac{\alpha^2}{ab}\right)^k - \left(\frac{\beta^2}{ab}\right)^k\right], & k \in \mathbb{Z}. \end{cases}$$

Hence

$$\begin{array}{rcl} x &=& \left(\frac{-1}{2} - \frac{s-2}{\sqrt{\Delta}}\right)\left(\frac{\alpha^2}{ab}\right)^k + \left(\frac{-1}{2} + \frac{s-2}{\sqrt{\Delta}}\right)\left(\frac{\beta^2}{ab}\right)^k \\ &=& \frac{1}{\alpha}\left(\frac{\alpha^2}{ab}\right)^k + \frac{1}{\beta}\left(\frac{\beta^2}{ab}\right)^k \\ &=& \frac{\alpha^{2k-1}+\beta^{2k-1}}{(ab)^k} \\ &=& \frac{l_{2k-1}}{a} \end{array}$$

and

$$\begin{array}{rcl} y &=& \left[\frac{1}{2} + \frac{s}{\sqrt{\Delta}}\right]\left(\frac{\alpha^2}{ab}\right)^k + \left[\frac{1}{2} - \frac{s}{\sqrt{\Delta}}\right]\left(\frac{\beta^2}{ab}\right)^k \\ &=& \frac{-\beta}{\sqrt{\Delta}}\left(\frac{\alpha^2}{ab}\right)^k + \frac{\alpha}{\sqrt{\Delta}}\left(\frac{\beta^2}{ab}\right)^k \\ &=& \frac{1}{\sqrt{\Delta}}\left[\frac{\alpha^{2k-1}-\beta^{2k-1}}{(ab)^{k-1}}\right] \\ &=& q_{2k-1}. \end{array}$$

Proceeding in the same way for the "$-$" sign, we get that all solutions are $(x, y) = \pm\left(\frac{l_{2k-1}}{a}, q_{2k-1}\right)$.

(3) If $\gcd(t, t-4) = 4$, let $s = \frac{t}{4}$, then equation (68) becomes

$$sx^2 - (s-1)y^2 = 1. \tag{72}$$

Let us have the following change of variables:

$$\begin{cases} u &= -sx-(s-1)y \\ v &= x+y \end{cases} \Longleftrightarrow \begin{cases} x &= -u-(s-1)v \\ y &= u+sv \end{cases}.$$

Thus the equation (72) becomes

$$u^2 - s(s-1)v^2 = 1. \tag{73}$$

The fundamental solution of (73) is $(u_0, v_0) = (2s-1, 2)$. That is, if $(u, 1)$ is a solution of (73), then

$$u^2 = s^2 - s + 1 = \left(s - \frac{1}{2}\right)^2 + \frac{3}{4} \Longrightarrow (2u)^2 - (2s-1)^2 = 3,$$

i.e., $(2u-2s+1)(2u+2s-1) = 3$, which leads to $s = 0$ or 1, this contradicts the hypothesis $\Delta = 16s(s-1) > 0$. Thus, according to Remark 1, the solutions of equation (73) verify

$$u + \sqrt{s(s-1)}v = \pm\left(1 - 2s + 2\sqrt{s(s-1)}\right)^k, \quad k \in \mathbb{Z}.$$

Thus we get for the "$+$" sign

$$\begin{cases} u + \sqrt{s(s-1)}v = (\alpha+1)^k = \left(\frac{\alpha^2}{ab}\right)^k, & k \in \mathbb{Z}, \\ u - \sqrt{s(s-1)}v = (\beta+1)^k = \left(\frac{\beta^2}{ab}\right)^k, & k \in \mathbb{Z}, \end{cases}$$

which is equivalent to

$$\begin{cases} u = \frac{1}{2}\left[\left(\frac{\alpha^2}{ab}\right)^k + \left(\frac{\beta^2}{ab}\right)^k\right], & k \in \mathbb{Z}, \\ v = \frac{2}{\sqrt{\Delta}}\left[\left(\frac{\alpha^2}{ab}\right)^k - \left(\frac{\beta^2}{ab}\right)^k\right], & k \in \mathbb{Z}. \end{cases}$$

Hence

$$\begin{aligned} x &= \left[\frac{-1}{2} - \frac{2(s-1)}{\sqrt{\Delta}}\right]\left(\frac{\alpha^2}{ab}\right)^k + \left[\frac{-1}{2} + \frac{2(s-1)}{\sqrt{\Delta}}\right]\left(\frac{\beta^2}{ab}\right)^k \\ &= \frac{1}{\alpha}\left(\frac{\alpha^2}{ab}\right)^k + \frac{1}{\beta}\left(\frac{\beta^2}{ab}\right)^k \\ &= \frac{\alpha^{2k-1}+\beta^{2k-1}}{(ab)^k} \\ &= \frac{l_{2k-1}}{a} \end{aligned}$$

and

$$\begin{aligned} y &= \left[\tfrac{1}{2}+\tfrac{2s}{\sqrt{\Delta}}\right]\left(\tfrac{\alpha^2}{ab}\right)^k+\left[\tfrac{1}{2}-\tfrac{2s}{\sqrt{\Delta}}\right]\left(\tfrac{\beta^2}{ab}\right)^k \\ &= \tfrac{-\beta}{\sqrt{\Delta}}\left(\tfrac{\alpha^2}{ab}\right)^k+\tfrac{\alpha}{\sqrt{\Delta}}\left(\tfrac{\beta^2}{ab}\right)^k \\ &= \tfrac{1}{\sqrt{\Delta}}\left[\tfrac{\alpha^{2k-1}-\beta^{2k-1}}{(ab)^{k-1}}\right] \\ &= q_{2k-1}. \end{aligned}$$

Proceeding in the same way for the "−" sign, we get that all solutions are $(x, y) = \pm\left(\frac{l_{2k-1}}{a}, q_{2k-1}\right)$.

Conversely, from (12) we have

$$\begin{aligned} \left(\tfrac{b}{a}\right) l_{2k-1}^2-\left(\tfrac{\Delta}{ab}\right) q_{2k-1}^2=-4 \quad &\Longleftrightarrow \quad ab\left(\tfrac{l_{2k-1}}{a}\right)^2-\left[\tfrac{(ab)^2+4ab}{ab}\right] q_{2k-1}^2=-4, \\ &\Longleftrightarrow \quad ab\left(\tfrac{l_{2k-1}}{a}\right)^2-(ab+4)q_{2k-1}^2=-4, \end{aligned}$$

which means that $\pm\left(\frac{l_{2k-1}}{a}, q_{2k-1}\right)$ are solutions of (67). □

Theorem 1. *Let n be an integer.*

(1) *All integer solutions (x, y) of*

$$x^2+abs_nxy-aby^2=q_{2n+1}^2 \tag{74}$$

are $\pm\left(q_{2(n-m)+1}, \frac{q_{2m}}{a}\right)$, where $m \in \mathbb{Z}$.

(2) *All integer solutions (x, y) of*

$$x^2-\left(\frac{\Delta}{ab}\right) q_{2n+1}xy+\left(\frac{\Delta}{ab}\right) y^2=s_n^2 \tag{75}$$

are $\pm\left(\frac{l_{2(n+m)+1}}{a}, \frac{q_{2m}}{a}\right)$, where $m \in \mathbb{Z}$.

Proof.

(1) Assume that

$$x^2+abs_nxy-aby^2=q_{2n+1}^2.$$

Then

$$(2x+abs_ny)^2-ab(4+abs_n^2)y^2=4q_{2n+1}^2.$$

Using (12), we obtain

$$(2x+abs_ny)^2-\Delta q_{2n+1}^2y^2=4q_{2n+1}^2. \tag{76}$$

We deduce that $q_{2n+1} \mid (2x+abs_ny)$. Thus the equation (76) is equivalent to

$$z^2-\Delta y^2=4, \tag{77}$$

where $z = \frac{2x+abs_ny}{q_{2n+1}}$. Therefore, according to Lemma 1, the solutions of equation (77) are $(z,y) = \pm(l_{2m}, t_m)$, $m \in \mathbb{Z}$.
For the " + " sign we obtain

$$\begin{cases} z = \frac{2x+abs_ny}{q_{2n+1}} = l_{2m}, \\ y = t_m. \end{cases} \iff \begin{cases} 2x = l_{2m}\, q_{2n+1} - abs_ny, \\ y = t_m. \end{cases}$$

From (9), we get $x = q_{2(n-m)+1}$.

Proceeding in the same way for the " − " sign, we find that all solutions are $(x,y) = \pm\left(q_{2(n-m)+1}, \frac{q_{2m}}{a}\right)$, $m \in \mathbb{Z}$.

Conversely, replacing in identity (18) n by $(2n+k+1)$ and m by $(2m-2n-k-1)$ for $k \in \mathbb{Z}$, we obtain

$$q_{2m-2n-1}^2 + \left(\frac{b}{a}\right) l_{2n+1} q_{2m-2n-1} q_{2m} - \left(\frac{b}{a}\right) q_{2m}^2 = q_{2n+1}^2.$$

From (5), we get

$$q_{2(n-m)+1}^2 + \left(\frac{b}{a}\right) l_{2n+1} q_{2(n-m)+1} q_{2m} - \left(\frac{b}{a}\right) q_{2m}^2 = q_{2n+1}^2,$$

which means that $\pm\left(q_{2(n-m)+1}, \frac{q_{2m}}{a}\right)$, $m \in \mathbb{Z}$, are solutions of (74).

(2) Assume that

$$x^2 - \left(\frac{\Delta}{ab}\right) q_{2n+1} xy + \left(\frac{\Delta}{ab}\right) y^2 = s_n^2.$$

Then

$$\left[2x - \left(\frac{\Delta}{ab}\right) q_{2n+1} y\right]^2 - \left(\frac{\Delta}{ab}\right)\left[\left(\frac{\Delta}{ab}\right) q_{2n+1}^2 - 4\right] y^2 = 4s_n^2.$$

Using (12), we obtain

$$\left[2x - \left(\frac{\Delta}{ab}\right) q_{2n+1} y\right]^2 - \Delta s_n^2 y^2 = 4s_n^2. \tag{78}$$

We deduce that $s_n \mid \left[2x - \left(\frac{\Delta}{ab}\right) q_{2n+1} y\right]$. Thus the equation (78) is equivalent to

$$z^2 - \Delta y^2 = 4. \tag{79}$$

where $z = \frac{2x - \left(\frac{\Delta}{ab}\right) q_{2n+1} y}{s_n}$. Therefore, according to Lemma 1, the solutions of equation (79) are $(z,y) = \pm(l_{2m}, t_m)$, $\quad m \in \mathbb{Z}$.

For the " + " sign we obtain

$$\begin{cases} z = l_{2m}, \\ y = t_m. \end{cases} \iff \begin{cases} 2x = l_{2m} s_n + \left(\frac{\Delta}{ab}\right) q_{2n+1} t_m, \\ y = t_m. \end{cases}$$

From (8), we get $x = \frac{l_{2(n+m)+1}}{a} = s_{2(n+m)+1}$.

Proceeding in the same way for the " $-$ " sign, we find that all solutions are $(x,y) = \pm\left(\frac{l_{2(n+m)+1}}{a}, \frac{q_{2m}}{a}\right), m \in \mathbb{Z}$.

Conversely, replacing in (45) n by $(2n+k+1)$ and m by $(2m-k)$ for $k \in \mathbb{Z}$, we get

$$l_{2(n+m)+1}^2 - \left(\frac{\Delta}{ab}\right) q_{2m} l_{2(n+m)+1} q_{2n+1} + \left(\frac{\Delta}{ab}\right) q_{2m}^2 = l_{2n+1}^2,$$

which means that $\pm\left(\frac{l_{2(n+m)+1}}{a}, \frac{q_{2m}}{a}\right), m \in \mathbb{Z}$, are solutions of (75).

□

Theorem 2. *Let n be an integer and assume that ab is a square-free integer.*

(1) *All integer solutions (x,y) of*

$$x^2 - l_{2n}xy + y^2 = -(ab)t_n^2 \tag{80}$$

are $\pm(q_{2(n+m)-1}, q_{2m-1})$ with $m \in \mathbb{Z}$, if $n \neq 0$ and (x,x) with $x \in \mathbb{Z}$, if $n = 0$.

(2) *All integer solutions (x,y) of*

$$x^2 - \left(\frac{\Delta}{ab}\right) q_{2n+1}xy + \left(\frac{\Delta}{ab}\right) y^2 = -(ab)s_n^2 \tag{81}$$

are $\pm(l_{2(n+m)}, q_{2m-1})$, where $m \in \mathbb{Z}$.

(3) *All integer solutions (x,y) of*

$$x^2 - \Delta t_n xy - \Delta y^2 = -(ab)l_{2n}^2 \tag{82}$$

are $\pm(bl_{2(n+m)-1}, q_{2m-1})$, where $m \in \mathbb{Z}$.

Proof.

(1) Assume that $n \neq 0$ and

$$x^2 - l_{2n}xy + y^2 = -(ab)t_n^2.$$

Then

$$(2x - l_{2n}y)^2 - (l_{2n}^2 - 4)y^2 = -4(ab)t_n^2.$$

Using (12), we obtain

$$(2x - l_{2n}y)^2 - \Delta t_n^2 y^2 = -4(ab)t_n^2. \tag{83}$$

We deduce that $t_n \mid (2x - l_{2n}y)$. Thus, the equation (83) is equivalent to

$$z^2 - \Delta y^2 = -4(ab), \tag{84}$$

where $z = \frac{2x - l_{2n}y}{t_n}$. Since (ab) is a square-free integer and $(ab) \mid \Delta$, then $(ab) \mid z$. Therefore, the equation (84) is equivalent to

$$(ab)w^2 - \left(\frac{\Delta}{ab}\right) y^2 = -4, \tag{85}$$

where $w = \frac{z}{ab}$. According to Lemma 3, the solutions of equation (85) are $(w, y) = \pm (s_{m-1}, q_{2m-1})$, $m \in \mathbb{Z}$.

For the " + " sign, we obtain

$$\begin{cases} w = \frac{2x - l_{2n} y}{(ab) t_n} = & s_{m-1}, \\ y = q_{2m-1}. \end{cases} \iff \begin{cases} 2x = (ab) t_n s_{m-1} + l_{2n} y, \\ y = q_{2m-1}. \end{cases}$$

From (11), we get $x = q_{2(n+m)-1}$.

Proceeding in the same way for the " − " sign, we find that all solutions are $(x, y) = \pm \left(q_{2(n+m)-1}, q_{2m-1}\right), m \in \mathbb{Z}$.

Conversely, replacing in (22) n by $(2n + k)$ and m by $(2m - k - 1)$ for $k \in \mathbb{Z}$, we obtain

$$q_{2(n+m)-1}^2 - l_{2n} q_{2(n+m)-1} q_{2m-1} + q_{2m-1}^2 = -\left(\frac{b}{a}\right) q_{2n}^2,$$

which means that $\pm \left(q_{2(n+m)-1}, q_{2m-1}\right), m \in \mathbb{Z}$, are solutions of (80).

If $n = 0$, the equation (80) becomes $(x - y)^2 = 0$.

(2) Assume that

$$x^2 - \left(\frac{\Delta}{ab}\right) q_{2n+1} xy + \left(\frac{\Delta}{ab}\right) y^2 = -(ab) s_n^2.$$

Then

$$\left[2x - \left(\frac{\Delta}{ab}\right) q_{2n+1} y\right]^2 + \left[4\left(\frac{\Delta}{ab}\right) - \left(\frac{\Delta}{ab}\right)^2 q_{2n+1}^2\right] y^2 = -4(ab) s_n^2.$$

Using (12), we obtain

$$\left[2x - \left(\frac{\Delta}{ab}\right) q_{2n+1} y\right]^2 - \Delta s_n^2 y^2 = -4(ab) s_n^2. \tag{86}$$

We deduce that $s_n \mid \left[2x - \left(\frac{\Delta}{ab}\right) q_{2n+1} y\right]$. Thus, the equation (86) gives

$$z^2 - \Delta y^2 = -4(ab), \tag{87}$$

where $z = \frac{2x - \left(\frac{\Delta}{ab}\right) q_{2n+1} y}{s_n}$. Since (ab) is a square-free integer and $(ab) \mid \Delta$, then $(ab) \mid z$. Therefore, the equation (87) is equivalent to

$$(ab) w^2 - \left(\frac{\Delta}{ab}\right) y^2 = -4, \tag{88}$$

where $w = \frac{z}{ab}$. According to Lemma 3, the solutions of equation (88) are $(w, y) = \pm (s_{m-1}, q_{2m-1})$, $m \in \mathbb{Z}$.

For the " + " sign, we obtain

$$\begin{cases} w = \frac{2x - \left(\frac{\Delta}{ab}\right) q_{2n+1} y}{(ab) s_n}, \\ w = s_{m-1}, \\ y = q_{2m-1}. \end{cases} \iff \begin{cases} 2x = (ab) s_n s_{m-1} + \left(\frac{\Delta}{ab}\right) q_{2n+1} y, \\ y = q_{2m-1}. \end{cases}$$

From (8), we get $x = l_{2(n+m)}$.

Proceeding in the same way for the " − " sign, we find that all solutions are $(x, y) = \pm \left(l_{2(n+m)}, q_{2m-1}\right), m \in \mathbb{Z}$.

Conversely, replacing in (37) n by $(2n+k+1)$ and m by $(2m-k-1)$ for $k \in \mathbb{Z}$, we obtain

$$l_{2(n+m)}^2 - \left(\frac{\Delta}{ab}\right) q_{2n+1} l_{2(n+m)} q_{2m-1} + \left(\frac{\Delta}{ab}\right) q_{2m-1}^2 = -\left(\frac{b}{a}\right) l_{2n+1}^2,$$

which means that $\pm \left(l_{2(n+m)}, q_{2m-1}\right), m \in \mathbb{Z}$, are solutions of (81).

(3) Assume that

$$x^2 - \Delta t_n xy - \Delta y^2 = -(ab) l_{2n}^2.$$

Then

$$(2x - \Delta t_n y)^2 - \left(\Delta^2 t_n^2 + 4\Delta\right) y^2 = -4(ab) l_{2n}^2.$$

Using (12), we obtain

$$(2x - \Delta t_n y)^2 - \Delta l_{2n}^2 y^2 = -4(ab) l_{2n}^2. \tag{89}$$

We deduce that $l_{2n} \mid (2x - \Delta t_n y)$. Thus the equation (89) is equivalent to

$$z^2 - \Delta y^2 = -4(ab), \tag{90}$$

where $z = \frac{2x - \Delta t_n y}{l_{2n}}$. Since (ab) is a square-free integer and $(ab) \mid \Delta$, then $(ab) \mid z$. Therefore, the equation (90) is equivalent to

$$(ab) w^2 - \left(\frac{\Delta}{ab}\right) y^2 = -4, \tag{91}$$

where $w = \frac{z}{ab}$. Thus, according to Lemma 3, the solutions of Equation (91) are $(w, y) = \pm (s_{m-1}, q_{2m-1})$, $m \in \mathbb{Z}$.

For the " + " sign, we obtain

$$\begin{cases} w = \frac{2x - \Delta t_n y}{(ab) l_{2n}} = s_{m-1}, \\ y = q_{2m-1}. \end{cases} \iff \begin{cases} 2x = (ab) l_{2n} s_{m-1} + \Delta t_n y, \\ y = q_{2m-1}. \end{cases}$$

From (8), we get $x = b l_{2(n+m)-1}$.

Proceeding in the same way for the " − " sign, we find that all solutions are $(x, y) = \pm \left(b l_{2(n+m)-1}, q_{2m-1}\right), m \in \mathbb{Z}$.

Conversely, replacing in (41) n by $(2n+k)$ and m by $(2m-k-1)$ for $k \in \mathbb{Z}$, we obtain

$$l_{2(n+m)-1}^2 - \left(\frac{\Delta}{ab}\right) q_{2n} l_{2(n+m)-1} q_{2m-1} - \left(\frac{\Delta}{b^2}\right) q_{2m-1}^2 = -\left(\frac{a}{b}\right) l_{2n}^2,$$

which means that $\pm\left(bl_{2(n+m)-1}, q_{2m-1}\right), m \in \mathbb{Z}$, are solutions of (82).

□

Theorem 3. *Let n be an integer and assume that $ab+4$ is a square-free integer.*

(1) *All integer solutions (x, y) of*

$$x^2 - (ab)s_n xy - (ab)y^2 = \left(\frac{\Delta}{ab}\right) q_{2n+1}^2 \tag{92}$$

are $\pm\left(l_{2(n+m)}, \frac{l_{2m-1}}{a}\right)$, where $m \in \mathbb{Z}$.

(2) *All integer solutions (x, y) of*

$$x^2 - l_{2n} xy + y^2 = \left(\frac{\Delta}{ab}\right) t_n^2 \tag{93}$$

are $\pm\left(\frac{l_{2(n+m)-1}}{a}, \frac{l_{2m-1}}{a}\right)$ with $m \in \mathbb{Z}$, if $n \neq 0$ and (x, x) with $x \in \mathbb{Z}$, if $n = 0$.

Proof.

(1) Assume that

$$x^2 - (ab)s_n xy - (ab)y^2 = \left(\frac{\Delta}{ab}\right) q_{2n+1}^2.$$

Then

$$[2x - (ab)s_n y]^2 - \left[(ab)^2 s_n^2 + 4(ab)\right] y^2 = 4\left(\frac{\Delta}{ab}\right) q_{2n+1}^2.$$

Using (12), we obtain

$$[2x - (ab)s_n y]^2 - \Delta q_{2n+1}^2 y^2 = 4\left(\frac{\Delta}{ab}\right) q_{2n+1}^2. \tag{94}$$

We deduce that $q_{2n+1} \mid [2x - (ab)s_n y]$. Thus, the equation (94) is equivalent to

$$z^2 - \Delta y^2 = 4(ab+4), \tag{95}$$

where $z = \frac{2x-(ab)s_n y}{q_{2n+1}}$. Since $(ab+4)$ is a square-free integer and $(ab+4) \mid \Delta$, we have $(ab+4) \mid z$. Therefore, the equation (95) is equivalent to

$$(ab)y^2 - \left(\frac{\Delta}{ab}\right) w^2 = -4, \tag{96}$$

where $w = \frac{z}{ab+4}$. According to Lemma 3, the solutions of equation (96) are $(y, w) = \pm (s_{m-1}, q_{2m-1})$, $m \in \mathbb{Z}$.

For the " + " sign, we obtain

$$\begin{cases} y = s_{m-1}, \\ w = \frac{2x-(ab)s_n y}{(ab+4)q_{2n+1}}, \\ w = q_{2m-1}. \end{cases} \iff \begin{cases} y = s_{m-1}, \\ 2x = \left(\frac{\Delta}{ab}\right) q_{2n+1}q_{2m-1} + (ab)s_n y. \end{cases}$$

From (8), we get $x = l_{2(n+m)}$.

Proceeding in the same way for the " − " sign, we find that all solutions are $(x, y) = \pm \left(l_{2(n+m)}, \frac{l_{2m-1}}{a}\right), m \in \mathbb{Z}$.

Conversely, replacing in (54) n by $(2n+k+1)$ and m by $(2m-k-1)$ for $k \in \mathbb{Z}$, we obtain

$$l^2_{2(n+m)} - \left(\frac{b}{a}\right) l_{2n+1}l_{2(n+m)}l_{2m-1} - \left(\frac{b}{a}\right) l^2_{2m-1} = \left(\frac{\Delta}{ab}\right) q^2_{2n+1},$$

which means that $\pm \left(l_{2(n+m)}, \frac{l_{2m-1}}{a}\right), m \in \mathbb{Z}$, are solutions of (92).

(2) Assume that $n \neq 0$ and

$$x^2 - l_{2n}xy + y^2 = \left(\frac{\Delta}{ab}\right) t^2_n.$$

Then

$$(2x - l_{2n}y)^2 - \left(l^2_{2n} - 4\right) y^2 = 4 \left(\frac{\Delta}{ab}\right) t^2_n.$$

Using (12), we obtain

$$(2x - l_{2n}y)^2 - \Delta t^2_n y^2 = 4 \left(\frac{\Delta}{ab}\right) t^2_n. \tag{97}$$

We deduce that $t_n \mid (2x - l_{2n}y)$. Thus, the equation (97) is equivalent to

$$z^2 - \Delta y^2 = 4(ab + 4), \tag{98}$$

where $z = \frac{2x-l_{2n}y}{t_n}$. Since $(ab + 4)$ is a square-free and $(ab + 4) \mid \Delta$, we have $(ab + 4) \mid z$. Therefore, the equation (98) is equivalent to

$$(ab)y^2 - \left(\frac{\Delta}{ab}\right) w^2 = -4, \tag{99}$$

where $w = \frac{z}{ab+4}$. Thus, according to Lemma 3, the solutions of equation (99) are $(y, w) = \pm (s_{m-1}, q_{2m-1})$, $m \in \mathbb{Z}$.

For the " + " sign, we obtain

$$\begin{cases} y = s_{m-1}, \\ w = \frac{2x-l_{2n}y}{(ab+4)t_n} = q_{2m-1}. \end{cases} \iff \begin{cases} y = s_{m-1}, \\ 2x = \left(\frac{\Delta}{ab}\right) t_n q_{2m-1} + l_{2n}y. \end{cases}$$

From (8), we get $x = \frac{l_{2(n+m)-1}}{a}$.

Proceeding in the same way for the " − " sign, we find that all solutions are $(x, y) = \pm\left(\frac{l_{2(n+m)-1}}{a}, \frac{l_{2m-1}}{a}\right), m \in \mathbb{Z}$.

Conversely, replacing in (58) n by $(2n+k)$ and m by $(2m-k-1)$ for $k \in \mathbb{Z}$, we obtain

$$l_{2(n+m)-1}^2 - l_{2n}l_{2(n+m)-1}l_{2m-1} + l_{2m-1}^2 = \left(\frac{\Delta}{ab}\right) q_{2n}^2,$$

which means that $\pm\left(\frac{l_{2(n+m)-1}}{a}, \frac{l_{2m-1}}{a}\right), m \in \mathbb{Z}$, are solutions of (93).

If $n = 0$, the equation (93) becomes $(x - y)^2 = 0$.

□

4. Concluding Remarks

Propositions 1, 2 and 3 give rise to twelve identities, these identities suggest us to study twelve Diophantine equations, but only seven of them are studied in Theorems 1, 2 and 3. The remaining ones rise from the identities (13),(27), (32), (49), (62) and are

$$x^2 - (ab)s_nxy - (ab)y^2 = q_{2n+1}^2, \tag{100}$$

$$-(ab)x^2 + (ab)s_nxy + y^2 = \left(\frac{\Delta}{ab}\right) q_{2n+1}^2, \tag{101}$$

$$x^2 - l_{2n}xy + y^2 = t_n^2, \tag{102}$$

$$x^2 - \Delta t_nxy - \Delta y^2 = l_{2n}^2, \tag{103}$$

$$x^2 - l_{2n}xy + y^2 = -\Delta t_n^2. \tag{104}$$

Equation (100) follows from equation (74) by replacing x by $-x$ and equation (101) follows from equation (92) by replacing (x, y) by $(-y, x)$.

Thanks to Binet's formulas, we have $t_n = u_n$ and $l_{2n} = v_n$, where $(u_n)_n$ and $(v_n)_n$ are defined respectively by $u_n = (ab+2)u_{n-1} - u_{n-2}$ with initial values $u_0 = 0, u_1 = 1$ and $v_n = (ab+2)v_{n-1} - v_{n-2}$ with initial values $v_0 = 2, v_1 = ab+2$. It turns out that the equations (102), (103) and (104) are solved in [6].

Acknowledgements

Research activities were carried out in LATN Laboratory, Faculty of Mathematics, USTHB, BP 32, El Alia, 16111, Bab Ezzouar, Algiers, Algeria.

References

[1] L. Ait-Amrane and D. Behloul, *On some Diophantine equations involving generalized Fibonacci and Lucas numbers*, Colloq. Math. **150** (2017), 257–268.

[2] T. Andreescu and D. Andrica, *Quadratic Diophantine equations. With a foreword by Preda Mihăilescu. Developments in Mathematics* **40**, Springer, New York, 2015.

[3] G. Bilgici, *Two generalizations of Lucas sequence*, Appl. Math. Comput. **245** (2014), 526–538.

[4] L. Carlitz, *A note on Fibonacci numbers*, Fibonacci Quart. **2** (1964), 15–28.

[5] H. Cohen, *Number Theory I, Tools and Diophantine Equations*, Springer, 2007.

[6] B. Demirtürk and R. Keskin, *On some Diophantine equations*, J. Inequal. Appl. **2013**(162) (2013), 12 pp.

[7] B. Demirtürk and R. Keskin, *Integer solutions of some Diophantine equations via Fibonacci and Lucas numbers*, J. Integer Seq. **12**(8) (2009), Article 09.8.7.

[8] M. Edson and O. Yayenie, *A new generalization of Fibonacci sequences and extended Binet's formula*, Integers **9** (2009), 639–654.

[9] V. E. Hoggatt Jr., *Fibonacci and Lucas Numbers*, Houghton Mifflin Company, Boston, 1969.

[10] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, Fibonacci Quart. **3** (1965), 161–176.

[11] J. P. Jones, *Diophantine representation of the Lucas numbers*, Fibonacci Quart. **14**(2) (1976), 134.

[12] J. P. Jones, *Representation of solutions of Pell equations using Lucas sequences, dedicated to the memory of Professor Dr. Péter Kiss.*, Acta Acad. Paedagog. Agriensis Sect. Math. (N.S.) **30** (2003), 75–86.

[13] S. P. Jun and K. H. Choi, *Some properties of the generalized Fibonacci sequence* $\{q_n\}$ *by matrix methods*, Korean J. Math. **24** (2016), 681–691.

[14] R. Keskin and B. Demirtürk, *Solutions of some Diophantine equations using generalized Fibonacci and Lucas sequences*, Ars Combin. **111** (2013), 161–179.

[15] T. Koshy, *Fibonacci and Lucas Numbers with Application*, Wiley-Interscience, 2001.

[16] W. L. McDaniel, *Diophantine representation of Lucas sequences*, Fibonacci Quart. **33** (1995), 59–63.

[17] P. Ribenboim, *The little Book of Big Primes*, Springer-Verlag, 1991.

[18] D. Savin, *About a Diophantine equation*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **17** (2009), 241–250.

[19] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Ellis Horwood Ltd., 1989.

[20] O. Yayenie, *A note on generalized Fibonacci sequences*, Appl. Math. Comput. **217** (2011), 5603–5611.

Ecole nationale Supérieure d'Informatique (ESI), BP 68M, Oued Smar, 16309, El Harrach, Algiers, Algeria

E-mail address: l_ait_amrane@esi.dz; lyesait@gmail.com

Faculty of Computer Science, USTHB, BP 32, El Alia, 16111, Bab Ezzouar, Algiers, Algeria

E-mail address: dbehloul@yahoo.fr

Faculty of Mathematics, USTHB, BP 32, El Alia, 16111 Bab Ezzouar, Algiers, Algeria

E-mail address: a.djoumakh@gmail.com