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About the transitivity of the property of being Segal topological algebra

Mart Abel

Abstract. We show that if (A, f, B) and (B, g, C) are left (right or two-sided) Segal topological algebras for which $g(f(A)) \subseteq g(B)g(f(A))$ $(g(f(A)) \subseteq g(f(A))g(B)$ or $g(f(A)) \subseteq g(B)g(f(A)) \cap g(f(A))g(B)$, respectively), then $(A, g \circ f, C)$ is also a left (right or two-sided, respectively) Segal topological algebra.

1. Introduction

The notion of a (general) Segal topological algebra was first introduced in 2018 in [1]. Since then, the author of [1] has published more than 15 papers on Segal topological algebras and has given several talks in different countries and at different conferences.

The main question of the present paper is the following.

Let (A, f, B) and (B, g, C) be two left (right or two-sided) Segal topological algebras. Is it true that then $(A, g \circ f, C)$ is also a left (respectively, right or two-sided) Segal topological algebra, i.e., is the property of being Segal topological algebra transitive?

The author has been asked this question several times when giving talks about the properties of Segal topological algebras. It has not been (even partially) answered so far. In this paper we give an affirmative answer for some special classes of Segal topological algebras, leaving the question still open in general.

The main result of this paper is Theorem 1, where a sufficient condition for affirmative answer of the main question is provided. Since there are some

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widely known classes of algebras, for which the condition of Theorem 1 is fulfilled, we provide results for these classes as corollaries.

All algebras in this paper are algebras over the field $\mathbb{K}$, which may stand for the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. There is no assumption of the existence of unit element in our algebras, but all algebras are considered to be associative.

Let us first recall some definitions. A topological algebra (A, τ_A) is a left (right or two-sided) **Segal topological algebra** in a topological algebra (B, τ_B) via an algebra homomorphism $f : A \to B$, if

1) $\mathrm{cl}_B(f(A)) = B$;
2) $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$, i.e., f is continuous;
3) $f(A)$ is a left (respectively, right or two-sided) ideal of B.

In what follows, a left (right or two-sided) Segal topological algebra will be denoted shortly by a triple (A, f, B).

As one can see from the definition of a Segal topological algebra, there are 3 conditions that need to be checked in order to show that the property of being Segal topological algebra is transitive. Checking the transitivity of conditions 1) and 2) will be easy. The problem is with checking the transitivity of the third condition. One could go for the easy way and put an assumption that for Segal topological algebras (A, f, B) and (B, g, C) the set $g(f(A))$ is a left (right or two-sided) ideal of C, but this would not get us any closer to the solution. Instead of this condition, we have found some (probably) stronger conditions, that imply the transitivity of the condition 3) and are perhaps more easily checked.

We would like to point out that we have checked the transitivity of each condition in the definition of the Segal topological algebra separately and independently of other conditions (except for the condition 1) for which we used the condition 2) for g, but not for f). Perhaps assuming all 3 conditions together would somehow imply the transitivity of the condition 3), but, as of now, we have no idea how to use the conditions 1) and/or 2) in showing that $g(f(A))$ is a left (right) or two-sided ideal of C. We neither have any counterexamples of the cases when (A, f, B) and (B, g, C) are left (right or two-sided) Segal topological algebras but $g(f(A))$ is not a left (right, or two-sided, respectively) ideal of C. Therefore, we finish the paper with an open question that might be solved by other mathematicians in the future.

2. Main results

We start this section with showing the transitivity of the condition 1) using an extra assumption of the continuity of the map g.

Lemma 1. *Let* $(A,\tau_A),(B,\tau_B),(C,\tau_C)$ *be topological spaces and* $f: A\to B$, $g: B\to C$ *maps. If* g *is continuous,* $\mathrm{cl}_B(f(A)) = B$ *and* $\mathrm{cl}_C(g(B)) = C$*, then* $\mathrm{cl}_C(g(f(A))) = C$.

Proof. From $f(A) \subseteq B$ it follows that $g(f(A)) \subseteq g(B)$. As $\mathrm{cl}_B(f(A)) = B$, then equivalently, the only closed subset of B that contains $f(A)$ is B. Similarly, as $\mathrm{cl}_C(g(B)) = C$, then equivalently, the only closed subset of C that contains $g(B)$ is C.

I) If $g(f(A))$ is closed in C, then $Z = g^{-1}(g(f(A)))$ is a closed subset of B, as g is a continuous map. Notice that $f(A) \subseteq g^{-1}(g(f(A))) = Z$. As B is the only closed subset of B that contains $f(A)$, we must have $Z = B$. But then, as g is onto $g(f(A))$, we obtain $g(f(A)) = g(Z) = g(B)$ and $\mathrm{cl}_C(g(f(A))) = \mathrm{cl}_C(g(B)) = C$, as wanted.

II) Suppose that $g(f(A))$ is not closed and $X = \mathrm{cl}_C(g(f(A))) \neq C$. Then X is a closed proper subset of C, containing $g(f(A))$. As $X \neq C$, then X does not contain $g(B)$. Hence, there exists $b_X \in B$ such that $g(b_X) \notin X$, which means that $Y = g^{-1}(X) = \{b \in B : g(b) \in X\} \neq B$.

Notice that Y is a closed proper subset of B, because g is continuous and $b_X \in B\setminus Y$. Moreover, Y contains $f(A)$. But this is a contradiction with the condition that B is the only closed subset of B that contains $f(A)$. Hence, in the case $g(f(A))$ is not closed in C, $X = \mathrm{cl}_C(g(f(A))) = C$, as well. $\square$

Let us recall that for any algebra A over the field $\mathbb{K}$ and its subsets $K, L \subseteq A$, the set KL is defined as follows:

$$KL = \Big\{\sum_{i=1}^{n} \lambda_i k_i l_i : k_1,\ldots,k_n \in K, l_1,\ldots,l_n \in L, \lambda_1,\ldots,\lambda_n \in \mathbb{K}, n \in \mathbb{Z}^+\Big\}.$$

Recall that a semigroup G is *factorisable* if for every $g \in G$ there exist $g_1, \overline{g}_1 \in G$ such that $g = g_1\overline{g}_1$. As every ring and every algebra is also a multiplicative semigroup, then we can talk also about factorisable rings and factorisable algebras, which are factorisable as multiplicative semigroups. Now it is clear that if A is a factorisable algebra, then $A \subseteq AA$.

One special class of factorizable algebras consists of all such algebras A, where all elements are idempotent, i.e., where $a^2 = a$ for each $a \in A$. These algebras are closely connected with the Boolean rings (i.e., rings R, where $r^2 = r$ for each $r \in R$), when we look at the algebra as a ring, and generalized (non-unital) Boolean algebras (where also all elements are idempotent). Every algebra, where all elements are idempotent, is a factorisable algebra, hence, satisfies the condition $A \subseteq AA$.

A two-sided ideal I of a ring R is called a *von Neumann regular ideal* if for each $i \in I$ there exists $j \in I$ such that $i = iji$. It is easy to see that every von Neumann regular ideal of an algebra is factorisable.

In what follows, we will be interested in ideals I of algebras A which satisfy the condition $I \subseteq AI$ or $I \subseteq IA$. As we noticed before, all von

Neumann regular ideals, ideals that are Boolean algebras and ideals that are factorizable as semigroups, represent some of the special cases of ideals with the property $I \subseteq AI$ and/or $I \subseteq IA$.

Notice that I is a left ideal of an algebra A if and only if $i_1+i_2, \lambda i_1, ai_1 \in I$ for all $i_1, i_2 \in I, \lambda \in \mathbb{K}, a \in A$. The last condition is equivalent to the condition $AI \subseteq I$. Similar conditions are easily obtained for right and two-sided ideals.

Notice that the transitivity of the condition 2) is trivial since the composition of two continuous algebra homomorphisms is always a continuous algebra homomorphism. The next result includes some cases of algebras for which the transitivity of the condition 3) is fulfilled.

Lemma 2. *Let A, B, C be algebras over $\mathbb{K}$ and $f : A \to B, g : B \to C$ any maps.*

a) *If $g(f(A))$ is a left (right or two-sided) ideal of C, then $g(f(A))g(f(A)) \subseteq g(f(A))$.*

b) *If $f(A)$ is a left (right or two-sided) ideal of B, then $f(A)f(A) \subseteq f(A)$. Moreover, if g is an algebra homomorphism, then $g(f(A))g(f(A)) \subseteq g(f(A))$ and $g(f(A))$ is a left (right or two-sided) ideal of $g(B)$.*

c) *If $f(A)$ is a left (right or two-sided) ideal of B, $g(B)$ is a left (respectively, right or two-sided) ideal of C, g is an algebra homomorphism and $g(f(A)) \subseteq g(B)g(f(A))$ (respectively, $g(f(A)) \subseteq g(f(A))g(B)$ or $g(f(A)) \subseteq g(f(A))g(B) \cap g(B)g(f(A))$), then $g(f(A))$ is a left (respectively, right or two-sided) ideal of C and $g(f(A)) = g(B)g(f(A))$ (respectively, $g(f(A)) = g(f(A))g(B)$ or $g(f(A)) = g(f(A))g(B) \cap g(B)g(f(A))$).*

d) *If $f(A)$ is a left (right or two-sided) ideal of B, $g(B)$ is a left (respectively, right or two-sided) ideal of C, g is an algebra homomorphism and $g(f(A)) \subseteq g(f(A))g(f(A))$, then $g(f(A))$ is a left (respectively, right or two-sided) ideal of C and $g(f(A)) = g(f(A))g(f(A))$.*

e) *If $f(A)$ is a left (right or two-sided) ideal of B, $g(B)$ is a left (right or two-sided, respectively) ideal of C, g is an algebra homomorphism and $f(A) \subseteq Bf(A)$ (respectively, $f(A) \subseteq f(A)B$ or $f(A) \subseteq f(A)B \cap Bf(A)$), then $g(f(A))$ is a left (respectively, right or two-sided) ideal of C and $g(f(A)) = g(B)g(f(A))$ (respectively, $g(f(A)) = g(f(A))g(B)$ or $g(f(A)) = g(f(A))g(B) \cap g(B)g(f(A))$).*

f) *If $f(A)$ is a left (right or two-sided) ideal of B and either*

(i) *$g(B)$ is a left (respectively, right or two-sided) ideal of C and $g(f(A)) = g(B)$*

or

(ii) *$g(B) = C$ and g is an algebra homomorphism,*

then $g(f(A))$ is a left (respectively, right or two-sided) ideal of C.

Proof. We will prove these claims for left ideals. The proofs in the case of right ideals or in the case of two-sided ideals are similar.

a) If $g(f(A))$ is a left ideal of C, then $Cg(f(A)) \subseteq g(f(A))$. Since $g(f(A)) \subseteq C$, we obtain immediately that

$$g(f(A))g(f(A)) \subseteq Cg(f(A)) \subseteq g(f(A)).$$

b) If $f(A)$ is a left ideal of B, then $Bf(A) \subseteq f(A)$. Since $f(A) \subseteq B$, then we obtain that $f(A)f(A) \subseteq Bf(A) \subseteq f(A)$.

Take any $n \in \mathbb{Z}^+$, $a_1, \ldots, a_n, \overline{a}_1, \ldots, \overline{a}_n \in A, \lambda_1, \ldots, \lambda_n \in \mathbb{K}$. Then

$$\sum_{i=1}^{n} \lambda_i g(f(a_i))g(f(\overline{a}_i)) = g\Big(\sum_{i=1}^{n} \lambda_i f(a_i)f(\overline{a}_i)\Big) \in g(f(A)f(A)) \subseteq g(f(A)),$$

because g is an algebra homomorphism. As it holds for arbitrary $n \in \mathbb{Z}^+$, $a_1, \ldots, a_n, \overline{a}_1, \ldots, \overline{a}_n \in A, \lambda_1, \ldots, \lambda_n \in \mathbb{K}$, then $g(f(A))g(f(A)) \subseteq g(f(A))$.

Similarly we obtain that $g(B)g(f(A)) = g(Bf(A)) \subseteq g(f(A))$. Hence, $g(f(A))$ is a left ideal of $g(B)$.

c) As $f(A)$ is a left ideal of B and $g(B)$ is a left ideal of C, then we have $Bf(A) \subseteq f(A)$ and $Cg(B) \subseteq g(B)$. Since g is an algebra homomorphism, $g(B)g(f(A)) = g(Bf(A)) \subseteq g(f(A))$. As $g(f(A)) \subseteq g(B)g(f(A))$, we obtain

$$Cg(f(A)) \subseteq Cg(B)g(f(A)) \subseteq g(B)g(f(A)) \subseteq g(f(A)),$$

which means that $g(f(A))$ is a left ideal of C. Notice that every ideal is also a subalgebra, which means that it is closed with respect to the algebraic operations of addition, multiplication and scalar multiplication. Therefore, it is easy to see that $g(B)g(f(A)) \subseteq g(f(A))$, which, together with the assumption $g(f(A)) \subseteq g(B)g(f(A))$, gives us the desired equality.

d) Notice that the inclusion $g(f(A)) \subseteq g(f(A))g(f(A)) \subseteq g(B)g(f(A))$ follows from $g(f(A)) \subseteq g(B)$. Hence, the claim is true by the part c) of the proof.

e) Since $f(A) \subseteq Bf(A)$ and g is an algebra homomorphism, we obtain $g(f(A)) \subseteq g(Bf(A)) = g(B)g(f(A))$. Hence, the claim follows by the proof of part c) of Lemma 2.

f) If $g(f(A)) = g(B)$ and $g(B)$ is a left ideal of C, then

$$Cg(f(A)) = Cg(B) \subseteq g(B) = g(f(A)).$$

If $g(B) = C$ and g is an algebra homomorphism, then $g(f(A))$ is a left ideal of $g(B) = C$ by part b) of Lemma 2. Hence, $g(f(A))$ is a left ideal of C in both cases. $\square$

Now we will state and prove similar results for some particular classes of algebras or ideals, described before Lemma 2. We start with factorisable algebras, because they are more widely known and used.

Corollary 1. *Let A, B, C be algebras over $\mathbb{K}$ and let $f : A \to B$, $g : B \to C$ be algebra homomorphisms. If $f(A)$ is a left (right or two-sided) ideal of B, $g(B)$ is a left (respectively, right or two-sided) ideal of C and $g(f(A))$ is factorisable, then $g(f(A))$ is a left (respectively, right or two-sided) ideal of C and $g(f(A)) = g(f(A))g(f(A))$.*

Proof. Notice that $g(f(A)) \subseteq g(f(A))g(f(A))$, because $g(f(A))$ is factorisable. Hence, the result follows from part d) of Lemma 2. $\square$

For Boolean rings we have the following result.

Corollary 2. *Let A, B, C be algebras over $\mathbb{K}$ and let $f : A \to B$, $g : B \to C$ be algebra homomorphisms. If $f(A)$ is a left (right or two-sided) ideal of B, $g(B)$ is a left (respectively, right or two-sided) ideal of C and $g(f(A))$ is a Boolean ring, then $g(f(A))$ is a left (respectively, right or two-sided) ideal of C and $g(f(A)) = g(f(A))g(f(A))$.*

Proof. Notice that $g(f(A)) \subseteq g(f(A))g(f(A))$, because every algebra, which is also a Boolean ring, is factorisable. Hence, the result follows from Corollary 1. $\square$

Next, we move to the results describing some "transitivity" properties of Segal topological algebras.

Theorem 1. *Let (A, f, B) and (B, g, C) be left (right or two-sided) Segal topological algebras. If $g(f(A)) \subseteq g(B)g(f(A))$ ($g(f(A)) \subseteq g(f(A))g(B)$ or $g(f(A)) \subseteq g(B)g(f(A)) \cap g(f(A))g(B)$, respectively), then $(A, g \circ f, C)$ is a left (right or two-sided, respectively) Segal topological algebra.*

Proof. Since both (A, f, B) and (B, g, C) are Segal topological algebras, $\mathrm{cl}_B(f(A)) = B$, $\mathrm{cl}_C(g(B)) = C$, $f(A)$ is a left (right or two-sided, respectively) ideal of B, $g(B)$ is a left (right or two-sided, respectively) ideal of C and the maps f, g are continuous algebra homomorphisms. Hence, $\mathrm{cl}_C((g \circ f)(A)) = C$ by Lemma 1, $g(f(A))$ is a left (respectively, right or two-sided) ideal of C by part c) of Lemma 2, and the composition $f \circ g$ is a continuous algebra homomorphism. This proves that $(A, g \circ f, C)$ is a left (respectively, right or two-sided) Segal topological algebra. $\square$

Using part d) of Lemma 2 instead of part c) of Lemma 2, we obtain another result.

Corollary 3. *Let (A, f, B) and (B, g, C) be left (right or two-sided) Segal topological algebras. If $g(f(A)) \subseteq g(f(A))g(f(A))$, then $(A, g \circ f, C)$ is a left (right or two-sided) Segal topological algebra.*

Proof. Notice that from $g(f(A)) \subseteq g(f(A))g(f(A))$ it follows that $g(f(A)) \subseteq g(B)g(f(A)) \cap g(f(A))g(B)$. So, the result is true by Theorem 1. $\square$

Using part e) of Lemma 2 instead of part c) of Lemma 2, we obtain the following.

Corollary 4. *Let (A, f, B) and (B, g, C) be left (right or two-sided) Segal topological algebras. If $f(A) \subseteq Bf(A)$ (respectively, $f(A) \subseteq f(A)B$ or $f(A) \subseteq Bf(A) \cap f(A)B$), then $(A, g \circ f, C)$ is a left (right or two-sided) Segal topological algebra.*

Using part f) of Lemma 2 instead of parts c) or d) of Lemma 2, we obtain another result.

Corollary 5. *Let (A, f, B) and (B, g, C) be left (right or two-sided) Segal topological algebras. If $g(f(A)) = g(B)$ or $g(B) = C$, then $(A, g \circ f, C)$ is a left (right or two-sided) Segal topological algebra.*

Proof. Since (B, g, C) is a Segal topological algebra, g is an algebra homomorphism. Now, by part e) of Lemma 2, $g(f(A))$ is a left (right or two-sided) ideal of C. By Lemma 1, we know that $g(f(A))$ is dense in C. As f and g are algebra homomorphisms, $g \circ f$ is also an algebra homomorphism. Thus, $(A, g \circ f, C)$ is a left (right or two-sided) Segal topological algebra. □

Using Corollaries 1 and 2 instead of Lemma 2, we obtain the following result.

Corollary 6. *Let (A, f, B) and (B, g, C) be left (right or two-sided) Segal topological algebras. If $g(f(A))$ is factorisable or a Boolean ring, then $(A, g \circ f, C)$ is a left (right or two-sided) Segal topological algebra.*

We end this part with the result about von Neumann regular ideals.

Corollary 7. *Let (A, f, B) and (B, g, C) be left (right or two-sided) Segal topological algebras. If either $g(f(A))$ is a von Neumann regular ideal of $g(B)$ or $g(B)$ is a von Neumann regular ideal of C, then $(A, g \circ f, C)$ is a left (respectively, right or two-sided) Segal topological algebra.*

Proof. If $g(f(A))$ is a von Neumann regular ideal of $g(B)$, then $g(f(A)) \subseteq g(f(A))g(f(A))$ and the result follows by Corollary 5.

Suppose that $g(f(A))$ is a von Neumann regular ideal of $g(B)$. Take any $c \in g(f(A))$. Then there exists $x \in g(f(A)) \subseteq g(B)$ such that $c = cxc$.

Notice that $cx \in g(B)$, which means that $g(f(A)) \subseteq g(B)g(f(A))$ and the claim follows by Theorem 1. □

We finish the paper with an open problem.

Open problem. Do there exist Segal topological algebras (A, f, B) and (B, g, C) such that $(A, g \circ f, C)$ is not a Segal topological algebra?

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School of Digital Technologies, Tallinn University, 25 Narva Str., 10120 Tallinn, Estonia; Institute of Mathematics and Statistics, University of Tartu, 18 Narva Str., 51009 Tartu, Estonia

E-mail address: `mart.abel@tlu.ee, mart.abel@ut.ee`