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Central part interpolation schemes for a class of fractional initial value problems

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Abstract. We consider an initial value problem for linear fractional integro-differential equations with weakly singular kernels. Using an integral equation reformulation of the underlying problem, a collocation method based on the central part interpolation by continuous piecewise polynomials on the uniform grid is constructed and analysed. Optimal convergence order of the proposed method is established and confirmed by numerical experiments.

1. Introduction

Differential equations with derivatives of fractional (non-integer) order have proved to be valuable tools in the modelling of many real-life processes, especially when modelling phenomena with memory properties [21]. Therefore theoretical and numerical analysis of equations involving fractional differential operators has become a very important area of research for many scientists. First of all, we refer here to the works [1, 5, 8, 19, 20, 24]. Some recent results concerning the numerical solution of fractional differential and integro-differential equations can be found in [4, 6, 7, 9, 16, 18, 26].

In the present paper we introduce and justify a high-order method for the numerical solution of a class of initial value problems for fractional integro-differential equations involving a Caputo fractional differential operator of order $\alpha \in (0, 1)$. First, the problem is reformulated as a weakly singular Volterra integral equation of the second kind. Then a smoothing change of variables is used to improve the boundary behaviour of the exact solution of the underlying problem. After that, a collocation method based on the central part interpolation by continuous piecewise polynomials on the uniform

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grid is constructed and analysed. The central part interpolation approach was first introduced in [12] for solving Fredholm integral equations of the second kind. The obtained numerical schemes show accuracy and numerical stability advantages compared to standard piecewise polynomial collocation methods, including collocation at Chebyshev knots [13]. In the present paper we will use some ideas and results of [12, 13].

Our paper is organized as follows. In Section 2 the exact problem setting is given. In Section 3 a result about the smoothness of the exact solution of the underlying problem is presented. Later this result will play a key role in the convergence analysis of the proposed method. In Section 4 a smoothing transformation is introduced and some of its properties are given. In Sections 5 and 6 a description of central part interpolation by polynomials and piecewise polynomials is presented. In Section 7 the attainable order of the proposed approach is studied and in Section 8 its matrix form is presented. Finally, in Section 9 the theoretical results are tested by some numerical experiments.

2. Problem setting

By $\mathbb{N}$ we denote the set of all positive integers $\{1, 2, \dots\}$, by $\mathbb{N}_0$ we define the set of all non-negative integers $\{0, 1, \dots\}$, by $\mathbb{Z}$ the set of all integers and by $\mathbb{R}$ the set of all real numbers $(-\infty, \infty)$. By $L^\infty(0,1)$ we denote the Banach space of measurable functions $u : [0,1] \to \mathbb{R}$ such that

$$\|u\|_{L^\infty(0,1)} = \inf_{\mathrm{meas}(\Omega)=0} \sup_{x\in(0,1)\setminus\Omega} |u(x)| < \infty,$$

where "meas$(\Omega) = 0$" means $\Omega \subset (0,1)$ is a measurable set with measure zero. By $C^m[0,1]$ we denote the set of m times ($m \in \mathbb{N}_0$, for $m = 0$ we set $C^0[0,1] = C[0,1]$) continuous functions $u : [0,1] \to \mathbb{R}$. By $C[0,1]$ we denote the Banach space of continous functions $u : [0,1] \to \mathbb{R}$ with the norm

$$\|u\|_{C[0,1]} = \|u\|_\infty = \max_{0\le x\le 1} |u(x)|.$$

We consider the following fractional initial value problem:

$$(D^\alpha_{Cap}u)(x) + r(x)\,u(x) + \int_0^x (x-y)^{-\kappa} K(x,y)u(y)dy = f(x), \quad 0 \le x \le 1, \tag{1}$$

$$u(0) = u_0. \tag{2}$$

Here $0 < \alpha < 1$, $0 \le \kappa < 1$, $u_0 \in \mathbb{R}$, $r, f \in C[0,1]$, $K \in C(\Delta)$,

$$\Delta := \{(x,y) \ : \ 0 \le y \le x \le 1\} \tag{3}$$

and $D^\alpha_{Cap}u$ is the α order Caputo fractional derivative of the unknown function $u = u(x)$. The Caputo derivative $D^\delta_{Cap}u$ of order $\delta \in (0,1)$ for

$u \in C[0,1]$ is defined by formula (see, e.g., [5])

$$(D^{\delta}_{Cap}u)(x) := (D^{\delta}_{RL}[u - u(0)])(x), \quad 0 < x \leq 1.$$

Here $D^{\delta}_{RL}u$ is the Riemann–Liouville fractional derivative of u:

$$(D^{\delta}_{RL}u)(x) := \frac{d}{dx}(J^{1-\delta}u)(x), \quad 0 < x \leq 1, \quad \delta \in (0,1),$$

with J^{δ}, the Riemann–Liouville integral operator, defined by

$$(J^{\delta}u)(x) := \frac{1}{\Gamma(\delta)} \int_0^x (x-y)^{\delta-1}\, u(y)\, dy, \quad x > 0, \quad \delta > 0; \quad J^0 := I, \quad (4)$$

where I is the identity mapping and

$$\Gamma(\delta) = \int_0^{\infty} y^{\delta-1} e^{-y} dy \quad (\delta > 0)$$

is the Euler gamma function. In [24], Vainikko has derived necessary and sufficient conditions for the existence of continuous functions $D^{\delta}_{RL}u$ and $D^{\delta}_{Cap}u$. In the following we are interested in solutions $u \in C[0,1]$ of the problem (1) – (2) such that $D^{\alpha}_{Cap}u \in C[0,1]$, $0 < \alpha < 1$.

It is well known (see, e.g., [3]) that J^{δ} ($\delta > 0$) is linear, bounded and compact as an operator from $L^{\infty}(0,1)$ into $C[0,1]$, and we have, for any $u \in L^{\infty}(0,1)$, that (see, e.g., [8])

$$J^{\delta}u \in C[0,1], \quad (J^{\delta}u)(0) = 0, \quad \delta > 0, \quad (5)$$

$$D^{\delta}_{RL}J^{\eta}u = D^{\delta}_{Cap}J^{\eta}u = J^{\eta-\delta}u, \quad 0 < \delta \leq \eta.$$

Using an integral equation reformulation of problem (1) – (2), we first study the existence, uniqueness and regularity of the exact solution u and its Caputo derivative $D^{\alpha}_{Cap}u$. We observe that (usual) derivatives of u may be unbounded near the left endpoint of the interval of integration $[0,1]$, even if r, f and K are infinitely differentiable on $[0,1]$ and Δ, respectively (see Theorem 1 below). Due to the lack of regularity of the exact solution, spline collocation methods on uniform grids for solving the underlying integral equation will show poor convergence behaviour. A better convergence can be established by using polynomial splines on special non-uniform grids, where the grid points are more densely clustered near the left endpoint of $[0,1]$, see, e.g., [2, 10]; in the case of fractional differential equations we refer to [11, 14, 15, 25]. A problem which may arise with strongly non-uniform grids is that they can create significant round-off errors in the calculations and lead to numerical instability. It is our aim, in the present paper, to construct and analyze a high order numerical method for problem (1) – (2) which does not need strongly graded grids. A suitable smoothing transformation will be introduced so that central part interpolation together with collocation techniques can be applied to the transformed integral equation on a uniform grid. We will study the attainable order of the proposed algorithm in a

situation where the higher order (usual) derivatives of $r(x)$ and $f(x)$ may be unbounded at $x = 0$.

3. Existence, uniqueness and smoothness of the solution

In what follows we use an integral equation reformulation of (1) – (2). Let $u \in C[0,1]$ be an arbitrary function such that $D^{\alpha}_{Cap}u \in C[0,1]$, where $0 < \alpha < 1$. Denote $z := D^{\alpha}_{Cap}u$. Then (cf. [5])

$$u(x) = (J^{\alpha}z)(x) + c, \tag{6}$$

where J^{α} is defined in (4) and c is a constant. Due to (5) a function of the form (6) satisfies the condition (2) if and only if

$$c = u_0,$$

that is, if $u(x)$ is determined by formula

$$u(x) = (J^{\alpha}z)(x) + u_0, \quad 0 \le x \le 1. \tag{7}$$

Let now $u \in C[0,1]$ be a solution of problem (1) – (2) such that $z = D^{\alpha}_{Cap}u \in C[0,1]$. From (1) it follows that

$$z(x) = f(x) - r(x)u(x) - \int_0^x (x-y)^{-\kappa}K(x,y)u(y)dy \tag{8}$$

and by substituting (8) into (7) we get

$$\begin{aligned} u(x) &= (J^{\alpha}[f - r\,u])(x) \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}\int_0^y (y-\tau)^{-\kappa}K(y,\tau)u(\tau)d\tau dy + u_0. \end{aligned}$$

By changing the order of integration we can write

$$\begin{aligned} &\int_0^x (x-y)^{\alpha-1}\int_0^y (y-\tau)^{-\kappa}K(y,\tau)u(\tau)d\tau dy \\ &\qquad = \int_0^x u(y)\int_y^x (x-\tau)^{\alpha-1}(\tau-y)^{-\kappa}K(\tau,y)d\tau dy. \end{aligned}$$

Using a change of variables $\tau = (x-y)\sigma + y$ we obtain

$$\begin{aligned} &\int_y^x (x-\tau)^{\alpha-1}(\tau-y)^{-\kappa}K(\tau,y)d\tau \\ &= (x-y)^{\alpha-\kappa}\int_0^1 \sigma^{-\kappa}(1-\sigma)^{\alpha-1}K((x-y)\sigma+y,y)d\sigma. \end{aligned}$$

Therefore

$$\int_0^x (x-y)^{\alpha-1}\int_0^y (y-\tau)^{-\kappa}K(y,\tau)u(\tau)d\tau dy = \int_0^x (x-y)^{\alpha-\kappa}L(x,y)u(y)dy,$$

where

$$L(x,y)=\int_0^1 \sigma^{-\kappa}(1-\sigma)^{\alpha-1}K((x-y)\sigma+y,y)d\sigma,\quad 0\le y\le x\le 1. \tag{9}$$

Thus u, the solution of (1) – (2), is also a solution of an integral equation in the form

$$u=Tu+g, \tag{10}$$

where, for $0\le x\le 1$, we have

$$(Tu)(x)=-\frac{1}{\Gamma(\alpha)}\int_0^x \left[(x-y)^{\alpha-1}r(y)+(x-y)^{\alpha-\kappa}L(x,y)\right]u(y)dy \tag{11}$$

and

$$g(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}f(y)dy+u_0. \tag{12}$$

Conversely, it is easy to see that if $u\in C[0,1]$ is a solution to (10), then u is also a solution to (1) – (2) such that $D^{\alpha}_{Cap}u\in C[0,1]$. In this sense equation (10) is equivalent to problem (1) – (2).

In order to study the existence and regularity properties of the exact solution of problem (1) – (2) we first establish some auxiliary results.

For given $q\in\mathbb{N}$ and $\nu\in\mathbb{R}$, $\nu<1$, by $C^{q,\nu}(0,1]$ we denote the set of continuous functions $u:[0,1]\to\mathbb{R}$ which are q times continuously differentiable in $(0,1]$ and such that, for all $x\in(0,1]$ and $i=1,\dots,q$, the following estimates hold (cf. [3, 23]):

$$\left|u^{(i)}(x)\right|\le c\begin{cases}1 & \text{if}\quad i<1-\nu,\\ 1+|\log x| & \text{if}\quad i=1-\nu,\\ x^{1-\nu-i} & \text{if}\quad i>1-\nu.\end{cases}$$

Here $c=c(u)$ is a positive constant. In other words, $u\in C^{q,\nu}(0,1]$ if $u\in C[0,1]\cap C^q(0,1]$ and

$$|u|_{q,\nu}:=\sum_{i=1}^{q}\sup_{0<x\le 1}\omega_{i-1+\nu}(x)\left|u^{(i)}(x)\right|<\infty,$$

where, for $x>0$, $\lambda\in\mathbb{R}$,

$$\omega_\lambda(x):=\begin{cases}1 & \text{if}\quad \lambda<0,\\ \frac{1}{1+|\log x|} & \text{if}\quad \lambda=0,\\ x^{\lambda} & \text{if}\quad \lambda>0.\end{cases}$$

Equipped with the norm $\|u\|_{C^{q,\nu}(0,1]}:=\|u\|_\infty+|u|_{q,\nu}$, the set $C^{q,\nu}(0,1]$ becomes a Banach space. Note that

$$C^q[0,1]\subset C^{q,\nu}(0,1]\subset C^{m,\mu}(0,1]\subset C[0,1],\quad q\ge m\ge 1,\quad \nu\le\mu<1.$$

In particular, a function of the form $u(x)=g_1(x)\,x^{\mu}+g_2(x)$ is included in $C^{q,\nu}(0,1]$ if $\mu\ge 1-\nu>0$ and $g_j\in C^q[0,1]$, $j=1,2$.

Next two lemmas follow from the corresponding results of [3].

Lemma 1. *If $v_1, v_2 \in C^{q,\nu}(0,1]$, $q \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$, then $v_1 v_2 \in C^{q,\nu}(0,1]$, and*

$$\|v_1 v_2\|_{C^{q,\nu}(0,1]} \le c\|v_1\|_{C^{q,\nu}(0,1]}\|v_2\|_{C^{q,\nu}(0,1]},$$

with a constant c which is independent of v_1 and v_2.

Lemma 2. *Let $\eta \in \mathbb{R}$, $\eta < 1$ and let $K \in C(\Delta)$ with Δ given by (3). Then operator S defined by*

$$(Sv)(x) = \int_0^x (x-y)^{-\eta} K(x,y)v(y)dy,\ x \in [0,1],$$

is compact as an operator from $L^\infty(0,1)$ into $C[0,1]$. If, in addition, $K \in C^q(\Delta), q \in \mathbb{N}$, then S is compact as an operator from $C^{q,\nu}(0,1]$ into $C^{q,\nu}(0,1]$, where $\eta \le \nu < 1$.

The existence, uniqueness and regularity of the solution to (1) – (2) (of the solution of equation (10)) can be characterized by the following theorem (cf. [15]).

Theorem 1. *Assume that $0 < \alpha < 1$, $0 \le \kappa < 1$, $r, f \in C[0,1]$ and $K \in C(\Delta)$. Then problem (1) – (2) possesses a unique solution $u \in C[0,1]$ such that $D^\alpha_{Cap} u \in C[0,1]$.*

Moreover, if $K \in C^q(\Delta)$, $r, f \in C^{q,\mu}(0,1]$, $q \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\mu < 1$, then u and its derivative $D^\alpha_{Cap} u$ belong to $C^{q,\nu}(0,1]$, where

$$\nu := \max\{1-\alpha, \mu, \kappa\}. \tag{13}$$

Proof. In order to prove that problem (1) – (2) possesses a unique solution $u \in C[0,1]$, we show that equation (10) is uniquely solvable in $C[0,1]$. We first note that due to $f \in C[0,1]$ and Lemma 2 the forcing function g of equation $u = Tu + g$ (see (10) and (12)) belongs to $C[0,1]$. Further, due to (11) operator T can be rewritten in the form

$$T = -J^\alpha R - T_{\alpha,\kappa,L}, \tag{14}$$

with R and $T_{\alpha,\kappa,L}$ defined by the following formulas:

$$(Rv)(x) = r(x)v(x), \quad x \in [0,1],$$

$$(T_{\alpha,\kappa,L}v)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-\kappa} L(x,y)v(y)dy, \quad x \in [0,1],$$

where the function L is given by the formula (9). Clearly, R is bounded as an operator from $C[0,1]$ into $C[0,1]$. It follows from Lemma 2 that J^α and $T_{\alpha,\kappa,L}$ are compact as operators from $C[0,1]$ into $C[0,1]$. Therefore T is compact as an operator from $C[0,1]$ into $C[0,1]$. Since the homogeneous equation $u = Tu$ has in $C[0,1]$ only the trivial solution $u = 0$, it follows from the Fredholm alternative theorem that equation (10) has in $C[0,1]$ a unique solution u. From this we obtain by equation (1) that $D^\alpha_{Cap} u \in C[0,1]$.

Assume now that $K \in C^q(\Delta)$, $r, f \in C^{q,\mu}(0,1]$, $q \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\mu < 1$. Let us prove that the solution u then belongs to $C^{q,\nu}(0,1]$ with ν given by the formula (13). With the help of Lemma 2 we observe that $g = J^\alpha f + u_0$, the forcing function of equation $u = Tu + g$, belongs to $C^{q,\nu}(0,1]$, since $\mu \le \nu < 1$ and hence $f \in C^{q,\mu}(0,1] \subset C^{q,\nu}(0,1]$. Further, we have also that $r \in C^{q,\mu}(0,1] \subset C^{q,\nu}(0,1]$. Therefore, by Lemma 1 we obtain that R is bounded as an operator from $C^{q,\nu}(0,1]$ into $C^{q,\nu}(0,1]$. Since $1 - \alpha \le \nu$, it follows from Lemma 2 that J^α is compact as an operator from $C^{q,\nu}(0,1]$ into $C^{q,\nu}(0,1]$. Thus, $J^\alpha R$ is linear and compact as an operator from $C^{q,\nu}(0,1]$ into $C^{q,\nu}(0,1]$. Moreover, since $\kappa - \alpha < 1 - \alpha \le \nu$, it follows from the definition of operator $T_{\alpha,\kappa,L}$ that it is compact in $C^{q,\nu}(0,1]$ (see Lemma 2). Therefore T defined by (14) is linear and compact as an operator from $C^{q,\nu}(0,1]$ into $C^{q,\nu}(0,1]$ and it follows from the Fredholm alternative theorem that equation $u = Tu + g$ has a unique solution $u \in C^{q,\nu}(0,1]$.

Finally, since $u, r, f \in C^{q,\nu}(0,1]$ and $K \in C^q(\Delta)$, we obtain with the help of equation (1) and Lemma 2 that $D^\alpha_{Cap} u \in C^{q,\nu}(0,1]$. □

We note that if $K = 0$ in Theorem 1, we actually have

$$\nu = \max\{1 - \alpha, \mu\}.$$

4. Smoothing transformation

The possible boundary singularities of the solution $u \in C^{q,\nu}(0,1]$ of equation (10) are generic, they occur for most free terms g even if g has no boundary singularities. To suppress the singularities of the solution we perform in equation (10) a change of variables using a suitable transformation φ. More precisely, let $\varphi : [0,1] \to [0,1]$ be defined by the formula

$$\begin{aligned} \varphi(t) &= \frac{1}{c_p} \int_0^t \sigma^{p-1}(1-\sigma)^{p-1} d\sigma, \quad 0 \le t \le 1, \quad p \in \mathbb{N}, \\ c_p &= \int_0^1 \sigma^{p-1}(1-\sigma)^{p-1} d\sigma = \frac{[(p-1)!]^2}{(2p-1)!}. \end{aligned} \tag{15}$$

We can see that φ is a polynomial,

$$\varphi(t) = \frac{1}{c_p} t^p \sum_{k=0}^{p-1} (-1)^k \frac{1}{k+p} \begin{pmatrix} p-1 \\ k \end{pmatrix} t^k.$$

Moreover, we see that $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi'(t) > 0$ for $0 < t < 1$. Thus φ is strictly increasing and there exists a unique continuous inverse $\varphi^{-1} : [0,1] \to [0,1]$.

If $p = 1$, then $\varphi(t) = t$ for $0 \le t \le 1$. We are interested in transformations (15) with $p > 1$ since then the transformation (15) possesses a smoothing

property for functions $u(x)$ with singularities of derivatives of $u(x)$ at $x = 0$. Namely, using some ideas and results of [17, 22] we can prove Lemma 3 below.

Lemma 3. *Let $q \in \mathbb{N}$ and $\nu \in \mathbb{R}$, $\nu < 1$. Let $u \in C^{q,\nu}(0,1]$ and $v(t) = u(\varphi(t))$, $t \in [0,1]$, where φ is defined by (15) with the parameter $p \in \mathbb{N}$ satisfying*

$$\begin{array}{ll} p > q & \text{for} \quad \nu \le 0\,, \\ p > \frac{q}{1-\nu} & \text{for} \quad 0 < \nu < 1. \end{array} \tag{16}$$

Then $v \in C^q[0,1]$ and

$$v^{(j)}(0) = v^{(j)}(1) = 0, \quad j = 1, \dots, q.$$

5. Central part interpolation by polynomials

Given an interval $[a,b]$ $(a < b)$ and $m \in \mathbb{N}$, introduce the uniform grid consisting of m points

$$x_i = a + \left(i - \frac{1}{2}\right) h, \quad i = 1, \dots, m, \quad h = \frac{b-a}{m}. \tag{17}$$

Denote by $\mathcal{P}_{m-1}$ the set of polynomials of degree not exceeding $m-1$ and by Π_m the Lagrange interpolation projection operator assigning to any $g \in C[a,b]$ the polynomial $\Pi_m g \in \mathcal{P}_{m-1}$ that interpolates g at points (17):

$$(\Pi_m g)(x) = \sum_{j=1}^{m} g(x_j) \prod_{\substack{k=1 \\ k \ne j}}^{m} \frac{x - x_k}{x_j - x_k}, \quad a \le x \le b, \quad m \ge 2,$$

$$(\Pi_1 g)(x) = g(x_1), \quad a \le x \le b.$$

The proof of the following lemma is given in [13], see also [12].

Lemma 4. *In the case of interpolation knots (17) with $m \in \mathbb{N}$, for $g \in C^m[a,b]$ it holds*

$$\max_{a \le x \le b} |g(x) - (\Pi_m g)(x)| \le \theta_m h^m \max_{a \le x \le b} \left| g^{(m)}(x) \right|, \tag{18}$$

with

$$\theta_m = \frac{(2m)!}{2^{2m}\,(m!)^2} \cong (\pi m)^{-\frac{1}{2}},$$

where $\theta_m \cong \epsilon_m$ means that $\theta_m/\epsilon_m \to 1$ as $m \to \infty$.

Further, for $m = 2k$, $k \ge 1$, the non-improvable estimate

$$\max_{x_k \le x \le x_{k+1}} |g(x) - (\Pi_m g)(x)| \le \vartheta_m h^m \max_{a \le x \le b} \left| g^{(m)}(x) \right| \tag{19}$$

holds with

$$\vartheta_m = 2^{-2m} \frac{m!}{((m/2)!)^2} \cong \sqrt{2/\pi}\; m^{-\frac{1}{2}} 2^{-m}, \tag{20}$$

whereas for $m = 2k+1$, $k \geq 1$, the non-improvable estimate

$$\max_{x_k \leq x \leq x_{k+2}} |g(x) - (\Pi_m g)(x)| \leq \vartheta_m h^m \max_{a \leq x \leq b} \left| g^{(m)}(x) \right| \tag{21}$$

holds with

$$\vartheta_m = \frac{2\sqrt{3}}{9} \frac{(k!)^2}{(2k+1)!} \cong \frac{2\sqrt{6\pi}}{9}\, m^{-\frac{1}{2}} 2^{-m}. \tag{22}$$

Comparing estimates (18) – (22) we observe that in the underlying central parts of $[a, b]$, the estimates for the error $g - \Pi_m g$ are approximately 2^m times more precise than on the whole interval $[a, b]$.

6. Central part interpolation by piecewise polynomials

Introduce in $\mathbb{R}$ the uniform grid

$$\{jh : j \in \mathbb{Z}\}, \quad h = \frac{1}{n}, \quad n \in \mathbb{N}.$$

Let $m \in \mathbb{N}$, $m \geq 2$ be fixed. Given a function $g \in C[-\delta, 1+\delta]$, $\delta > 0$, we define a piecewise polynomial interpolant $\Pi_{h,m} g \in C[0,1]$ for $h = \frac{1}{n} \leq \frac{2\delta}{m}$ as follows. On every subinterval $[jh, (j+1)h]$, $0 \leq j \leq n-1$, the function $\Pi_{h,m} g$ is defined independently from other subintervals as a polynomial $\Pi_{h,m}^{[j]} g \in \mathcal{P}_{m-1}$ of degree $\leq m-1$ by the conditions

$$\Pi_{h,m}^{[j]} g(lh) = g(lh), \quad l = j - \frac{m}{2} + 1, \ldots, j + \frac{m}{2} \quad \text{if } m \text{ is even},$$

$$\Pi_{h,m}^{[j]} g(lh) = g(lh), \quad l = j - \frac{m-1}{2}, \ldots, j + \frac{m-1}{2} \quad \text{if } m \text{ is odd}.$$

A unified writing form of these interpolation conditions is

$$\Pi_{h,m}^{[j]} g(lh) = g(lh), \quad \text{for} \quad l \in \mathbb{Z} \quad \text{such that} \quad l - j \in \mathbb{Z}_m, \tag{23}$$

where

$$\mathbb{Z}_m = \left\{ k \in \mathbb{Z} : -\frac{m}{2} < k \leq \frac{m}{2} \right\}.$$

For an "interior" knot jh, $1 \leq j \leq n-1$, interpolation conditions (23) contain the condition $\left(\Pi_{h,m}^{[j-1]} g\right)(jh) = g(jh)$ as well as the condition $\left(\Pi_{h,m}^{[j]} g\right)(jh) = g(jh)$, thus $\Pi_{h,m} g$ is uniquely defined at interior knots and $\Pi_{h,m} g$ is continuous on $[0,1]$. The one side derivatives of the interpolant $\Pi_{h,m} g$ at the interior knots may be different.

Introduce the Lagrange fundamental polynomials $L_{k,m} \in \mathcal{P}_{m-1}$, $k \in \mathbb{Z}_m$, satisfying $L_{k,m}(l) = \delta_{k,l}$ for $l \in \mathbb{Z}_m$, where $\delta_{k,l}$ is the Kronecker symbol, $\delta_{k,l} = 0$ for $k \neq l$ and $\delta_{k,k} = 1$. An explicit formula for $L_{k,m}$ is given by

$$L_{k,m}(t) = \prod_{l \in \mathbb{Z}_m \setminus \{k\}} \frac{t-l}{k-l}, \quad k \in \mathbb{Z}_m. \tag{24}$$

We claim for $0 \leq j \leq n-1$ that

$$\left(\Pi_{h,m}^{[j]} g\right)(t) = \sum_{k \in \mathbb{Z}_m} g\left((j+k)\,h\right) L_{k,m}\left(nt - j\right), \quad t \in [jh, (j+1)h]. \tag{25}$$

Indeed, $\Pi_{h,m}^{[j]} g$ defined by (25) is really a polynomial of degree $\leq m-1$ and it satisfies interpolation conditions (23): for l with $l - j \in \mathbb{Z}_m$, it holds that

$$\left(\Pi_{h,m}^{[j]} g\right)(lh) = \sum_{k \in \mathbb{Z}_m} g\left((j+k)\,h\right) L_{k,m}\left(l - j\right)$$

$$= \sum_{k \in \mathbb{Z}_m} g\left((j+k)\,h\right) \delta_{k,l-j} = g\left((j + (l-j))\,h\right) = g\left(lh\right).$$

For $m = 2$, the interpolant $\Pi_{h,2} g$ is the usual piecewise linear function joining for $0 \leq j \leq n-1$ the pair of points

$$(jh, g\left(jh\right)) \in \mathbb{R}^2 \text{ and } \left((j+1)\,h, g\left((j+1)\,h\right)\right) \in \mathbb{R}^2$$

by a straight line and $\Pi_{h,2} g$ does not use the values of g outside $[0,1]$.

For $m \geq 3$, $\Pi_{h,m} g$ uses values of g outside of $[0,1]$. For $g \in C\left[0,1\right]$, $\Pi_{h,m} g$ obtains a sense after an extension of g onto $[-\delta, 1+\delta]$ with $\delta \geq \frac{m}{2} h$. In our work we will consider the functions $g \in C^m\left[0,1\right]$ that satisfy the boundary conditions (recall Lemma 3)

$$g^{(j)}\left(0\right) = g^{(j)}\left(1\right) = 0, \; j = 1, \ldots, m.$$

Then the simplest extension operator

$$E_\delta : C\left[0,1\right] \to C\left[-\delta, 1+\delta\right], \left(E_\delta g\right)(t) = \begin{Bmatrix} g\left(0\right) & \text{for} & -\delta \leq t \leq 0 \\ g\left(t\right) & \text{for} & 0 \leq t \leq 1 \\ g\left(1\right) & \text{for} & 1 \leq t \leq 1+\delta \end{Bmatrix}$$

maintains the smoothness of g. Therefore, we can define an operator $P_{h,m} : C\left[0,1\right] \to C\left[0,1\right]$, $m \geq 2$, as follows:

$$P_{h,m} := \Pi_{h,m} E_\delta. \tag{26}$$

We see that $P_{h,m}$ is well defined and $P_{h,m}^2 = P_{h,m}$, i.e., $P_{h,m}$ is a projector in $C\left[0,1\right]$.

For $w_h \in \mathcal{R}\left(P_{h,m}\right)$ (the range of $P_{h,m}$) we have $w_h = P_{h,m} w_h = \Pi_{h,m} E_\delta w_h$ and due to (25) we get for $t \in [jh, (j+1)h]$ $(j = 0, \ldots, n-1)$ that

$$w_h(t) = \sum_{k \in \mathbb{Z}_m} (E_\delta w_h)((j+k)h) L_{k,m}(nt - j), \tag{27}$$

where

$$(E_\delta w_h)(ih) = \begin{Bmatrix} w_h(ih) & \text{for} & i = 0, \ldots, n \\ w_h(0) & \text{for} & i < 0 \\ w_h(1) & \text{for} & i > n \end{Bmatrix}.$$

Thus $w_h \in \mathcal{R}\left(P_{h,m}\right)$ is uniquely determined on $[0,1]$ by its knot values

$w_h(ih)$, $i = 0, \ldots, n$. We conclude that $\dim \mathcal{R}(P_{h,m}) = n + 1$. It is also clear that for a $w_h \in \mathcal{R}(P_{h,m})$ we have $w_h = 0$ if and only if $w_h(ih) = 0$, $i = 0, \ldots, n$.

For $g \in C[-\delta, 1+\delta]$, the interpolant $\Pi_{h,m} g$ is closely related to the central part interpolation of g on the uniform grid treated in Section 5. On $[jh, (j+1)h]$, the interpolant $\Pi_{h,m} g = \Pi_{h,m}^{[j]} g$ coincides with the polynomial interpolant $\Pi_m g$ constructed for g on the interval $[a_j, b_j]$ where

$$a_j = \left(j - \frac{m-1}{2}\right) h, \quad b_j = \left(j + \frac{m+1}{2}\right) h \quad \text{(in the case of even } m\text{)},$$

$$a_j = \left(j - \frac{m}{2}\right) h, \quad b_j = \left(j + \frac{m}{2}\right) h \quad \text{(in the case of odd } m\text{)}.$$

Moreover, $[jh, (j+1)h]$ is contained in the central part of $[a_j, b_j]$ on which the interpolation error can be estimated by (19) in the case of even m and by (21) in the case of odd m. It follows from [13] that we have the following results.

Lemma 5. *Let $m \in \mathbb{N}, m \geq 2, h = \frac{1}{n}, n \in \mathbb{N}$. Let the operator $P_{h,m}$ be defined by the formula (26). Then we have for any $g \in C[0,1]$ that*

$$\|g - P_{h,m} g\|_\infty \to 0 \text{ as } n \to \infty.$$

Moreover, for $g \in C^m[0,1]$, $g^{(j)}(0) = g^{(j)}(1) = 0$, $j = 1, \ldots, m$, we have

$$\|g - P_{h,m} g\|_\infty \leq \vartheta_m h^m \max_{0 \leq t \leq 1} \left| g^{(m)}(t) \right|, \tag{28}$$

where ϑ_m is defined by (20) for even m and by (22) for odd m.

7. Collocation method based on the central part interpolation

Let E be a Banach space. In what follows by $\mathcal{L}(E) = \mathcal{L}(E, E)$ we denote the Banach space of linear bounded operators $A : E \to E$ with the norm

$$\|A\|_{\mathcal{L}(E)} = \sup\{\|Ax\|_E : x \in E, \|x\|_E \leq 1\}.$$

Let $p \in \mathbb{N}$ and let $\varphi : [0,1] \to [0,1]$ be defined by the formula (15). After a change of variables

$$x = \varphi(t), \quad y = \varphi(s), \quad 0 \leq t \leq 1, \quad 0 \leq s \leq 1,$$

equation (10) takes the form

$$v(t) = \int_0^t k_\varphi(t,s) v(s) ds + g_\varphi(t), \quad 0 \leq t \leq 1, \tag{29}$$

with

$$g_\varphi(t) = g(\varphi(t)), \quad 0 \leq t \leq 1,$$

$$k_\varphi(t,s) = -\frac{1}{\Gamma(\alpha)}\Big[(\varphi(t)-\varphi(s))^{\alpha-1} r(\varphi(s)) + (\varphi(t)-\varphi(s))^{\alpha-\kappa} L(\varphi(t),\varphi(s))\Big]\varphi'(s),$$

where $0 \le s < t \le 1$ and L is given by the formula (9). Since $f \in C[0,1]$, it follows from (12) and Lemma 2 that $g_\varphi \in C[0,1]$. We rewrite equation (29) in the form

$$v = T_\varphi v + g_\varphi,$$

where the underlying Volterra integral operator T_φ is defined by

$$(T_\varphi v)(t) = \int_0^t k_\varphi(t,s)v(s)ds, \quad 0 \le t \le 1.$$

Observe that under the stated assumptions on φ, r, K, α and κ the kernel $k_\varphi(t,s)$ is a continuous function for $0 \le s < t \le 1$. Moreover, we have

$$|k_\varphi(t,s)| \le c\,(t-s)^{\alpha-1}, \quad 0 \le s < t \le 1,$$

where c is a positive constant. Therefore $T_\varphi \in \mathcal{L}(C[0,1])$ is a compact Volterra integral operator and thus the homogeneous equation

$$v = T_\varphi v$$

corresponding to equation (29) has in $C[0,1]$ only the trivial solution $v = 0$, implying that equation (29) has in $C[0,1]$ a unique solution $v \in C[0,1]$.

The solutions of (10) (of (1)-(2)) and (29) are related by the equalities

$$v(t) = u(\varphi(t)), \quad u(x) = v(\varphi^{-1}(x)). \tag{30}$$

Using the interpolation projector $P_{h,m}$ defined in (26), we approximate equation (29) by equation

$$v_h = P_{h,m}T_\varphi v_h + P_{h,m}g_\varphi. \tag{31}$$

This is the operator form of our method for finding approximate solutions to equation (29). The matrix form of (31) is given in Section 8.

The approximation u_h for u, the exact solution of equation (10) (of problem (1) – (2)), is defined by the formula (see (30))

$$u_h(x) := v_h(\varphi^{-1}(x)), \quad 0 \le x \le 1. \tag{32}$$

Theorem 2. (i) *Assume that* $0 < \alpha < 1$, $0 \le \kappa < 1$, $K \in C(\Delta)$, $r, f \in C[0,1]$. *Let the smoothing transformation* $\varphi : [0,1] \to [0,1]$ *be defined by* (15) *and let* $m \in \mathbb{N}$, $m \ge 2$. *Then there exists an* $n_0 \in \mathbb{N}$ *such that for* $n \ge n_0$ *the collocation equation* (31) *has a unique solution* $v_h \in C[0,1]$ *such that*

$$\|u - u_h\|_\infty = \|v - v_h\|_\infty \to 0 \quad \textit{for} \quad n \to \infty, \tag{33}$$

where $u \in C[0,1]$ *is the solution of* (10) *(of problem* (1) – (2)*),* $v(t) = u(\varphi(t))$ $(0 \le t \le 1)$ *is the solution of* (29) *and* $u_h \in C[0,1]$ *is defined by the formula* (32).

(ii) *In addition to* (i), *let* $K \in C^m(\Delta)$, $r, f \in C^{m,\mu}(0,1]$, $m \geq 2$, $\mu \in \mathbb{R}$, $\mu < 1$ *and let the parameter* $p \in \mathbb{N}$ *of the smoothing transformation* $\varphi : [0,1] \to [0,1]$ *defined by* (15) *satisfy* (16) *with* $q = m$ *and* ν *defined by* (13). *Then the following error estimate holds:*

$$\|u - u_h\|_\infty = \|v - v_h\|_\infty \leq c\,\vartheta_m\, h^m \|v^{(m)}\|_\infty, \quad h = \frac{1}{n}, \quad n \geq n_0. \tag{34}$$

Here c *is defined by the formula* (36) *below, and* ϑ_m *is defined by* (20) *for even* m *and by* (22) *for odd* m.

Proof. Since $T_\varphi \in \mathcal{L}(C[0,1])$ is compact and the homogeneous equation $v = T_\varphi v$ has in $C[0,1]$ only the trivial solution $v = 0$, the bounded inverse $(I - T_\varphi)^{-1} : C[0,1] \to C[0,1]$ exists by the Fredholm alternative (here I is the identity mapping in $C[0,1]$). The compactness of T_φ together with the pointwise convergence of $P_{h,m}$ to I in $C[0,1]$ (see Lemma 5) implies the following norm convergence:

$$\|P_{h,m}T_\varphi - T_\varphi\|_{\mathcal{L}(C[0,1])} \to 0 \quad \text{as } n \to \infty \ (\text{as } h = \frac{1}{n} \to 0).$$

We pick an $n_0 \in \mathbb{N}$ such that

$$\delta_{n_0} := \sup_{n \in \mathbb{N}, n \geq n_0} \|P_{h,m}T_\varphi - T_\varphi\|_{\mathcal{L}(C[0,1])} < \frac{1}{\|(I - T_\varphi)^{-1}\|_{\mathcal{L}(C[0,1])}}.$$

It follows from this that the inverse

$$\left[I + (I - T_\varphi)^{-1}(T_\varphi - P_{h,m}T_\varphi)\right]^{-1} : C[0,1] \to C[0,1]$$

exists and is uniformly bounded for $n \geq n_0$:

$$\|[I + (I - T_\varphi)^{-1}(T_\varphi - P_{h,m}T_\varphi)]^{-1}\|_{\mathcal{L}(C[0,1])} \leq \frac{1}{1 - \delta_{n_0}\|(I - T_\varphi)^{-1}\|_{\mathcal{L}(C[0,1])}}.$$

Further, we have

$$I - P_{h,m}T_\varphi = (I - T_\varphi)\left[I + (I - T_\varphi)^{-1}(T_\varphi - P_{h,m}T_\varphi)\right]$$

and therefore we get for $n \geq n_0$ that the operators $I - P_{h,m}T_\varphi$ are invertible in $C[0,1]$ and the norms of $(I - P_{h,m}T_\varphi)^{-1}$ are uniformly bounded:

$$\|(I - P_{h,m}T_\varphi)^{-1}\|_{\mathcal{L}(C[0,1])} \leq \frac{\|(I - T_\varphi)^{-1}\|_{\mathcal{L}(C[0,1])}}{1 - \delta_{n_0}\,\|(I - T_\varphi)^{-1}\|_{\mathcal{L}(C[0,1])}}.$$

This proves the unique solvability of the collocation equation (31) for $n \geq n_0$:

$$v_h = (I - P_{h,m}T_\varphi)^{-1}P_{h,m}g_\varphi, \quad n \geq n_0.$$

Let v and v_h be the solutions of (29) and (31), respectively. Then

$$(I - P_{h,m}T_\varphi)(v - v_h) = v - P_{h,m}T_\varphi v - P_{h,m}g_\varphi = v - P_{h,m}v,$$
$$v - v_h = (I - P_{h,m}T_\varphi)^{-1}(v - P_{h,m}v), \quad n \geq n_0.$$

Thus we can write

$$\|u-u_h\|_\infty = \|v-v_h\|_\infty \le c\,\|v-P_{h,m}v\|_\infty, \quad h=\frac{1}{n},\ n\ge n_0, \tag{35}$$

where

$$c := \frac{\|(I-T_\varphi)^{-1}\|_{\mathcal{L}(C[0,1])}}{1-\delta_{n_0}\,\|(I-T_\varphi)^{-1}\|_{\mathcal{L}(C[0,1])}}. \tag{36}$$

Due to Lemma 5 the convergence (33) now follows from (35).

In the case of (ii) it follows from Theorem 1 that the solution u of (10) (of (1)–(2)) belongs to $C^{m,\nu}(0,1]$; by Lemma 3 we have for $v(t)=u(\varphi(t))$ that $v\in C^m[0,1]$ and $v^{(j)}(0)=v^{(j)}(1)=0$, $j=1,\dots,m$; by Lemma 5 (see (28)) we obtain that $\|v-P_{h,m}v\|_\infty \le \vartheta_m h^m \|v^{(m)}\|_\infty$. This together with (35) yields (34). □

8. Matrix form of the method (31)

The solution v_h of equation (31) belongs to $\mathcal{R}(P_{h,m})$, so the knot values $v_h(ih)$ $(i=0,\dots,n)$ determine v_h uniquely. Equation (31) is equivalent to a system of linear algebraic equations with respect to $v_h(ih)$, $i=0,\dots,n$, and our task is to write down this system.

For $w_h\in\mathcal{R}(P_{h,m})$ we have $w_h=0$ if and only if $w_h(ih)=0$, $i=0,\dots,n$. Since $(P_{h,m}w_h)(ih)=w_h(ih)$, $i=0,\dots,n$, equation (31) is equivalent to the conditions

$$v_h(ih)=(T_\varphi v_h)(ih)+g_\varphi(ih), \quad i=0,\dots,n,$$

i.e. $v_h\in\mathcal{R}(P_{h,m})$ satisfies equation (31) (equation (29)) at the knots ih, $i=0,\dots,n$. Clearly, $v_h(0)=g_\varphi(0)$. Using for v_h the representation (27) we obtain for $1\le i\le n$ that

$$(T_\varphi v_h)(ih)=\int_0^{ih} k_\varphi(ih,s)v_h(s)ds=\sum_{j=0}^{i-1}\int_{jh}^{(j+1)h} k_\varphi(ih,s)v_h(s)ds$$

$$=\sum_{j=0}^{i-1}\sum_{k\in\mathbb{Z}_m}\int_{jh}^{(j+1)h} k_\varphi(ih,s)L_{k,m}(ns-j)ds\,(E_\delta v_h)((j+k)h)$$

$$=\sum_{j=0}^{i-1}\sum_{k\in\mathbb{Z}_m}\alpha_{i,j,k}\cdot\left\{\begin{array}{lll} v_h(0) & \text{for} & j+k\le 0\\ v_h((j+k)h) & \text{for} & 1\le j+k\le n-1\\ v_h(1) & \text{for} & j+k\ge n\end{array}\right\}.$$

Here

$$\alpha_{i,j,k}=\int_{jh}^{(j+1)h} k_\varphi(ih,s)L_{k,m}(ns-j)ds,\ i=1,\dots,n,\ j=0,\dots,i-1,\ k\in\mathbb{Z}_m.$$

We denote

$$b_{i,l} = \left\{ \begin{array}{ll} \sum_{k\in\mathbb{Z}_m} \sum_{\{j:0\le j\le i-1, j+k\le 0\}} \alpha_{i,j,k}, & \text{for} \quad l=0 \\ \sum_{k\in\mathbb{Z}_m} \sum_{\{j:0\le j\le i-1, j+k=l\}} \alpha_{i,j,k}, & \text{for} \quad 1\le l\le n-1 \\ \sum_{k\in\mathbb{Z}_m} \sum_{\{j:0\le j\le i-1, j+k\ge n\}} \alpha_{i,j,k}, & \text{for} \quad l=n \end{array} \right\}$$

for $i = 1, \ldots, n,\ l = 0, \ldots, n$, and

$$b_{0,l} = 0, \text{ for } l = 0, \ldots, n.$$

Thus,

$$(T_\varphi v_h)(ih) = \sum_{l=0}^{n} b_{i,l} v_h(lh),\ i = 0, \ldots, n.$$

We see that the matrix form for finding v_h by method (31) is equivalent to finding the solution to the following linear algebraic system:

$$v_h(ih) = \sum_{l=0}^{n} b_{i,l} v_h(lh) + g_\varphi(ih), \quad i = 0, \ldots, n. \tag{37}$$

Having determined $v_h(ih)$ $(i = 0, \ldots, n)$ by solving the system (37), the collocation solution $v_h(t)$ at any intermediate point $t \in [jh, (j+1)h]$ $(j = 0, \ldots, n-1)$, is given by

$$v_h(t) = \sum_{k\in\mathbb{Z}_m} \left\{ \begin{array}{cc} v_h(0) & \text{for} \quad j+k\le 0 \\ v_h((j+k)h) & \text{for} \quad 1\le j+k\le n-1 \\ v_h(1) & \text{for} \quad j+k\ge n \end{array} \right\} L_{k,m}(nt-j),$$

with $L_{k,m}$ $(k \in \mathbb{Z}_m)$ defined by (24). Finally, the approximate solution u_h of u, the solution to (10) (to problem (1)–(2)), is determined by the formula (32).

9. Numerical experiments

Example 1. Consider the following initial value problem:

$$(D^{0.4}_{Cap} u)(x) + x^{0.6} u(x) = f(x), \quad x \in [0,1], \quad u(0) = 1, \tag{38}$$

with

$$f(x) = \frac{\Gamma(1.9)}{\Gamma(1.5)} x^{0.5} + x^{0.6} + x^{1.5}, \quad x \in [0,1].$$

We see that this is a special problem of (1)–(2) with

$$\alpha = 0.4,\ u_0 = 1;\ r(x) = x^{0.6},\ x \in [0,1]; \quad K(x,y) = 0, \quad 0 \le y \le x \le 1.$$

Clearly $K \in C^m(\Delta)$, $r \in C^{m,0.4}(0,1] \subset C^{m,\mu}(0,1]$, $f \in C^{m,\mu}(0,1]$ with $\mu = 0.5$ and for all $m \in \mathbb{N}$. The exact solution of problem (38) is given by the formula

$$u(x) = x^{0.9} + 1, \quad x \in [0,1].$$

In order to find a numerical solution to problem (38) for $m \geq 2$, we assemble and solve the system (37). The smoothing parameter p in the definition (15) of φ is chosen to be greater than $\frac{m}{1-\nu}$, where $\nu = \max\{1-\alpha, \mu\} = 0.6$. Thus we have to take $p > 2.5m$ to achieve the expected convergence order $O(h^m)$ given by Theorem 2.

In Table 1, the errors

$$\varepsilon_n := \max_{0 \leq j \leq 10n} \left\{ u\left(\frac{j}{10n}\right) - u_h\left(\frac{j}{10n}\right) \right\} \quad (n = 2^k,\ k = 3, \ldots, 8)$$

are presented. Here u is the exact solution of problem (38) and u_h, the approximate solution to (38), is obtained by the method proposed above (see (32)). Additionally, the quotients $\varepsilon_{n/2}/\varepsilon_n$ for different values of m, n and p are presented. Due to Theorem 2, the expected limit value of $\varepsilon_{n/2}/\varepsilon_n$ is 2^m. These values are given in the last row of Table 1. As we can see, the obtained numerical results are in good agreement with the theoretical estimates.

TABLE 1. Numerical results for problem (38).

	$m = 2$, $p = 6$		$m = 3$, $p = 8$		$m = 4$, $p = 11$	
n	ε_n	$\varepsilon_{n/2}/\varepsilon_n$	ε_n	$\varepsilon_{n/2}/\varepsilon_n$	ε_n	$\varepsilon_{n/2}/\varepsilon_n$
8	$1.96 \cdot 10^{-2}$		$1.40 \cdot 10^{-2}$		$1.14 \cdot 10^{-2}$	
16	$4.85 \cdot 10^{-3}$	4.03	$1.94 \cdot 10^{-3}$	7.21	$9.55 \cdot 10^{-4}$	11.96
32	$1.25 \cdot 10^{-3}$	3.88	$2.62 \cdot 10^{-4}$	7.41	$6.51 \cdot 10^{-5}$	14.66
64	$3.11 \cdot 10^{-4}$	4.02	$3.27 \cdot 10^{-5}$	8.01	$4.08 \cdot 10^{-6}$	15.95
128	$7.80 \cdot 10^{-5}$	3.98	$4.09 \cdot 10^{-6}$	7.99	$2.56 \cdot 10^{-7}$	15.93
256	$1.94 \cdot 10^{-5}$	4.00	$5.10 \cdot 10^{-7}$	8.03	$1.60 \cdot 10^{-8}$	16.00
		4		8		16

Example 2. Consider the following initial value problem:

$$(D^{0.5}_{Cap}u)(x) - xu(x) + \int_0^x (x-y)^{-0.3}u(y)dy = f(x),\ x \in [0,1], \quad u(0) = -1, \tag{39}$$

where

$$f(x) = \frac{\Gamma(2.1)}{\Gamma(1.6)}x^{0.6} + x - x^{2.1} + \frac{\Gamma(0.7)\Gamma(2.1)}{\Gamma(2.8)}x^{1.8} - \frac{10}{7}x^{0.7}, \quad x \in [0,1].$$

We see that this is a special problem of (1) – (2) with

$$\alpha = 0.5,\ \kappa = 0.3,\ u_0 = -1$$

and

$$r(x) = -x, \quad 0 \leq x \leq 1; \quad K(x,y) = 1, \quad 0 \leq y \leq x \leq 1.$$

Clearly $K \in C^m(\Delta)$, $r \in C^m[0,1] \subset C^{m,\mu}(0,1]$, $f \in C^{m,\mu}(0,1]$ with $\mu = 0.4$ and for all $m \in \mathbb{N}$. The exact solution of problem (39) is given by the formula

$$u(x) = x^{1.1} - 1, \quad x \in [0,1].$$

In order to find a numerical solution to problem (39) for $m \geq 2$, we assemble and solve the system (37). The smoothing parameter p in the definition (15) of φ is chosen to be greater than $\frac{m}{1-\nu}$, where $\nu = \max\{1 - \alpha, \mu, \kappa\} = 0.5$. Thus we have to take $p > 2\,m$ to achieve the expected convergence order $O(h^m)$ given by Theorem 2.

In Table 2, the errors ε_n are presented similarly as in Table 1. Due to Theorem 2, the expected limit value of $\varepsilon_{n/2}/\varepsilon_n$ is 2^m. These values are given in the last row of Table 2. As we can see, the obtained numerical results agree with the theoretical estimates.

TABLE 2. Numerical results for problem (39).

	$m = 2$, $p = 5$		$m = 3$, $p = 7$		$m = 4$, $p = 9$	
n	ε_n	$\varepsilon_{n/2}/\varepsilon_n$	ε_n	$\varepsilon_{n/2}/\varepsilon_n$	ε_n	$\varepsilon_{n/2}/\varepsilon_n$
8	$2.58 \cdot 10^{-2}$		$1.82 \cdot 10^{-2}$		$1.10 \cdot 10^{-2}$	
16	$6.79 \cdot 10^{-3}$	3.80	$2.46 \cdot 10^{-3}$	7.37	$1.11 \cdot 10^{-3}$	9.85
32	$1.73 \cdot 10^{-3}$	3.90	$3.14 \cdot 10^{-4}$	7.85	$7.87 \cdot 10^{-5}$	14.20
64	$4.40 \cdot 10^{-4}$	3.94	$3.93 \cdot 10^{-5}$	7.97	$5.06 \cdot 10^{-6}$	15.55
128	$1.10 \cdot 10^{-4}$	3.97	$4.95 \cdot 10^{-6}$	7.95	$3.19 \cdot 10^{-7}$	15.83
256	$2.77 \cdot 10^{-5}$	3.98	$6.19 \cdot 10^{-7}$	7.99	$2.01 \cdot 10^{-8}$	15.88
		4		8		16

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References

[1] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, World Scientific, Boston, 2016.

[2] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge University Press, Cambridge, 2004.

[3] H. Brunner, A. Pedas, and G. Vainikko, *Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels*, SIAM J. Numer. Anal. **30** (2001), 957–982.

[4] A. Cardone, D. Conte, R. D'Ambrosio, and B. Paternoster, *Multivalue collocation methods for ordinary and fractional differential equations*, Mathematics **10**, 185 (2022), https://doi.org/10.3390/math10020185.

[5] K. Diethelm. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer-Verlag, Berlin, 2010.

[6] N. J. Ford, M. L. Morgado, and M. Rebelo, *A nonpolynomial collocation method for fractional terminal value problems*, J. Comput. Appl. Math. **275** (2015), 392–402.
[7] R. Garrappa, *Numerical solution of fractional differential equations: A survey and a software tutorial*, Mathematics **6**, 16 (2018), https://doi.org/10.3390/math6020016.
[8] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
[9] M. Kolk, A. Pedas, and E. Tamme, *Smoothing transformation and spline collocation for linear fractional boundary value problems*, Appl. Math. Comput. **283** (2016), 234–250.
[10] M. Kolk, A. Pedas, and G. Vainikko, *High-order methods for Volterra integral equations with general weak singularities*, Numer.Funct.Anal.Optim. **30** (2009), 1002–1024.
[11] H. Liang and M. Stynes, *Collocation methods for general Caputo two-point boundary value problems*, J. Sci. Comput. **76** (2018), 390–425.
[12] K. Orav-Puurand and G. Vainikko, *Central part interpolation schemes for integral equations*, Numer. Funct. Anal. Optim. **30** (2009), 352–370.
[13] K. Orav-Puurand, A. Pedas, and G. Vainikko, *Central part interpolation schemes for integral equations with singularities*, J. Integral Equations Appl. **29** (2017), 401–440.
[14] A. Pedas and E. Tamme, *Spline collocation for nonlinear fractional boundary value problems*, Appl. Math. Comput. **244** (2014), 502–513.
[15] A. Pedas, E. Tamme, and M. Vikerpuur, *Spline collocation for fractional weakly singular integro-differential equations*, Appl. Numer. Math. **110** (2016), 204–214.
[16] A. Pedas, E. Tamme, and M. Vikerpuur, *Numerical solution of linear fractional weakly singular integro-differential equations with integral boundary conditions*, Appl. Numer. Math. **149** (2020), 124–140.
[17] A. Pedas and G. Vainikko, *Smoothing transformation and piecewise polynomial collocation for weakly singular Volterra integral equations*, Computing **73** (2004), 271–293.
[18] A. Pedas and M. Vikerpuur, *Spline collocation for multi-term fractional integro-differential equations with weakly singular kernels*, Fractal Fract. **5**, 90 (2021), https://doi.org/10.3390/fractalfract5030090.
[19] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[20] S. Samko, A. A. Kilbas, and O. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science Publishers, Yverdon, 1993.
[21] H. G. Sun, Y. Zhang, D. Baleanu, W. Chen, and Y. Q. Chen, *A new collection of real world applications of fractional calculus in science and engineering*, Commun. Nonlinear Sci. Numer. Simul. **64** (2018), 213–231.
[22] E. Vainikko and G. Vainikko, *A spline product quasi-interpolation method for weakly singular Fredholm integral equations*, SIAM J. Numer. Anal. **46** (2008), 1799–1820.
[23] G. Vainikko, *Multidimensional Weakly Singular Integral Equations*, Springer-Verlag, Berlin, 1993.
[24] G. Vainikko, *Which functions are fractionally differentiable?*, Z. Anal. Anwend. **35** (2016), 465–487.
[25] M. Vikerpuur, *Two collocation type methods for fractional differential equations with non-local boundary conditions*, Math. Model. Anal. **22** (2017), 654–670.
[26] J. Zhao, J. Xiao, and N. J. Ford, *Collocation methods for fractional integro-differential equations with weakly singular kernels*, Numer. Algorithms **65** (2014), 723–743.

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