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Properties of θ-$\mathcal{H}$-compact sets in hereditary m-spaces

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Abstract. Let $(X, m, \mathcal{H})$ be a hereditary m-space. A subset A of X is said to be θ-$\mathcal{H}$-compact relative to X if for every cover $\mathcal{U}$ of A by $m(\theta)$-open sets of X, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \setminus \cup \mathcal{U}_0 \in \mathcal{H}$. We obtain several properties of these sets. Also, we define and investigate two kinds of strong forms of "θ-$\mathcal{H}$-compact relative to X".

1. Introduction

In 1967, Newcomb [9] introduced the notion of compactness modulo an ideal. Rančin [12] and Hamlett and Janković [4] further investigated this notion and obtained some more properties of compactness modulo an ideal. Jafari et al. [5] introduced and studied compactness via ideals called θ-I-compactness. Császár [3] introduced the notion of hereditary classes as a generalization of ideals. In [10], a minimal structure and a minimal space (X, m) are introduced and investigated.

In this paper, we define a subset A of a hereditary m-space $(X, m, \mathcal{H})$ to be θ-$\mathcal{H}$-compact relative to X if for every cover $\mathcal{U}$ of A by $m(\theta)$-open sets of X, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \setminus \cup\, \mathcal{U}_0 \in \mathcal{H}$. We obtain several properties of these sets. For example, if A is θ-$\mathcal{H}$-compact relative to X and B is $m(\theta)$-closed, then $A \cap B$ is θ-$\mathcal{H}$-compact relative to X (Theorem 4). And also, we define and investigate two kinds of strong forms of "θ-$\mathcal{H}$-compact relative to X". Moreover, for a function $f : (X, m, \mathcal{H}) \to (Y, n)$ we define a hereditary class $J_H = \{B \subset Y : f^{-1}(B) \in \mathcal{H}\}$ and by using hereditary classes $f(\mathcal{H})$ and J_H on Y we obtain several preservation theorems. For example, if $f : (X, m, \mathcal{H}) \to (Y, n)$ is a quasi $m(\theta)$-continuous function and A is super

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θ-$\mathcal{H}$-compact relative to X, then $f(A)$ is super θ-$\mathcal{J}_H$-compact relative to Y (Theorem 19). Also papers [1, 2] have introduced some property related to θ-$\mathcal{H}$-compact sets in hereditary m-spaces.

2. Preliminaries

Definition 1. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [10] on X if $\emptyset \in m$ and $X \in m$.

By (X, m) we denote a nonempty set X with a minimal structure m on X and call it an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*. For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in m\}$ is denoted by $m(x)$.

Definition 2. Let (X, m) be an m-space and A a subset of X. The *m-closure* $\mathrm{mCl}(A)$ of A [8] is defined by $\mathrm{mCl}(A) = \cap\{F \subset X : A \subset F, X \setminus F \in m\}$.

Lemma 1 (Maki et al. [8]). *Let X be a nonempty set and m a minimal structure on X. For subsets A and B of X, the following properties hold:*
(1) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mCl}(A) = A$ *if A is m-closed,*
(2) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$,
(3) *if* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$,
(4) $\mathrm{mCl}(A) \cup \mathrm{mCl}(B) \subset \mathrm{mCl}(A \cup B)$,
(5) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$.

Definition 3. A minimal structure m of a set X is said to have *property $\mathcal{B}$* [8] if the union of any collection of elements of m is an element of m.

Lemma 2 (Popa and Noiri [10]). *Let (X, m) be an m-space and A a subset of X.*
(1) $x \in \mathrm{mCl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m(x)$.
(2) *Let m have property $\mathcal{B}$. Then the following properties hold:*
(*i*) *A is m-closed if and only if* $\mathrm{mCl}(A) = A$,
(*ii*) $\mathrm{mCl}(A)$ *is m-closed.*

Definition 4. Let A be a subset of (X, m). A point $x \in X$ is called an *$m(\theta)$-cluster* point of A if $\mathrm{mCl}(U) \cap A \neq \emptyset$ for every m-open set U containing x.

The set of all $m(\theta)$-cluster points of A is called the *$m(\theta)$-closure* of A and is denoted by $\mathrm{mCl}_\theta(A)$. If $A = \mathrm{mCl}_\theta(A)$, then A is said to be *$m(\theta)$-closed*. The complement of an $m(\theta)$-closed set is said to be *$m(\theta)$-open*.

Lemma 3 (Popa and Noiri [11]). *Let A be a subset of (X, m). Then the following properties hold.*
(1) *If A is m-open in (X, m), then* $\mathrm{mCl}(A) = \mathrm{mCl}_\theta(A)$.

(2) *If* (X, m) *satisfies the property* $\mathcal{B}$*, then an* $m(\theta)$*-open set is* m*-open.*

(3) *Let* $m(\theta)$ *be the family of all* $m(\theta)$*-open sets of* (X, m)*, then* $m(\theta)$ *is a minimal structure with property* $\mathcal{B}$*.*

Definition 5. A nonempty subfamily $\mathcal{H}$ of $\mathcal{P}(X)$ is called a *hereditary class* on X [3] if it satisfies the following properties: $A \in \mathcal{H}$ and $B \subset A$ implies $B \in \mathcal{H}$. A hereditary class $\mathcal{H}$ is called an *ideal* [7, 13] if it satisfies the additional condition: $A \in \mathcal{H}$ and $B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$.

A minimal space (X, m) with a hereditary class $\mathcal{H}$ on X is called a *hereditary minimal space* (briefly *hereditary m-space*) and is denoted by $(X, m, \mathcal{H})$. The notion of ideals has been introduced in [7] and [13] and further investigated in [6].

Lemma 4 ([9])**.** *For a function* $f : (X, \tau) \to (Y, \sigma)$ *and ideals* I *and* J*, the following properties hold:*

(1) *if* f *is surjective and* I *is an ideal on* X*, then* $f(I) = \{f(A) : A \in I\}$ *is an ideal on* Y*,*

(2) *if* f *is injective and* J *is an ideal on* Y*, then* $f^{-1}(J) = \{f^{-1}(B) : B \in J\}$ *is an ideal on* X*.*

Lemma 5. *Let* $(X, m, \mathcal{H})$ *be a hereditary m-space,* $f : (X, m, \mathcal{H}) \to (Y, n)$ *a function and* $J_H = \{B \subset Y : f^{-1}(B) \in \mathcal{H}\}$*. Then the following properties hold:*

(1) J_H *is a hereditary class on* Y*,*

(2) *if* f *is injective, then* $\mathcal{H} \subset f^{-1}(J_H)$*,*

(3) *if* f *is surjective, then* $J_H \subset f(\mathcal{H})$*,*

(4) *if* f *is bijective, then* $J_H = f(\mathcal{H})$*.*

Proof. (1) Let $A \subset B$ and $B \in J_H$, then $f^{-1}(A) \subset f^{-1}(B) \in \mathcal{H}$. Hence $f^{-1}(A) \in \mathcal{H}$ and $A \in J_H$. Therefore, J_H is a hereditary class on Y.

(2) Since f is injective, for any $A \in \mathcal{H}$, $f^{-1}(f(A)) = A \in \mathcal{H}$ and $f(A) \in J_H$. Therefore, $A \in f^{-1}(J_H)$ and $\mathcal{H} \subset f^{-1}(J_H)$.

(3) For any $B \in J_H, f^{-1}(B) \in \mathcal{H}$. Since f is surjective, $B = f(f^{-1}(B)) \in f(\mathcal{H})$ and hence $J_H \subset f(\mathcal{H})$.

(4) The proof is obvious by (2) and (3). □

Definition 6. Let (X, m) be an m-space. A subset A of X is said to be *$m(\theta)$-compact relative to X* if for each cover $\{U_\alpha : \alpha \in \Delta\}$ of A by $m(\theta)$-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$.

Definition 7. An m-space (X, m) is said to be *$m(\theta)$-compact* if the set X is $m(\theta)$-compact relative to X.

Definition 8. A function $f : (X, m) \to (Y, n)$ is said to be

(1) *quasi $m(\theta)$-continuous* if $f^{-1}(V)$ is $m(\theta)$-open in (X, m) for every $n(\theta)$-open set V in (Y, n),

(2) *M-$m(\theta)$-open* if $f(U)$ is $n(\theta)$-open in (Y, n) for every $m(\theta)$-open set U of (X, m).

3. θ-$\mathcal{H}$-compact sets

Definition 9. Let $(X, m, \mathcal{H})$ be a *hereditary m-space.*

(1) A subset A of X is said to be *θ-$\mathcal{H}$-compact relative to X* if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by $m(\theta)$-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$.

(2) $(X, m\ \mathcal{H})$ is called a *hereditary m-space* if X is θ-$\mathcal{H}$-compact relative to X.

Theorem 1. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. For a subset A of X, the following properties are equivalent:*

(1) *A is θ-$\mathcal{H}$-compact relative to X;*

(2) *for every family $\{F_\alpha : \alpha \in \Delta\}$ of $m(\theta)$-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) = \emptyset$, there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta_0\}) \in \mathcal{H}$.*

Proof. (1) ⇒ (2): Let $\{F_\alpha : \alpha \in \Delta\}$ be a family of $m(\theta)$-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) = \emptyset$. Then $A \subset X \setminus (\cap\{F_\alpha : \alpha \in \Delta\}) = \cup\{X \setminus F_\alpha : \alpha \in \Delta\}$. Since $X \setminus F_\alpha$ is $m(\theta)$-open for each $\alpha \in \Delta$, by (1) there exists a finite subset Δ_0 of Δ such that $A \setminus (\cup\{X \setminus F_\alpha : \alpha \in \Delta_0\}) \in \mathcal{H}$. Therefore, we have

$$\begin{aligned} A \cap (\cap\{F_\alpha : \alpha \in \Delta_0\}) =& A \cap [X \setminus \cup\{(X \setminus F_\alpha : \alpha \in \Delta_0\})] \\ =& A \setminus (\cup\{X \setminus F_\alpha : \alpha \in \Delta_0\}) \in \mathcal{H}. \end{aligned}$$

(2) ⇒ (1): Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of A by $m(\theta)$-open sets of X. Then $A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\}) = \emptyset$. Since $X \setminus U_\alpha$ is $m(\theta)$-closed for each $\alpha \in \Delta$, by (2) there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta_0\}) \in \mathcal{H}$. Therefore, we have

$$\begin{aligned} A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta_0\}) =& A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\}) \\ =& A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}. \end{aligned}$$

This shows that A is $m(\theta)$-$\mathcal{H}$-compact relative to X. □

Corollary 1. *For a hereditary m-space $(X, m, \mathcal{H})$, the following properties are equivalent:*

(1) *$(X, m, \mathcal{H})$ is θ-$\mathcal{H}$-compact;*

(2) *for every family $\{F_\alpha : \alpha \in \Delta\}$ of $m(\theta)$-closed sets of X such that $\cap\{F_\alpha : \alpha \in \Delta\} = \emptyset$, there exists a finite subset Δ_0 of Δ such that $\cap\{F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$.*

Definition 10. Let $(X, m, \mathcal{H})$ be a hereditary m-space. A subset A of X is said to be

(1) *$\mathcal{H}\theta g$-closed* if $\mathrm{mCl}_\theta(A) \subset U$ whenever $A \setminus U \in \mathcal{H}$ and U is $m(\theta)$-open,
(2) *θg-closed* if $\mathrm{mCl}_\theta(A) \subset U$ whenever $A \subset U$ and U is $m(\theta)$-open.

Theorem 2. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and A, B subsets of X such that $A \subset B \subset \mathrm{mCl}_\theta(A)$. Then the following properties hold:*

(1) *if A is θ-$\mathcal{H}$-compact relative to X and $\mathcal{H}\theta g$-closed, then B is $m(\theta)$-compact relative to X,*

(2) *if B is θ-$\mathcal{H}$-compact relative to X and θg-closed, then A is θ-$\mathcal{H}$-compact relative to X.*

Proof. (1): Suppose that A is θ-$\mathcal{H}$-compact relative to X and $\mathcal{H}\theta g$-closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of B by $m(\theta)$-open sets of X. Then $\{U_\alpha : \alpha \in \Delta\}$ is a cover of A by $m(\theta)$-open sets of X. Since A is θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Since A is $\mathcal{H}\theta g$-closed, $mCl_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since $B \subset \mathrm{mCl}_\theta(A)$, we have $B \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Therefore, B is $m(\theta)$-compact relative to X.

(2): Suppose that B is θ-$\mathcal{H}$-compact relative to X and θg-closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of A by $m(\theta)$-open sets of X. Since A is θg-closed, we have $B \subset \mathrm{Cl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since B is θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Now $A \subset B$ implies $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Therefore, A is θ-$\mathcal{H}$-compact relative to X. ☐

Corollary 2. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. If A is $\mathcal{H}\theta g$-closed and $A \subset B \subset \mathrm{mCl}_\theta(A)$, then the following properties are equivalent:*
(1) *A is θ-$\mathcal{H}$-compact relative to X;*
(2) *B is θ-$\mathcal{H}$-compact relative to X.*

Theorem 3. *Let $(X, m, \mathcal{H})$ be an ideal m-space. If subsets A and B of X are θ-$\mathcal{H}$-compact relative to X, then $A \cup B$ is θ-$\mathcal{H}$-compact relative to X.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be any cover of $A \cup B$ by $m(\theta)$-open sets of X. Then $\mathcal{U}$ is a cover of A and B by $m(\theta)$-open sets of X. Since A and B are θ-$\mathcal{H}$-compact relative to X, there exist finite subsets Δ_A and Δ_B of Δ and subsets H_A and H_B of $\mathcal{H}$ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_A\} \cup H_A$ and $B \subset \cup\{U_\alpha : \alpha \in \Delta_B\} \cup H_B$. Hence we have $A \cup B \subset \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \cup (H_A \cup H_B)$. Since $\mathcal{H}$ is an ideal, we have $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \in \mathcal{H}$. This shows that $A \cup B$ is θ-$\mathcal{H}$-compact relative to X. ☐

Theorem 4. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, and A, B be subsets of X. If A is θ-$\mathcal{H}$-compact relative to X and B is $m(\theta)$-closed, then $A \cap B$ is θ-$\mathcal{H}$-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a cover of $A \cap B$ by $m(\theta)$-open sets of X. Then $A \setminus B \subset X \setminus B$ and $X \setminus B$ is $m(\theta)$-open. Then $\{U_\alpha : \alpha \in \Delta\} \cup \{X \setminus B\}$ is a cover of A by $m(\theta)$-open sets of X. Since A is θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \subset (\cup\{U_\alpha : \alpha \in \Delta_0\}) \cup \{X \setminus B\} \cup H_0$, where $H_0 \in \mathcal{H}$. Then we have

$$(A \cap B) \subset (\cup\{U_\alpha \cap B : \alpha \in \Delta_0\}) \cup (H_0 \cap B) \subset \cup\{U_\alpha : \alpha \in \Delta_0\} \cup H_0.$$

Therefore, $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \subset H_0 \in \mathcal{H}$. This shows that $A \cap B$ is θ-$\mathcal{H}$-compact relative to X. □

Corollary 3. *If a hereditary m-space* $(X, m, \mathcal{H})$ *is* θ-$\mathcal{H}$*-compact and* B *is* $m(\theta)$*-closed, then* B *is* θ-$\mathcal{H}$*-compact relative to* X*.*

Theorem 5. *If* $f : (X, m, \mathcal{H}) \to (Y, n, f(\mathcal{H}))$ *is a surjective quasi* $m(\theta)$*-continuous function and* A *is* θ-$\mathcal{H}$*-compact relative to* X*, then* $f(A)$ *is* θ-*f(*$\mathcal{H}$*)-compact relative to* Y*.*

Proof. Suppose that A is θ-$\mathcal{H}$-compact relative to X. Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of $f(A)$ by $n(\theta)$-open sets of Y. Since f is quasi $m(\theta)$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a cover of A by $m(\theta)$-open sets of X. Since A is θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Hence we have $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \cup H_0$, where $H_0 \in \mathcal{H}$ and $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta_0\} \cup f(H_0)$. Therefore, we have $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta_0\} \in f(\mathcal{H})$. This shows that $f(A)$ is θ-$f(\mathcal{H})$-compact relative to Y. □

Corollary 4. *If* $f : (X, m, \mathcal{H}) \to (Y, n, f(\mathcal{H}))$ *is a surjective quasi* $m(\theta)$*-continuous function and* $(X, m, \mathcal{H})$ *is* θ-$\mathcal{H}$*-compact, then* $(Y, n, \mathcal{H})$ *is* θ-$f(\mathcal{H})$*-compact.*

Theorem 6. *Let* $f : (X, m) \to (Y, n, \mathcal{J})$ *be an* M-$m(\theta)$*-open bijective function. If* B *is* θ-$\mathcal{J}$*-compact relative to* Y*, then* $f^{-1}(B)$ *is* θ-$f^{-1}(J)$*-compact relative to X.*

Proof. Since $f^{-1} : (Y, n, \mathcal{J}) \to (X, m)$ is a quasi-$m(\theta)$-continuous bijection, by Theorem 5 the proof is obvious. □

Corollary 5. *If* $f : (X, m) \to (Y, n, \mathcal{J})$ *is an* M-$m(\theta)$*-open bijective function and* $(Y, n, \mathcal{J})$ *is* θ-$\mathcal{J}$*-compact, then* $(X, m, f^{-1}(\mathcal{J}))$ *is* θ-$f^{-1}(\mathcal{J})$*-compact.*

Theorem 7. *If* $f : (X, m, \mathcal{H}) \to (Y, n)$ *is an injective quasi* $m(\theta)$*-continuous function and* A *is* θ-$\mathcal{H}$*-compact relative to* X*, then* $f(A)$ *is* θ-$\mathcal{J}_H$*-compact relative to Y.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of $f(A)$ by $n(\theta)$-open sets of Y. Then $A \subset f^{-1}(f(A)) \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ and $f^{-1}(V_\alpha)$ is $m(\theta)$-open in X for each $\alpha \in \Delta$. Since A is θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Hence $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \cup H_0$, where $H_0 \in \mathcal{H}$. Therefore, we have $f(A) \subset \cup\{f(f^{-1}(V_\alpha)) : \alpha \in \Delta_0\} \cup f(H_0) \subset \cup\{V_\alpha : \alpha \in \Delta_0\} \cup f(H_0)$. Since

f is injective, $f^{-1}(f(H_0)) = H_0 \in \mathcal{H}$ and $f(H_0) \in \mathcal{J}_H$. Consequently, we obtain $f(A)\setminus\cup\{V_\alpha : \alpha \in \Delta_0\} \in \mathcal{J}_H$. This shows that $f(A)$ is θ-$\mathcal{J}_H$-compact relative to Y. □

4. Strongly θ-$\mathcal{H}$-compact sets

Definition 11. Let $(X, m, \mathcal{H})$ be a hereditary m-space. A subset A of X is said to be *strongly θ-$\mathcal{H}$-compact relative to X* if for every family $\{U_\alpha : \alpha \in \Delta\}$ of $m(\theta)$-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$.

Definition 12. A hereditary m-space $(X, m, \mathcal{H})$ is said to be *strongly θ-$\mathcal{H}$-compact* if X is strongly θ-$\mathcal{H}$-compact relative to X

Theorem 8. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. For a subset A of X, the following properties are equivalent:*

(1) *A is strongly θ-$\mathcal{H}$-compact relative to X;*

(2) *for every family $\{F_\alpha : \alpha \in \Delta\}$ of $m(\theta)$-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta_0\}) \in \mathcal{H}$.*

Proof. (1) ⇒ (2): Let $\{F_\alpha : \alpha \in \Delta\}$ be a family of $m(\theta)$-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Then $A \setminus \cup\{X \setminus F_\alpha : \alpha \in \Delta\}) = A \setminus (X \setminus \cap\{F_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Since $X \setminus F_\alpha$ is $m(\theta)$-open for each $\alpha \in \Delta$ and A is strongly θ-$\mathcal{H}$-compact relative to X by (1), there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{X \setminus F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. This implies that$A \cap (\cap\{F_\alpha : \alpha \in \Delta_0\}) = A \setminus (X \setminus (\cap\{F_\alpha : \alpha \in \Delta_0\})) = A \setminus \cup\{X \setminus F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$.

(2) ⇒ (1): Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of $m(\theta)$-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of $m(\theta)$-closed sets of X and also $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} = A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta\}) = A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Thus by (2) there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta_0\}) \in \mathcal{H}$. Therefore, we have $A\setminus\cup\{U_\alpha : \alpha \in \Delta_0\} = A\cap(X\setminus\cup\{U_\alpha : \alpha \in \Delta_0\}) = A\cap(\cap\{X\setminus U_\alpha : \alpha \in \Delta_0\}) \in \mathcal{H}$. This shows that A is strongly θ-$\mathcal{H}$-compact relative to X. □

Corollary 6. *For a hereditary m-space $(X, m, \mathcal{H})$, the following properties are equivalent:*

(1) *$(X, m, \mathcal{H})$ is strongly θ-$\mathcal{H}$-compact;*

(2) *for every family $\{F_\alpha : \alpha \in \Delta\}$ of $m(\theta)$-closed sets of X such that $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $\cap\{F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$.*

Theorem 9. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. If A is $\mathcal{H}\theta g$-closed and $A \subset B \subset \mathrm{mCl}_\theta(A)$, then the following properties are equivalent:*

(1) *A is strongly θ-$\mathcal{H}$-compact relative to X;*

(2) *B is strongly θ-$\mathcal{H}$-compact relative to X.*

Proof. (1) $\Rightarrow$ (2): Suppose that A is strongly θ-$\mathcal{H}$-compact relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of $m(\theta)$-open sets of X such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A is strongly θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Since A is $\mathcal{H}\theta g$-closed, $\mathrm{mCl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since $B \subset \mathrm{mCl}_\theta(A)$, we have $B \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \subset \mathrm{mCl}_\theta(A) \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} = \emptyset \in \mathcal{H}$. Therefore, B is strongly θ-$\mathcal{H}$-compact relative to X.

(2) $\Rightarrow$ (1): Suppose that B is strongly θ-$\mathcal{H}$-compact relative to X. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of $m(\theta)$-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A is $\mathcal{H}\theta g$-closed, we have $B \subset \mathrm{mCl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since B is strongly θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Now $A \subset B$ implies $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Hence, A is strongly θ-$\mathcal{H}$-compact relative to X. □

Theorem 10. *Let $(X, m, \mathcal{H})$ be an ideal m-space. If subsets A and B of X are strongly θ-$\mathcal{H}$-compact relative to X, then $A \cup B$ is strongly θ-$\mathcal{H}$-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of $m(\theta)$-open sets of X such that $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$ and $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A and B are strongly θ-$\mathcal{H}$-compact relative to X, there exist finite subsets Δ_A and Δ_B of Δ and subsets H_A and H_B of $\mathcal{H}$ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_A\} \cup H_A$ and $B \subset \cup\{U_\alpha : \alpha \in \Delta_B\} \cup H_B$. Hence we have $(A \cup B) \subset \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \cup (H_A \cup H_B)$. Since $\mathcal{H}$ is an ideal, we have $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\} \in \mathcal{H}$. This shows that $A \cup B$ is strongly θ-$\mathcal{H}$-compact relative to X. □

Theorem 11. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and A, B be subsets of X. If A is strongly θ-$\mathcal{H}$-compact relative to X and B is $m(\theta)$-closed, then $A \cap B$ is strongly θ-$\mathcal{H}$-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of $m(\theta)$-open sets of X such that $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $\{U_\alpha : \alpha \in \Delta\} \cup \{X \setminus B\}$ is a family of $m(\theta)$-open sets of X such that $A \setminus [(X \setminus B) \cup (\cup\{U_\alpha : \alpha \in \Delta\})] \in \mathcal{H}$. Since A is strongly θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \subset [\cup\{U_\alpha : \alpha \in \Delta_0\}] \cup \{X \setminus B\} \cup H_0$, where $H_0 \in \mathcal{H}$. Then we have

$$(A \cap B) \subset [\cup\{U_\alpha \cap B : \alpha \in \Delta_0\}] \cup (H_0 \cap B) \subset [\cup\{U_\alpha : \alpha \in \Delta_0\}] \cup H_0.$$

Therefore, $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \subset H_0 \in \mathcal{H}$. This shows that $A \cap B$ is strongly θ-$\mathcal{H}$-compact relative to X. □

Corollary 7. *If a hereditary m-space $(X, m, \mathcal{H})$ is strongly θ-$\mathcal{H}$-compact and B is $m(\theta)$-closed, then B is strongly θ-$\mathcal{H}$-compact relative to X.*

Theorem 12. *If* $f : (X, m, \mathcal{H}) \to (Y, n)$ *is a bijective quasi* $m(\theta)$*-continuous function and A is strongly* θ*-*$\mathcal{H}$*-compact relative to X, then f(A) is strongly* θ*-f(*$\mathcal{H}$*)-compact relative to* Y*.*

Proof. Suppose that A is strongly θ-$\mathcal{H}$-compact relative to X. Let $\{V_\alpha : \alpha \in \Delta\}$ be any family of $n(\theta)$-open sets in Y such that $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta\} \in f(\mathcal{H})$. Then $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta\} \cup f(H_0)$ for some $H_0 \in \mathcal{H}$. Since f is bijective, $A = f^{-1}(f(A)) \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \cup H_0$ and hence $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \in \mathcal{H}$. Since f is quasi $m(\theta)$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a family of $m(\theta)$-open sets of X. Since A is strongly θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Hence we have $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \cup H_A$, where $H_A \in \mathcal{H}$ and $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta_0\} \cup f(H_A)$. Therefore, we have $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta_0\} \in f(\mathcal{H})$. This shows that $f(A)$ is strongly θ-$f(\mathcal{H})$-compact relative to Y. □

Corollary 8. *If* $f : (X, m, \mathcal{H}) \to (Y, n)$ *is a bijective quasi* $m(\theta)$*-continuous function and* $(X, m, \mathcal{H})$ *is strongly* θ*-*$\mathcal{H}$*-compact, then* $(Y, n, f(\mathcal{H}))$ *is strongly* θ*-*$f(\mathcal{H})$*-compact.*

Theorem 13. *If* $f : (X, m, \mathcal{H}) \to (Y, n)$ *is a quasi* $m(\theta)$*-continuous injective function and A is strongly* θ*-*$\mathcal{H}$*-compact relative to X, then f(A) is strongly* θ*-*J_H*-compact relative to Y.*

Proof. Suppose that A is strongly θ-$\mathcal{H}$-compact relative to X. Let $\{V_\alpha : \alpha \in \Delta\}$ be any family of $n(\theta)$-open sets in Y such that $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta\} \in J_H$. Then $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta\} \cup J_0$ for some $J_0 \in J_H$. Therefore $A \subset f^{-1}(f(A)) \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \cup f^{-1}(J_0)$ and hence $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \in \mathcal{H}$. Since f is quasi $m(\theta)$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a family of $m(\theta)$-open sets of X. Since A is strongly θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Hence we have $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\} \subset H_0$, where $H_0 \in \mathcal{H}$ and $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta_0\} \cup f(H_0)$. Since f is injective, we have $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta_0\} \in J_H$ and thus $f(A)$ is strongly θ-J_H-compact relative to Y. □

Corollary 9. *If* $f : (X, m, \mathcal{H}) \to (Y, n)$ *is a bijective quasi* $m(\theta)$*-continuous function and* $(X, m, \mathcal{H})$ *is strongly* θ*-*$\mathcal{H}$*-compact, then* (Y, n) *is strongly* θ*-*J_H*-compact.*

5. Super θ-$\mathcal{H}$-compact sets

Definition 13. Let $(X, m, \mathcal{H})$ be a hereditary m-space. A subset A of X is said to be *super* θ*-*$\mathcal{H}$*-compact relative to* X if for every family $\{U_\alpha : \alpha \in \Delta\}$ of $m(\theta)$-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$.

Definition 14. A hereditary m-space $(X, m, \mathcal{H})$ is said to be *super θ-$\mathcal{H}$-compact* if the set X is super θ-$\mathcal{H}$-compact relative to X.

Theorem 14. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. For a subset A of X, the following properties are equivalent:*

(1) *A is super θ-$\mathcal{H}$-compact relative to X;*

(2) *for every family $\{F_\alpha : \alpha \in \Delta\}$ of $m(\theta)$-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta_0\}) = \emptyset$.*

Proof. (1) $\Rightarrow$ (2): Let $\{F_\alpha : \alpha \in \Delta\}$ be a family of $m(\theta)$-closed sets of X such that $A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Then $\{X \setminus F_\alpha : \alpha \in \Delta\}$ is a family of $m(\theta)$-open sets of X. Then $A \setminus \cup\{(X \setminus F_\alpha) : \alpha \in \Delta\} = A \cap [X \setminus (X \setminus \cap\{F_\alpha : \alpha \in \Delta\})] = A \cap (\cap\{F_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Since $A \setminus \cup\{(X \setminus F_\alpha) : \alpha \in \Delta\} \in \mathcal{H}$, by (1) there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{(X \setminus F_\alpha) : \alpha \in \Delta_0\}$. This implies that $A \cap (\{F_\alpha : \alpha \in \Delta_0\}) = \emptyset$.

(2) $\Rightarrow$ (1): Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of $m(\theta)$-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $\{X \setminus U_\alpha : \alpha \in \Delta\}$ is a family of $m(\theta)$-closed sets of X and $A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta\}) = A \cap (X \setminus \cup\{U_\alpha : \alpha \in \Delta\}) \in \mathcal{H}$. Thus by (2) there exists a finite subset Δ_0 of Δ such that $A \cap (\cap\{X \setminus U_\alpha : \alpha \in \Delta_0\}) = \emptyset$; hence $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. This shows that $(X, m, \mathcal{H})$ is super θ-$\mathcal{H}$-compact. □

Corollary 10. *For a hereditary m-space $(X, m, \mathcal{H})$, the following properties are equivalent:*

(1) *$(X, m, \mathcal{H})$ is super θ-$\mathcal{H}$-compact;*

(2) *for every family $\{F_\alpha : \alpha \in \Delta\}$ of $m(\theta)$-closed sets of X such that $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $\cap\{F_\alpha : \alpha \in \Delta_0\} = \emptyset$.*

Theorem 15. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and A, B be subsets of X such that $A \subset B \subset \mathrm{mCl}_\theta(A)$. Then the following properties hold:*

(1) *if A is super θ-$\mathcal{H}$-compact relative to X and θg-closed, then B is super θ-$\mathcal{H}$-compact relative to X,*

(2) *if A is strongly θ-$\mathcal{H}$-compact relative to X and $\mathcal{H}\theta g$-closed, then B is super θ-$\mathcal{H}$-compact relative to X,*

(3) *if B is $m(\theta)$-compact relative to X and A is $\mathcal{H}\theta g$-closed, then A is super θ-$\mathcal{H}$-compact relative to X.*

Proof. (1): Suppose that A is super θ-$\mathcal{H}$-compact relative to X and θg-closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of $m(\theta)$-open sets of X such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A is super θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since A is θg-closed, $\mathrm{mCl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since $B \subset \mathrm{mCl}_\theta(A)$, we have $B \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Therefore, B is super θ-$\mathcal{H}$-compact relative to X.

(2): Suppose that A is strongly θ-$\mathcal{H}$-compact relative to X and $\mathcal{H}\theta g$-closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of $m(\theta)$-open sets of X such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A is strongly θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Since A is $\mathcal{H}\theta g$-closed, $\mathrm{mCl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since $B \subset \mathrm{mCl}_\theta(A)$, we have $B \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Therefore, B is super θ-$\mathcal{H}$-compact relative to X.

(3): Suppose that B is $m(\theta)$-compact relative to X and A is $\mathcal{H}\theta g$-closed. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of $m(\theta)$-open sets of X such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A is $\mathcal{H}\theta g$-closed, we have $B \subset \mathrm{mCl}_\theta(A) \subset \cup\{U_\alpha : \alpha \in \Delta\}$. Since B is $m(\theta)$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $B \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Now $A \subset B$ implies $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. Therefore, A is super θ-$\mathcal{H}$-compact relative to X. □

Corollary 11. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. If A is $\mathcal{H}\theta g$-closed and $A \subset B \subset \mathrm{mCl}_\theta(A)$, then the following properties are equivalent:*

(1) *A is super θ-$\mathcal{H}$-compact relative to X;*

(2) *B is super θ-$\mathcal{H}$-compact relative to X.*

Theorem 16. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. If subsets A and B of X are super θ-$\mathcal{H}$-compact relative to X, then $A \cup B$ is super θ-$\mathcal{H}$-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any family of $m(\theta)$-open sets of X such that $(A \cup B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $A \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$ and $B \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Since A and B are super θ-$\mathcal{H}$-compact relative to X, there exist finite subsets Δ_A and Δ_B of Δ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_A\}$ and $B \subset \cup\{U_\alpha : \alpha \in \Delta_B\}$. Hence we have $(A \cup B) \subset \cup\{U_\alpha : \alpha \in \Delta_A \cup \Delta_B\}$. This shows that $A \cup B$ is super θ-$\mathcal{H}$-compact relative to X. □

Theorem 17. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and A, B be subsets of X. If A is super θ-$\mathcal{H}$-compact relative to X and B is $m(\theta)$-closed, then $A \cap B$ is super θ-$\mathcal{H}$-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of $m(\theta)$-open sets of X such that $(A \cap B) \setminus \cup\{U_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $\{U_\alpha : \alpha \in \Delta\} \cup \{X \setminus B\}$ is a family of $m(\theta)$-open sets of X such that $A \subset [(X \setminus B) \cup (\cup\{U_\alpha : \alpha \in \Delta\})] \cup H_0$, where $H_0 \in \mathcal{H}$. Since A is super θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \subset [\cup\{U_\alpha : \alpha \in \Delta_0\}] \cup \{X \setminus B\}$. Then we have $(A \cap B) \subset \cup\{U_\alpha \cap B : \alpha \in \Delta_0\} \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$. This shows that $A \cap B$ is super θ-$\mathcal{H}$-compact relative to X. □

Corollary 12. *If a hereditary m-space $(X, m, \mathcal{H})$ is super θ-$\mathcal{H}$-compact and B is $m(\theta)$-closed, then B is super θ-$\mathcal{H}$-compact relative to X.*

Theorem 18. *If* $f : (X, m, \mathcal{H}) \to (Y, n)$ *is a bijective quasi* $m(\theta)$*-continuous function and* A *is super* θ*-*$\mathcal{H}$*-compact relative to* X*, then f(A) is super* θ*-f(*$\mathcal{H}$*)-compact relative to* Y*.*

Proof. Suppose that A is super θ-$\mathcal{H}$-compact relative to X. Let $\{V_\alpha : \alpha \in \Delta\}$ be any family of $n(\theta)$-open sets in Y such that $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta\} \in f(\mathcal{H})$. Then $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta\} \cup f(H_0)$ for some $H_0 \in \mathcal{H}$. Since f is bijective, $A = f^{-1}(f(A)) \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \cup H_0$ and hence $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \in \mathcal{H}$. Since f is quasi $m(\theta)$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a family of $m(\theta)$-open sets of X. Since A is super θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$. Hence we have $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta_0\}$ and thus $f(A)$ is super θ-$f(\mathcal{H})$-compact relative to Y. □

Corollary 13. *If* $f : (X, m, \mathcal{H}) \to (Y, n)$ *is a bijective quasi* $m(\theta)$*-continuous function and* $(X, m, \mathcal{H})$ *is super* θ*-*$\mathcal{H}$*-compact, then* $(Y, n, f(\mathcal{H}))$ *is super* θ*-*$f(\mathcal{H})$*-compact.*

Theorem 19. *If* $f : (X, m, \mathcal{H}) \to (Y, n)$ *is a quasi* $m(\theta)$*-continuous function and* A *is super* θ*-*$\mathcal{H}$*-compact relative to* X*, then* $f(A)$ *is super* θ*-*$\mathcal{J}_H$*-compact relative to* Y*.*

Proof. Suppose that A is super θ-$\mathcal{H}$-compact relative to X. Let $\{V_\alpha : \alpha \in \Delta\}$ be any family of $n(\theta)$-open sets in Y such that $f(A) \setminus \cup\{V_\alpha : \alpha \in \Delta\} \in \mathcal{J}_H$. Then $f(A) \subset \cup\{V_\alpha : \alpha \in \Delta\} \cup J_0$ for some $J_0 \in \mathcal{J}_H$. Therefore $A \subset f^{-1}(f(A)) \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \cup f^{-1}(J_0)$ and hence $A \setminus \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta\} \subset f^{-1}(J_0) \in \mathcal{H}$. Since f is quasi $m(\theta)$-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a family of $m(\theta)$-open sets of X. Since A is super θ-$\mathcal{H}$-compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$. Hence we have $f(A) \subset \cup\{f(f^{-1}(V_\alpha)) : \alpha \in \Delta_0\} \subset \cup\{V_\alpha : \alpha \in \Delta_0\}$. Therefore, $f(A)$ is super θ-$\mathcal{J}_H$-compact relative to Y. □

Corollary 14. *If* $f : (X, m, \mathcal{H}) \to (Y, n)$ *is a surjective quasi* $m(\theta)$*-continuous function and* $(X, m, \mathcal{H})$ *is super* θ*-*$\mathcal{H}$*-compact, then* (Y, n) *is super* θ*-*$\mathcal{J}_H$*-compact.*

Remark 1. We have the following relationships:

$$\begin{tikzcd}
\text{super } \theta\text{-}\mathcal{H}\text{-compact relative to } X \arrow[r] \arrow[d] & \text{strongly } \theta\text{-}\mathcal{H}\text{-compact relative to } X \arrow[d] \\
m(\theta)\text{-compact relative to } X \arrow[r] & \theta\text{-}\mathcal{H}\text{-compact relative to } X
\end{tikzcd}$$

Remark 2. The following examples show that "$m(\theta)$-compact relative to X" and "strongly θ-$\mathcal{H}$-compact relative to X" are independent of each other.

Example 1. Let $\mathcal{R}$ be the set of real numbers with the usual topology, $X = [1, 2]$ and $m = \{X \cap (a, b) : a < b, a, b \in \mathcal{R}\}$. Then it is clear that (X, m) is a regular space. Let $\mathcal{H} = \{\emptyset, \{1\}, \{2\}\}$. Observe that (X, m) is $m(\theta)$-compact relative to X but $(X, m, \mathcal{H})$ is not strongly θ-$\mathcal{H}$-compact relative to X. In fact if $U_n = (1 + \frac{1}{n}, 2]$ for all integer number $n > 1$, then $X \setminus \cup_{n>1} U_n = \{1\} \in \mathcal{H}$. If we take $N = \max\{n_1, n_2, \cdots, n_k\}$, $k \in \mathbf{Z}$, and $n_1, n_2, \cdots, n_k$ are integers then $X \setminus \cup_{i=1}^{k} U_{n_i} = X \setminus (1 + \frac{1}{N}, 2] = [1, 1 + \frac{1}{N}] \notin \mathcal{H}$.

Example 2. Let $\mathcal{R}$ be the set of real numbers with the usual topology τ. Let $X = (0, 1)$, m be the relative topology of τ on X and $\mathcal{H} = \{A : A \subseteq (0, 1)\}$. Then $(X, m, \mathcal{H})$ is strongly θ-$\mathcal{H}$-compact relative to X but (X, m) is not $m(\theta)$-compact relative to X. Because an $m(\theta)$-open cover $\{(\frac{1}{n+1}, 1 - \frac{1}{n+1}) : n \in \mathbf{Z}^{+}\}$ of X has no finite subcover.

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