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On Pellnomial coefficients and Pell–Catalan numbers

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Abstract. In this paper, we first give the Pascal's identity for Pellnomial coefficients and then we show that the Pellnomial coefficients are integers. We obtain that the product of r consecutive Pell numbers is divisible by the Pell analog of $r!$. Also, we introduce the divisibility theorems between Pell numbers and Pellnomial coefficients. Furthermore, we first define Pell–Catalan numbers and then we derive two formulas for presenting Pell–Catalan numbers.

1. Introduction

For any integer $n \geqslant 0$, the Pell numbers P_n are defined by the second-order linear recurrence sequence $P_{n+2} = 2P_{n+1} + P_n$, where $P_0 = 0$ and $P_1 = 1$ [13]. Various aspects of Pell numbers have been extensively studied in the literature. These include quaternionic aspects [5], the matrices considering their entries as Pell numbers [2], the sums involving Pell numbers [15, 17], symmetric matrix with harmonic Pell entries [1], the infinite sums of reciprocal Pell numbers [22] and others. For a full introduction to Pell and Pell–Lucas numbers see [13].

The nth Catalan number C_n is defined by the closed formula for $n \geq 0$:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

A very large literature concerns the Catalan numbers. Several books already present general survey for these numbers, see [16, 9, 20, 12] for the more recent ones.

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In combinatorics, the numbers $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$, where $\begin{pmatrix} n \\ 0 \end{pmatrix} = 1$, $\begin{pmatrix} n \\ n \end{pmatrix} = 1$ and $\begin{pmatrix} n \\ k \end{pmatrix} = 0$ with $k > n$, are called binomial coefficients. The theories of binomial coefficients in the classical algebra can be found in, e.g. [8, 6, 19]. Let $\{A_n\}$ be any sequence of real or complex numbers such that $A_0 = 0$ and $A_n \neq 0$ with $n \geqslant 1$. In 1915, a generalization of binomial coefficients, replacing the natural numbers by the terms of $\{A_n\}$, is defined by

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = (A_n A_{n-1} \cdots A_{n-k+1} / A_1 A_2 \cdots A_k).$$

Properties of this generalization and its history may be found in [7]. Since then, many authors have worked in this area and many interesting results have been discovered. Some of the results are analogs of results in binomial coefficients theory, but most of them are substantially different.

Fibonomial coefficients $\begin{pmatrix} n \\ m \end{pmatrix}_F$ correspond to the choice $A_n = F_n$, where F_n are the Fibonacci numbers defined by $F_{n+2} = F_n + F_{n+1}$, with $F_0 = 0, F_1 = 1$. Therefore, Fibonomial coefficients are defined by the relation for $n \geq m \geq 1$ as

$$\begin{pmatrix} n \\ m \end{pmatrix}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k}$$

with $\begin{pmatrix} n \\ 0 \end{pmatrix}_F = 1$ and $\begin{pmatrix} n \\ k \end{pmatrix}_F = 0$ for $n < k$. $F_n! = \prod_{k=1}^{n} F_k$ is the Fibonacci analog of $n!$. Then, Fibonomial coefficients $\begin{pmatrix} n \\ m \end{pmatrix}_F$, for $n \geq m \geq 1$, are also defined by

$$\begin{pmatrix} n \\ m \end{pmatrix}_F = \frac{F_n!}{F_m! F_{n-m}!}.$$

There exist numerous identities concerning the Fibonomial coefficients in the mathematical literature (see [14, 7, 10, 18] just for examples). Gould [7] recorded the following interesting one:

$$\begin{pmatrix} n \\ k \end{pmatrix}_F = F_{k+1} \begin{pmatrix} n-1 \\ k \end{pmatrix}_F + F_{n-k-1} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_F,$$

known as Pascal's identity for Fibonomial coefficients. For some background information on Fibonomial coefficients, see [14, 18, 21].

The focus of this paper is the study of Pellnomial coefficients and Pell–Catalan numbers. We first give the Pascal's identity for Pellnomial coefficients and then we show that the Pellnomial coefficients are integers. We obtain that the product of r consecutive Pell numbers is divisible by the Pell analog of $r!$. We present some interesting divisibility theorems between

Pell numbers and Pellnomial coefficients. Furthermore, we first define Pell–Catalan numbers and then we derive two formulas for Pell–Catalan numbers.

2. Main results

In this section, we first present several new properties of the Pellnomial coefficients and secondly, we define Pell–Catalan numbers and then we derive two formulas to present these numbers.

Let P_n be the nth Pell number. In [11], Pellnomial coefficients $\left(\begin{array}{c} n \\ m \end{array}\right)_P$, which correspond to the choice $A_n = P_n$, are defined, for $n \geq m \geq 1$, as

$$\left(\begin{array}{c} n \\ m \end{array}\right)_P = \frac{P_n P_{n-1} \cdots P_{n-m+1}}{P_1 \cdots P_m}$$

with $\left(\begin{array}{c} n \\ 0 \end{array}\right)_P = 1$ and $\left(\begin{array}{c} n \\ k \end{array}\right)_P = 0$ for $n < k$. The Pell analog of $n!$ is defined for $n \geq 0$ by

$$P_n! = \prod_{k=1}^{n} P_k. \tag{1}$$

In [11], according to the Pell analog of $n!$, Pellnomial coefficients $\left(\begin{array}{c} n \\ m \end{array}\right)_P$, for $n \geq m \geq 1$, are also defined by

$$\left(\begin{array}{c} n \\ m \end{array}\right)_P = \frac{P_n!}{P_m! P_{n-m}!}. \tag{2}$$

We give Pascal's identity for Pellnomials in the following theorem.

Theorem 1 (Pascal's identity for Pellnomials)**.** *Let n and r be any positive integers, where $1 \leq r \leq n$. Then*

$$\left(\begin{array}{c} n \\ r \end{array}\right)_P = P_{r+1} \left(\begin{array}{c} n-1 \\ r \end{array}\right)_P + P_{n-r-1} \left(\begin{array}{c} n-1 \\ r-1 \end{array}\right)_P.$$

Proof. We simplify the

$$\text{RHS} = P_{r+1} \left(\begin{array}{c} n-1 \\ r \end{array}\right)_P + P_{n-r-1} \left(\begin{array}{c} n-1 \\ r-1 \end{array}\right)_P$$

and show that it equals the LHS $= \left(\begin{array}{c} n \\ r \end{array} \right)_P$. By definitions (1) and (2), we get the following result for the RHS:

$$\begin{aligned} \text{RHS} &= P_{r+1}\frac{P_{(n-1)}!}{P_r!P_{(n-r-1)}!} + P_{n-r-1}\frac{P_{(n-1)}!}{P_{(r-1)}!P_{(n-r)}!} \\ &= P_{r+1}\frac{P_{(n-1)}!}{P_rP_{(r-1)}!P_{(n-r-1)}!} + P_{n-r-1}\frac{P_{(n-1)}!}{P_{(r-1)}!P_{n-r}P_{(n-r-1)}!} \\ &= \left(\frac{P_{r+1}}{P_r} + \frac{P_{n-r-1}}{P_{n-r}}\right)\frac{P_{(n-1)}!}{P_{(r-1)}!P_{(n-r-1)}!} \\ &= \left(\frac{P_{n-r}P_{r+1} + P_{n-r-1}P_r}{P_rP_{n-r}}\right)\frac{P_{(n-1)}!}{P_{(r-1)}!P_{(n-r-1)}!}. \end{aligned}$$

Changing m to $n-r$ and n to r in the addition formula for Pell numbers in the form [13, p. 158]

$$P_{m+n} = P_mP_{n+1} + P_{m-1}P_n$$

gives us

$$P_n = P_{n-r}P_{r+1} + P_{n-r-1}P_r.$$

Therefore, we obtain

$$\frac{P_{n-r}P_{r+1} + P_{n-r-1}P_r}{P_rP_{n-r}} = \frac{P_n}{P_rP_{n-r}}.$$

Thus, the proof of this theorem is completed. □

By the following theorem we show that the Pellnomial coefficients are integers.

Theorem 2. *The Pellnomial coefficient* $\left(\begin{array}{c} n \\ r \end{array} \right)_P$ *is an integer for* $0 \leq r \leq n$.

Proof. We use mathematical induction to prove this theorem. First, we prove that the theorem is true when $n = 0$. When $n = 0$, $\left(\begin{array}{c} 0 \\ r \end{array} \right)_P = 1$ is an integer. Now, we assume that it is true for all nonnegative integers $< k$. Then $\left(\begin{array}{c} k-1 \\ r-1 \end{array} \right)$ and $\left(\begin{array}{c} k-1 \\ r \end{array} \right)$ are integers by the inductive hypothesis. Hence, the sum

$$P_{r+1}\left(\begin{array}{c} k-1 \\ r \end{array} \right)_P + P_{k-r-1}\left(\begin{array}{c} k-1 \\ r-1 \end{array} \right)_P = \left(\begin{array}{c} k \\ r \end{array} \right)_P$$

is an integer by Pascal's identity for Pellnomials. Therefore, by induction, every Pellnomial coefficient $\left(\begin{array}{c} n \\ r \end{array} \right)_P$ is an integer. □

Corollary 1. *The product of r consecutive Pell numbers is divisible by $P_r!$.*

Proof. Let the product of r consecutive Pell numbers be $P_n P_{(n+1)} \cdots P_{(n+r-1)}$. Then, by definitions (1) and (2), we have

$$\frac{P_n P_{(n+1)} \cdots P_{(n+r-1)}}{P_r!} = \frac{P_{(n+r-1)}!}{P_r! P_{(n-1)}!} = \left(\begin{array}{c} n+r-1 \\ r \end{array} \right)_P .$$

By Theorem 2, $\frac{P_n P_{(n+1)} \cdots P_{(n+r-1)}}{P_r!}$ is an integer. So $P_n P_{(n+1)} \cdots P_{(n+r-1)}$ is divisible by $P_r!$. □

Throughout the paper, (a, b) denotes the greatest common divisor (gcd) of the positive integers a and b.

We present the first divisibility property between Pell numbers and Pellnomial coefficients in the following theorem.

Theorem 3. *Let $m, n \geq 1$. Then*

$$\frac{P_m}{(P_m, P_n)} \mid \left(\begin{array}{c} m \\ n \end{array} \right)_P .$$

Proof. Let $d = (P_m, P_n)$. Since d can always be expressed as a linear combination $AP_m + BP_n$, there exist integers A and B such that

$$d = AP_m + BP_n.$$

Multiplying both sides by $\left(\begin{array}{c} m \\ n \end{array} \right)_P$ yields

$$\begin{aligned}
d \left(\begin{array}{c} m \\ n \end{array} \right)_P &= AP_m \left(\begin{array}{c} m \\ n \end{array} \right)_P + BP_n \left(\begin{array}{c} m \\ n \end{array} \right)_P \\
&= AP_m \left(\begin{array}{c} m \\ n \end{array} \right)_P + BP_n \frac{P_m!}{P_n! P_{m-n}!} \\
&= AP_m \left(\begin{array}{c} m \\ n \end{array} \right)_P + BP_n \frac{P_m P_{m-1}!}{P_n P_{n-1}! P_{m-n}!} \\
&= P_m \left[A \left(\begin{array}{c} m \\ n \end{array} \right)_P + B \frac{P_{m-1}!}{P_{n-1}! P_{m-n}!} \right] \\
&= P_m \left[A \left(\begin{array}{c} m \\ n \end{array} \right)_P + B \left(\begin{array}{c} m-1 \\ n-1 \end{array} \right)_P \right] \\
&= P_m C, \qquad (3)
\end{aligned}$$

where $C = A \left(\begin{array}{c} m \\ n \end{array} \right)_P + B \left(\begin{array}{c} m-1 \\ n-1 \end{array} \right)_P$. Then, by Theorem 2, $\left(\begin{array}{c} m \\ n \end{array} \right)_P$ is an integer. According to (3), $\left(\begin{array}{c} m \\ n \end{array} \right)_P$ is divisible by $\frac{P_m}{d}$ since $d \mid P_m$. Thus, we obtain the assertion

$$\frac{P_m}{(P_m, P_n)} \mid \left(\begin{array}{c} m \\ n \end{array} \right)_P.$$

The proof of this theorem is completed. □

In order to prove the divisibility

$$\frac{P_{m-n+1}}{(P_{m+1}, P_n)} \mid \left(\begin{array}{c} m \\ n \end{array} \right)_P,$$

we introduce the following lemma.

Lemma 1. *If $m, n \geq 1$, then $(P_{m+1}, P_n) \mid P_{m+1-n}$.*

Proof. Let $d = (P_{m+1}, P_n)$. Then $d \mid P_{m+1}$ and $d \mid P_n$. It is well known that the addition formula for Pell numbers [13, p. 158] is

$$P_{m+n} = P_m P_{n+1} + P_{m-1} P_n. \tag{4}$$

We can use this addition formula in (4) to derive a formula for P_{m+1-n}. Changing m to $m+1$ and n to $-n$ yields

$$\begin{aligned} P_{m+1-n} &= P_{m+1} P_{-(n-1)} + P_{m+1-1} P_{-n} \\ &= (-1)^{n-2} P_{m+1} P_{n-1} + (-1)^{n-1} P_m P_n \\ &= (-1)^n \left(P_{m+1} P_{n-1} - P_m P_n \right) \end{aligned}$$

since $P_0 = 0$ and $P_{-n} = (-1)^{n-1} P_n$ [13, p. 117]. Since $d \mid P_{m+1}$ and $d \mid P_n$, it follows that $d \mid P_{m+1-n}$, as desired. □

We present the second divisibility property between Pell numbers and Pellnomial coefficients in the following theorem.

Theorem 4. *If $m, n \geq 1$, then*

$$\frac{P_{m-n+1}}{(P_{m+1}, P_n)} \mid \left(\begin{array}{c} m \\ n \end{array} \right)_P.$$

Proof. Let $d = (P_{m+1}, P_n)$. So

$$d = D P_{m+1} + E P_n$$

for some integers D and E. Multiplying both sides by $\frac{P_m!}{P_n! P_{m+1-n}!}$ yields

$$\begin{aligned} d \frac{P_m!}{P_n! P_{m+1-n}!} &= D P_{m+1} \frac{P_m!}{P_n! P_{m+1-n}!} + E P_n \frac{P_m!}{P_n! P_{m+1-n}!} \\ &= D \frac{P_{m+1}!}{P_n! P_{m+1-n}!} + E \frac{P_m!}{P_{n-1}! P_{m+1-n}!} \\ &= D \left(\begin{array}{c} m+1 \\ n \end{array} \right)_P + E \left(\begin{array}{c} m \\ n-1 \end{array} \right)_P. \end{aligned} \tag{5}$$

Let $F = D\left(\begin{array}{c} m+1 \\ n \end{array}\right)_P + E\left(\begin{array}{c} m \\ n-1 \end{array}\right)_P$. Then F is an integer by Theorem 2. Since $P_{m+1-n}! = P_{m+1-n}P_{m-n}!$, by (5), we obtain

$$\left(\begin{array}{c} m \\ n \end{array}\right)_P = \frac{P_{m+1-n}}{d}F. \tag{6}$$

Then, by Theorem 2, $\left(\begin{array}{c} m \\ n \end{array}\right)_P$ is an integer. According to (6), $\left(\begin{array}{c} m \\ n \end{array}\right)_P$ is divisible by $\frac{P_{m+1-n}}{d}$ since $d \mid P_{m+1-n}$. Thus, we obtain the assertion

$$\frac{P_{m-n+1}}{(P_{m+1}, P_n)} \mid \left(\begin{array}{c} m \\ n \end{array}\right)_P.$$

The proof of this theorem is completed. □

We need the following theorem and corollary given in [13] before we can prove the desired property:

$$P_{n+1} \mid \left(\begin{array}{c} 2n \\ n \end{array}\right)_P.$$

Theorem 5 ([13, p. 166])**.** $(P_m, P_n) = P_{(m,n)}$.

Corollary 2 ([13, p. 167])**.** $(P_m, P_n) = 1$ *if and only if* $(m, n) = 1$.

Theorem 6. *If* $n \geq 1$, *then* $P_{n+1} \mid \left(\begin{array}{c} 2n \\ n \end{array}\right)_P$.

Proof. Let $m = 2n$. Then, by Theorem 4, we have

$$\frac{P_{n+1}}{(P_{2n+1}, P_n)} \mid \left(\begin{array}{c} 2n \\ n \end{array}\right)_P.$$

The numbers $2n+1$ and n are relatively prime, that is, $(2n+1, n) = 1$. Hence $(P_{2n+1}, P_n) = P_{(2n+1,n)} = P_1$; so P_{2n+1} and P_n are relatively prime. Thus we have the result $P_{n+1} \mid \left(\begin{array}{c} 2n \\ n \end{array}\right)_P$, as desired. □

It follows from Theorem 6 that $\frac{1}{P_{n+1}}\left(\begin{array}{c} 2n \\ n \end{array}\right)_P$ is an integer. We define Pell–Catalan numbers by

$$C_{n,P} = \frac{1}{P_{n+1}}\left(\begin{array}{c} 2n \\ n \end{array}\right)_P \tag{7}$$

with $n \geq 1$. Thus, every Pell-Catalan number is an integer.

We present two ways of defining $C_{n,P}$ with the following theorem.

Theorem 7. *For $n \geq 0$, we have*

$$C_{n,P} = \frac{P_{(2n)}!}{P_n! P_{(n+1)}!},$$

and for $n \geq 1$, we have

$$C_{n,P} = \frac{1}{P_n + P_{n-1}} \left[\left(\begin{array}{c} 2n \\ n \end{array} \right)_P - \left(\begin{array}{c} 2n \\ n-1 \end{array} \right)_P \right].$$

Proof. By the definitions of the Pell analog of $n!$ in (1), the Pellnomials in (2) and the Pell–Catalan number in (7), we have

$$\begin{aligned} C_{n,P} &= \frac{1}{P_{n+1}} \left(\begin{array}{c} 2n \\ n \end{array} \right)_P \\ &= \frac{1}{P_{n+1}} \frac{P_{(2n)}!}{P_n! P_n!} \\ &= \frac{P_{(2n)}!}{P_n! P_{(n+1)}!}. \end{aligned} \tag{8}$$

Thus, the first result of this theorem is obtained. From (8), we obtain

$$\begin{aligned} (P_{n+1} - P_n)\, C_{n,P} &= (P_{n+1} - P_n) \frac{P_{(2n)}!}{P_n! P_{(n+1)}!} \\ &= \frac{P_{n+1}}{P_n! P_{(n+1)}!} P_{(2n)}! - \frac{P_n}{P_n! P_{(n+1)}!} P_{(2n)}! \\ &= \frac{P_{(2n)}!}{P_n! P_n!} - \frac{P_{(2n)}!}{P_{(n-1)}! P_{(n+1)}!} \\ &= \left(\begin{array}{c} 2n \\ n \end{array} \right)_P - \left(\begin{array}{c} 2n \\ n-1 \end{array} \right)_P. \end{aligned}$$

Since $P_{n+1} - P_n = P_n + P_{n-1}$, we reach the second result of this theorem. The proof of this theorem is completed. □

2.1. More results on divisibility. In this section, we introduce some results on divisibility properties including Pell numbers and Pellnomial coefficients.

Theorem 8. *For $n \geq 1$, we have $2 \mid \left(\begin{array}{c} 2n \\ n \end{array} \right)_P$.*

Proof. For $n \geq 1$, we obtain

$$\begin{aligned} \left(\begin{array}{c} 2n \\ n \end{array} \right)_P &= \frac{P_{2n}!}{P_n! P_n!} \\ &= \frac{P_{2n}}{P_n} \frac{P_{(2n-1)}!}{P_{(n-1)}! P_n!} \end{aligned}$$

$$= \frac{P_{2n}}{P_n} \left(\begin{array}{c} 2n-1 \\ n-1 \end{array} \right)_P$$
$$= \frac{P_{2n}}{P_n} \left(\begin{array}{c} 2n-1 \\ n-1 \end{array} \right)_P . \tag{9}$$

Since $\left(\begin{array}{c} 2n-1 \\ n-1 \end{array} \right)_P$ is an integer by Theorem 2 and $P_{2n} = 2P_nQ_n$ [13, p. 123] where Q_n is nth Pell–Lucas number defined by the same recurrence, with the initial conditions $Q_0 = Q_1 = 1$, it follows from (9) that

$$\left(\begin{array}{c} 2n \\ n \end{array} \right)_P = 2Q_n \left(\begin{array}{c} 2n-1 \\ n-1 \end{array} \right)_P ,$$

as desired. □

In [3], the superfactorial is defined by

$$n!! = \prod_{k=1}^{n} k! \text{ and } 0!! = 1$$

as the product of the first n factorials. The superfactorials of $n = 0, 1, 2, 3, 4$ and 5 are

$$\begin{aligned} 0!! &= 1, \\ 1!! &= 1, \\ 2!! &= 2, \\ 3!! &= 12, \\ 4!! &= 288, \\ 5!! &= 34560. \end{aligned}$$

Also, the superfactorial can be given by the formula $n!! = \prod_{0 \le i < j \le n} (j - i)$ which is the determinant of a Vandermonde matrix. Now, we define Pell analog of the superfactorial by

$$P_n!! = \prod_{k=1}^{n} P_k! \text{ and } P_0!! = 1$$

as the product of the first P_n factorials. The Pell-superfactorials of $n = 0, 1, 2$ and 3 are

$$\begin{aligned} P_0!! &= 1, \\ P_1!! &= P_1!P_0! = 1, \\ P_2!! &= P_2!P_1!P_0! = 2, \\ P_3!! &= P_3!P_2!P_1!P_0! = 240. \end{aligned}$$

By the following theorem, we now present an interesting confluence of superfactorials and Pell–Catalan numbers defined in (7).

Theorem 9. *Let $A_n = P_{(2n-1)}!!/[P_{(n-1)}!!]^4$, where n is a positive integer. Then, we have $P_{(2n-1)}! \mid A_n$.*

Proof. Let $B_n = \frac{A_n}{P_{(2n-1)}!}$. Then we obtain

$$\begin{aligned}
\frac{B_{n+1}}{P_{(n+1)}} &= \frac{1}{P_{(n+1)}} \cdot \frac{A_{n+1}}{P_{(2n+1)}!} \\
&= \frac{1}{P_{(n+1)}} \cdot \frac{P_{(2n+1)}!!}{(P_n!!)^4 P_{(2n+1)}!} \\
&= \frac{1}{P_{(n+1)}} \cdot \frac{P_{(2n-1)}!!}{[P_{(n-1)}!!]^4} \cdot \frac{P_{(2n)}!P_{(2n+1)}!}{(P_n!)^4 P_{(2n+1)}!} \\
&= \frac{1}{P_{(n+1)}} \cdot A_n \cdot \frac{P_{(2n)}!}{(P_n!)^4} \\
&= \frac{1}{P_{(n+1)}} \cdot \frac{A_n}{P_{(2n-1)}!} \cdot \frac{P_{(2n)}!P_{(2n-1)}!}{(P_n!)^4} \\
&= \frac{1}{P_{(n+1)}} \cdot \frac{B_n}{P_n} \cdot \frac{P_{(2n)}!P_{(2n-1)}!P_n}{(P_n!)^4} \\
&= \frac{B_n}{P_n} \cdot \frac{1}{P_{(n+1)}} \cdot \frac{P_{(2n)}!}{(P_n!)^2} \cdot \frac{P_{(2n-1)}!P_n}{P_n!P_n!} \\
&= \frac{B_n}{P_n} \cdot \frac{1}{P_{(n+1)}} \cdot \frac{P_{(2n)}!}{(P_n!)^2} \cdot \frac{P_{(2n-1)}!}{P_n!P_{(n-1)}!} \\
&= \frac{B_n}{P_n} C_{n,P} \begin{pmatrix} 2n-1 \\ n \end{pmatrix}_P .
\end{aligned}$$

Since

$$A_1 = P_1!!/[P_0!!]^4 = 1,$$

$$\frac{B_1}{P_1} = \frac{1}{P_1}\frac{A_1}{P_1} = 1,$$

and $\frac{B_2}{P_2} = \frac{B_1}{P_1} C_{1,P} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_P$ is an integer, it follows by induction that $\frac{B_n}{P_n}$ is an integer for every positive integer n. Then $B_n = P_n a$ for some ineteger a. Since $B_n = \frac{A_n}{P_{(2n-1)}!}$, we get

$$A_n = P_{(2n-1)}!P_n a$$

as desired. □

We now verify the divisibility $P_{(2n-1)}! \mid A_n$ for $n = 2$. Since $A_2 = P_3!!/[P_1!!]^4 = 240$ and $P_3! = 120$, we obtain $P_{(2n-1)}! \mid A_n$ for $n = 2$.

3. Conclusion

We first proved several new properties of the Pellnomial coefficients along with their corresponding Pell numbers and the Pell analog of $r!$. Secondly, we defined Pell–Catalan numbers and then we derived two formulas for their presentation. Finally, we introduced some results on divisibility properties including Pell numbers and Pellnomial coefficients.

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