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On some properties of $CV_{(0)}(X, A)$

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Abstract. Let X be a completely regular Hausdorff space and V a Nachbin family on X. For a locally convex algebra A, let $CV_{(0)}(X, A)$ be the algebra of all weighted vector-valued continuous functions with the topology given by the uniform seminorms induced by V. In this paper we study some properties of A that are inherited by $CV_{(0)}(X, A)$. These properties are related to the unit element, spectral seminorms and uniformly A-convex property.

1. Introduction

Algebras of continuous functions have been a very interesting topic in Functional Analysis, in particular in the field of Topological Algebras. The study of certain types of algebras of continuous functions has brought the attention of many mathematicians. One of the main interests of this subject is its algebraic and topological structures ([4, 5, 8, 11, 12, 28, 29, 30]). A particular case in this sense is the class of algebras of weighted continuous functions. These are algebras of complex-valued continuous functions defined on a completely regular Hausdorff space X, where a family V of upper semi-continuous functions on X acts as weights in the definition of these spaces. They are usually denoted by $CV_{(0)}(X)$. Oubbi has studied algebraic and topological properties of them in many papers ([16, 17, 20, 18]). $CV_{(0)}(X)$ is actually a generalization of algebras of continuous functions with different topologies.

Analogously, spaces of vector-valued continuous functions have been also extensively studied ([1, 2, 3, 6, 10, 13, 14]). The most thoroughly studied one is the space $C(X, A)$ of all continuous functions defined on a completely regular Hausdorff space X with values in A, where A is a topological vector

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space. Recently, some authors ([1, 2, 3, 10]) have obtained results concerning the algebra $C_b(X, A)$ of all bounded continuous functions with values in A, where A is a Banach algebra or a more general topological algebra. Oubbi [19, 21, 22] studied the case of algebras of weighted vector-valued continuous functions. These algebras are usually defined via a family V of upper semi-continuous functions on X and are denoted by $CV_{(0)}(X, A)$. Oubbi found many interesting algebraic and topological properties of these algebras.

Recently, García, Palacios, and C. Signoret [10] obtained several results concerning the algebra $C_b(X, A)$. They established conditions on X and/or A so that they can be inherited by $C_b(X, A)$.

The purpose of this paper is to establish properties on A that can be inherited by $CV_{(0)}(X, A)$. In order to do so, some appropriate conditions on X and $CV_{(0)}(X)$ are considered. We give conditions under which $CV_{(0)}(X, A)$ is a unital algebra. We also study spectral properties of $CV_{(0)}(X, A)$, when A has a spectral seminorm. Uniform A-convexity of this algebra is also studied.

Recall that a topological space X is *completely regular* if for each $x \in X$ and a closed subset F of X such that x does not belong to F, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$. Also, a completely regular space X is *pseudo-compact* if every continuous function $f : X \to \mathbb{R}$ is bounded (see [7]). Throughout this paper X will denote a completely regular or pseudocompact space and A a complex locally convex algebra.

2. Preliminaries

A *topological algebra* is a complex algebra A endowed with a Hausdorff linear topology τ in which the multiplication is separately continuous. If the multiplication is jointly continuous we say that A is a topological algebra *with jointly continuous multiplication.* A topological algebra (A, τ) is a *locally convex algebra* if τ is a locally convex topology. In this case, the topology is defined by a family of seminorms $\{\| \, \|_\alpha\}_{\alpha \in I}$. The family of seminorms is *separating* if for each $x \in A$, there exists $\alpha \in I$ such that $\|x\|_\alpha \neq 0$. In this case, we say that the algebra $(A, \{\| \, \|_\alpha\}_{\alpha \in I})$ is a *separating locally convex algebra.* Furthermore, if the multiplication is jointly continuous, then, for each $\alpha \in I$, there exists $\beta \in I$ such that

$$\|xy\|_\alpha \leq \|x\|_\beta \|y\|_\beta$$

for every $x, y \in A$. Also, a locally convex algebra A is said to be a *locally multiplicatively-convex algebra*, abbreviated *m-convex*, if there is a family of submultiplicative seminorms defining the topology, that is, for each $\alpha \in I$

$$\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha$$

for every $x, y \in A$.

If A is a topological algebra with unit e we denote by $G(A)$ the set of its invertible elements. We say that A is *Q-algebra* if $G(A)$ is an open subset of A. If A has no unit, an element $x \in A$ is said to be *quasi-invertible* if there is $y \in A$ such that $x + y - xy = 0 = y + x - yx$, and we denote by $G_q(A)$ the set of all its quasi-invertible elements. In this case we say that A is a *Q-algebra* if the set $G_q(A)$ is an open subset of A.

For a topological algebra A with unit e, if $x \in A$, the *spectrum* of x is defined as

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e \notin G(A)\}.$$

We define the *spectral radius* of x as $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$, when $\sigma(x) \neq \emptyset$, $r(x) = -\infty$ if $\sigma(x) = \emptyset$ and $r(x) = \infty$ if $\sigma(x)$ is not bounded.

If A does not have a unit and $x \in A$, we define the *spectrum* of x as

$$\sigma(x) = \{0\} \cup \left\{\lambda \in \mathbb{C} \setminus \{0\} : \frac{x}{\lambda} \notin G_q(A)\right\}.$$

We also define the *spectral radius* of x as $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$ and $r(x) = \infty$ if $\sigma(x)$ is not bounded.

Now, if A is a locally convex algebra with a family of seminorms $\{\| \ \|_\alpha\}_{\alpha \in I}$ defining its topology and $\mathcal{B}$ is a given family of subsets of A, we will say that A is *$\mathcal{B}$-hypotopological*, if, for each $\alpha \in I$ and each $B \in \mathcal{B}$, there is $\beta \in I$ and $M > 0$ such that

$$\max\{\|xy\|_\alpha, \|yx\|_\alpha\} \leq M\|y\|_\beta$$

for every $x \in B$ and every $y \in A$.

Let X be a completely regular Hausdorff space and V a family of upper semi-continuous non-negative functions on X. We say that V is a *Nachbin family* if it satisfies:

(1) for each $x \in X$ there is a $v \in V$ such that $v(x) > 0$,
(2) for each $v \in V$ and $\lambda > 0$ we have that $\lambda v \in V$,
(3) for each $v_1, v_2 \in V$ there is $v \in V$ such that $\max\{v_1, v_2\} \leq v$.

We will need that $\sup_{x \in X} v(x) > 0$ for every $v \in V$, when this supremum is finite, so additionally, we will suppose that for each $v \in V$ we have that $v \neq 0$. For each $v \in V$ we write $N_v = \{x \in X : v(x) > 0\}$, and for every $v \in V$ and $\epsilon > 0$ we define $N_{\epsilon,v} = \{x \in X : v(x) > \epsilon\}$. Also, we will use the following notations:

$$B(X) = \{f : X \to \mathbb{C} \mid f \text{ is bounded on } X\}$$

and

$$B_0(X) = \{f : X \to \mathbb{C} \mid f \text{ vanishes at } \infty\},$$

we write $B_{(0)}(X)$ to simplify the notation for both spaces.

Let X be a completely regular Hausdorff space and A a locally convex algebra. We define the following weighted spaces

$$CV(X, A) = \{f \in C(X, A) : v \cdot (\| \ \|_\alpha \circ f) \in B(X),\ \forall (v \in V, \alpha \in I)\}$$

and

$$CV_0(X,A) = \{f \in C(X,A) : v \cdot (\| \|_\alpha \circ f) \in B_0(X)\ ,\ \forall (v \in V, \alpha \in I)\}.$$

These are vector spaces with the usual operations for vector-valued functions. To simplify the notation, we write $CV_{(0)}(X,A)$ for both spaces.

In both cases we can define the following seminorms, for $\alpha \in I$ and $v \in V$:

$$\|f\|_{\alpha,v} = \sup_{x\in X} v(x)\|f(x)\|_\alpha$$

for every $f \in CV_{(0)}(X,A)$. So, we have that $(CV_{(0)}(X,A), \{\| \|_{\alpha,v}\}_{\alpha,v})$ is a locally convex space. When $A = \mathbb{C}$, it is usual to write $CV_{(0)}(X)$ instead of $CV_{(0)}(X,\mathbb{C})$ and the seminorms in this case are

$$\|f\|_v = \sup_{x\in X} v(x)|f(x)|$$

for every $f \in CV_{(0)}(X)$. Oubbi [17, 18, 20] has extensively studied these spaces of complex-valued continuous functions.

Oubbi [19] gave some conditions for X and A which imply that $CV_{(0)}(X,A)$ is a locally convex algebra. The conditions that Oubbi assumed for X and A are the following:

(a) for every $x \in X$ there is $v \in V$ such that the set N_v is a neighbourhood of x,

(b) A is $\mathcal{F}_{V_{(0)}}$-hypotopological, where

$$\mathcal{F}_{V_{(0)}} = \{\{f(x) : x \in N_v\} : f \in CV_{(0)}(X,A), v \in V\}.$$

Oubbi proved (see Propositions 4.1 and 4.2 in [19]) that if X and A satisfy (a) and (b), respectively, then $(CV_{(0)}(X,A), \{\| \|_{\alpha,v}\}_{\alpha,v})$ is a locally convex algebra. *Throughout this paper we will assume that X and A satisfy properties (a) and (b).*

3. Identity

We will start this section by giving some conditions for $CV_{(0)}(X,A)$ so that it has a unit. For an algebra A and an element $a \in A$, we define the constant function $\hat{a} : X \to A$ by $\hat{a}(x) = a$ for every $x \in X$.

Proposition 1. *Let X be a completely regular Hausdorff space and let $(A, \{\| \|_\alpha\}_{\alpha\in I})$ a separating locally convex algebra with unit e. Then the following conditions are equivalent:*

(1) $\mathbf{1} \in CV_{(0)}(X)$,

(2) *each $v \in V$ belongs to $B_{(0)}(X)$,*

(3) $\hat{e} \in CV_{(0)}(X,A)$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathbf{1} \in CV_{(0)}(X)$. From the definition of $CV_{(0)}(X)$ it is easy to see that, for each $v \in V$, we have that $v \in B_{(0)}(X)$.

(2) $\Rightarrow$ (3). Let us suppose now that every $v \in V$ belongs to $B_{(0)}(X)$. Consider $\alpha \in I$ and $v \in V$, then

$$\|\hat{e}\|_{\alpha,v} = \sup_{x\in X} v(x)\|e\|_\alpha = \|e\|_\alpha \left(\sup_{x\in X} v(x)\right) < \infty.$$

Also, if $x \in X$, then

$$v \cdot (\| \ \|_\alpha \circ \hat{e})(x) = v(x)\|\hat{e}(x)\|_\alpha = \|e\|_\alpha v(x).$$

Therefore, the function $v \cdot (\| \ \|_\alpha \circ \hat{e})$ vanishes at ∞. Hence $\hat{e} \in CV_{(0)}(X, A)$.

(3) $\Rightarrow$ (1). Let $\alpha \in I$ be such that $\|e\|_\alpha \neq 0$. We have two cases. On the one hand, if $\hat{e} \in CV(X, A)$, then

$$\sup_{x\in X} v(x) = \frac{1}{\|e\|_\alpha} \sup_{x\in X} v(x)\|e\|_\alpha < \infty,$$

for every $v \in V$. It follows that $\mathbf{1} \in CV(X)$. On the other hand, if $\hat{e} \in CV_0(X, A)$, then we have that

$$v(x) = \frac{1}{\|e\|_\alpha} v(x)\|e\|_\alpha = \frac{1}{\|e\|_\alpha} v(x)(\| \ \|_\alpha \circ \hat{e})(x)$$

vanishes at infinity, for every $v \in V$. This shows that $\mathbf{1} \in CV_0(X)$ and it follows that $\mathbf{1} \in CV_{(0)}(X)$. $\square$

Recall that a family $\mathcal{A}$ of complex-valued functions defined on a completely regular space X is *essential* if for each $x \in X$ there is an $f \in \mathcal{A}$ such that $f(x) \neq 0$.

Example. This is an example of an algebra of complex-valued continuous functions with unit that is non-essential. Let $X = [0, 1] \cup (\mathbb{Q} \cap [2, 3])$ and the Nachbin family $V = C^+(X)$, the family of all positive continuous functions on X. We claim that

$$CV_{(0)}(X) = \{f \in C(X) \mid f(x) = 0, \ \forall\, x \in \mathbb{Q} \cap [2, 3]\}.$$

If $f \in C(X)$ and vanishes on $\mathbb{Q} \cap [2, 3]$, then, since $[0, 1]$ is compact, it follows that for each $v \in V$ we have that $vf \in B_{(0)}(X)$. Consider $f \in CV_{(0)}(X)$ and $x \in \mathbb{Q} \cap [2, 3]$. If $f(x) \neq 0$, then there is $\epsilon > 0$ such that $|f(y)| > \frac{|f(x)|}{2}$ for every $y \in (x - \epsilon, x + \epsilon) \cap (\mathbb{Q} \cap [2, 3])$. Choose $z \in (x - \epsilon, x + \epsilon) \cap [2, 3]$ with $z \notin \mathbb{Q}$. Define $v_0 : X \to \mathbb{R}$ by

$$v_0(y) = \begin{cases} \frac{1}{(1-z)^2}, & \text{if } x \in [0, 1], \\ \frac{1}{(y-z)^2}, & \text{if } x \in \mathbb{Q} \cap [2, 3]. \end{cases}$$

Clearly $v_0 \in V$ and $v_0 f$ is not bounded on $(x - \epsilon, x + \epsilon) \cap (\mathbb{Q} \cap [2, 3])$. This contradicts the fact that $f \in CV_{(0)}(X)$, so $f(x) = 0$.

Note that $CV_{(0)}(X)$ has a unit $u : X \to \mathbb{R}$ given by

$$u(x) = \begin{cases} 1, & \text{if } x \in [0,1], \\ 0, & \text{if } x \in \mathbb{Q} \cap [2,3]. \end{cases}$$

Remark 1. Suppose that $CV_{(0)}(X)$ is essential and that $u \in CV_{(0)}(X)$ is a unit. If $x \in X$, then there is $f_0 \in CV_{(0)}(X)$ such that $f_0(x) \neq 0$. Therefore, from the fact that $f_0(x)u(x) = f_0(x)$, it follows that $u(x) = 1$, that is, $u = \mathbf{1}$. That is, $CV_{(0)}(X)$ has a unit if and only if $\mathbf{1} \in CV_{(0)}(X)$ and this happens if and only if $CV_{(0)}(X)$ contains the constant functions.

In the essential case we have the next result.

Proposition 2. *Let X be a completely regular Hausdorff space and let $(A, \{\| \|_\alpha\}_{\alpha \in I})$ a separating locally convex algebra with unit e. If $CV_{(0)}(X)$ is essential, then the following conditions are equivalent:*
(1) *$CV_{(0)}(X)$ has a unit,*
(2) *Every $v \in V$ belongs to $B_{(0)}(X)$,*
(3) *$CV_{(0)}(X, A)$ contains the constant functions,*
(4) *$CV_{(0)}(X, A)$ has a unit.*

Proof. Since we have proved Proposition 1 and Remark 1, we need to prove (4) $\Rightarrow$ (1). In order to do this, let us suppose that $u \in CV_{(0)}(X, A)$ is a unit. Fix $x \in X$, since $CV_{(0)}(X)$ is essential, there exists $f_0 \in CV_{(0)}(X)$ such that $f_0(x) \neq 0$. Define $f : X \to A$ by $f(y) = f_0(y)e$ for each $y \in X$. It follows that if $v \in V$ and $\alpha \in I$, then

$$v(y)\|f(y)\|_\alpha = v(y)\|f_0(y)e\|_\alpha = v(y)|f_0(y)|\|e\|_\alpha$$

for every $y \in X$. Therefore $f \in CV_{(0)}(X, A)$ and

$$u(x)f_0(x)e = u(x)f(x) = f(x) = f_0(x)e.$$

So, $f_0(x)(u(x) - e) = 0$, which implies that $u(x) = e$. Since x is arbitrary, we obtain that $u = \hat{e} \in CV_{(0)}(X, A)$. From Proposition 1 we conclude that $CV_{(0)}(X)$ has a unit. ☐

Remark 2. Due to [9] and its corollary, Propositions 1 and 2 can be proved without the separating property, assuming jointly continuous multiplication on A.

4. Spectral seminorms

This section is devoted to the study of spectral seminorms in $CV_{(0)}(X, A)$.

Proposition 3. *Let X be a pseudocompact Hausdorff space and let $(A, \{\| \|_\alpha\}_{\alpha \in I})$ a locally convex algebra with jointly continuous multiplication, with continuous inversion and with unit e. If $CV_{(0)}(X)$ is essential* *then*

$$\sigma(f) = \bigcup_{x \in X} \sigma(f(x))$$

for each $f \in CV_{(0)}(X, A)$*.*

Proof. First let us suppose that $\mathbf{1} \in CV_{(0)}(X)$. In this case by Remark 2 we have that every $v \in V$ belongs to $B_{(0)}(X)$ and from the fact that X is pseudocompact, it can be proved that $CV_{(0)}(X, A) = C(X, A)$. In this case it is well known that, if A has continuous inversion, then

$$\sigma(f) = \bigcup_{x \in X} \sigma(f(x))$$

for all $f \in C(X, A)$.

Now, assume that $\mathbf{1} \notin CV_{(0)}(X)$. Then by Remark 1 and Remark 2, we have that $CV_{(0)}(X, A)$ has no unit. Take $f \in CV_{(0)}(X, A)$. If $\lambda \notin \sigma(f)$, then $\lambda \neq 0$ and $\frac{f}{\lambda}$ is quasi-invertible, so there is $g \in CV_{(0)}(X, A)$ such that $\frac{f}{\lambda} + g = \frac{f}{\lambda} g$. Then, for each $x \in X$, we have that $\frac{f(x)}{\lambda} + g(x) = \frac{f(x)}{\lambda} g(x)$. Thus, for each $x \in X$, we have that $\lambda \notin \sigma(f(x))$.

Now take λ such that $\lambda \notin \sigma(f(x))$ for every $x \in A$. Then $\lambda \neq 0$ and $\frac{f(x)}{\lambda}$ is quasi-invertible. Therefore, for each $x \in A$, there is $g(x) \in A$ such that $\frac{f(x)}{\lambda} + g(x) = \frac{f(x)}{\lambda} g(x)$. The relation between invertible and quasi-invertible elements is given by the point-wise equality $e - G_q(A) = G(A)$. Then $f(x) - \lambda e$ is invertible and $g(x) = f(x)(f(x) - \lambda e)^{-1}$ for all $x \in A$. Since A has jointly continuous multiplication, g is a continuous function. Now, take $\alpha \in I$ and $v \in V$. Then there is $\beta \in I$ such that

$$\|xy\|_\alpha \leq \|x\|_\beta \|y\|_\beta$$

for every $x, y \in A$. Therefore, if $v \in V$ and $\alpha \in I$, we have

$$\begin{aligned} v(x)\|g(x)\|_\alpha &= v(x)\|f(x)(f(x) - \lambda e)^{-1}\|_\alpha \\ &\leq v(x)\|f(x)\|_\beta \|(f(x) - \lambda e)^{-1}\|_\beta. \end{aligned}$$

From this, because X is pseudocompact, the inversion is continuous and $f \in CV_{(0)}(X, A)$, it follows that $g \in CV_{(0)}(X, A)$, and since $\frac{f}{\lambda} + g = \frac{f}{\lambda} g$, we obtain that $\lambda \notin \sigma(f)$. □

Definition 1. Let A be an algebra in which a seminorm $\| \ \|$ is defined. We say that $\| \ \|$ is a *weakly spectral seminorm* on A if $r(a) \leq \|a\|$ for every $a \in A$. Furthermore, if $\| \ \|$ is a weakly spectral seminorm and $\| \ \|$ is submultiplicative, we say that $\| \ \|$ is a *spectral seminorm* on A.

Suppose that A is an algebra in which a seminorm $\| \ \|$ is defined, and consider A endowed with the topology generated by the seminorm $\| \ \|$. For a Nachbin family V, we consider

$$CV_{(0)}(X, A) = \{f \in C(X, A) : v \cdot (\| \ \| \circ f) \in B_{(0)}(X), \ \forall (v \in V)\}.$$

Then, for a fixed $v \in V$, we can consider the seminorm on $CV_{(0)}(X,A)$ given by

$$\|f\|_v = \sup_{x\in X} v(x)\|f(x)\|$$

for every $f \in CV_{(0)}(X,A)$.

We have the next result.

Proposition 4. *Let X be a pseudocompact Hausdorff space and A an algebra with unit e, endowed with the topology generated by a spectral seminorm $\| \ \|$. If $CV_{(0)}(X)$ is essential and a Q-algebra, then there is $v \in V$ such that $\| \ \|_v$ is a weakly spectral seminorm on $CV_{(0)}(X,A)$. Conversely, suppose that $\mathbf{1} \in CV_{(0)}(X)$ and $v \in V$ is such that $\| \ \|_v$ is a spectral seminorm on $CV_{(0)}(X,A)$, then $CV_{(0)}(X)$ is a Q-algebra.*

Proof. Take $f \in CV_{(0)}(X,A)$. Since $(A, \| \ \|)$ is a spectral algebra we have that

$$r(f(x)) \le \|f(x)\|$$

for every $x \in X$. Oubbi proved in [17] that, if $CV_{(0)}(X)$ is a Q-algebra, then there are $v \in V$ and $\epsilon > 0$ such that the set

$$N(\epsilon, v) = \{x \in X : v(x) > \epsilon\}$$

is dense in X. Consider now $x \in X$ and take $(x_\delta)_\delta$ a net contained in $N(\epsilon, v)$ such that $x_\delta \to x$. Then

$$\begin{aligned}\|f(x)\| &= \lim_\delta \|f(x_\delta)\| = \lim_\delta \|f(x_\delta)\| v(x_\delta)\frac{1}{v(x_\delta)} \\ &\le \lim_\delta \|f(x_\delta)\| v(x_\delta)\frac{1}{\epsilon} \le \|f\|_v \frac{1}{\epsilon} = \|f\|_{\frac{v}{\epsilon}}.\end{aligned}$$

So, if $\lambda \in \sigma(f)$, by Proposition 3 there is an element $x_0 \in X$ such that $\lambda \in \sigma(f(x_0))$. Therefore

$$|\lambda| \le r(f(x_0)) \le \sup_{x\in X} r(f(x)) \le \sup_{x\in X} \|f(x)\| \le \|f\|_{\frac{v}{\epsilon}}.$$

If $v' = \frac{v}{\epsilon} \in V$, then $r(f) \le \|f\|_{v'}$. This means that $\| \ \|_{v'}$ is a weakly spectral seminorm on $CV_{(0)}(X,A)$.

Now, suppose that $\mathbf{1} \in CV_{(0)}(X)$, this implies that $\hat{e} \in CV_{(0)}(X,A)$. Let $v \in V$ such that $\| \ \|_v$ is a spectral seminorm on $CV_{(0)}(X,A)$. Using Theorem 2.2.5 of [27] we have that $(CV_{(0)}(X,A), \| \ \|_v)$ is a Q-algebra. Then

$$\{F \in CV_{(0)}(X,A) \mid \|\hat{e} - F\|_v < 1\} \subseteq G(CV_{(0)}(X,A)).$$

Note that there is a natural embedding, which in fact is an isometry, of $CV_{(0)}(X)$ into $CV_{(0)}(X,A)$ in the following way: if $f \in CV_{(0)}(X)$, we define $\tilde{f} \in CV_{(0)}(X,A)$ by $\tilde{f}(x) = f(x)e$ for every $x \in X$. Now, let $f \in CV_{(0)}(X)$ be such that

$$\|\mathbf{1} - f\|_v < 1.$$

Using corollary from [9] we may suppose that the seminorm in A satisfies $\|e\| = 1$, then it is not difficult to see that

$$\|\hat{e} - \tilde{f}\|_v = \|\mathbf{1} - f\|_v < 1.$$

It follows that $\tilde{f} \in G(CV_{(0)}(X, A))$. In this way, there is $H \in CV_{(0)}(X, A)$ such that $\tilde{f}H = \hat{e}$. From this we have that $f(x)eH(x) = e$ for every $x \in X$. This implies that $f(x) \neq 0$ for every $x \in X$. Define $h : X \to \mathbb{C}$ by

$$h(x) = \frac{\overline{f(x)}}{|f(x)|}\|H(x)\|$$

for every $x \in X$. Then $h \in CV_{(0)}(X)$ and.

$$f(x)h(x) = \frac{f(x)\overline{f(x)}}{|f(x)|}\|H(x)\| = \frac{|f(x)|^2}{|f(x)|}\|H(x)\|$$
$$= |f(x)|\|H(x)\| = \|f(x)eH(x)\| = \|e\| = 1$$

for every $x \in X$. This means that $f \in G(CV_{(0)}(X))$. Again from Theorem 2.2.5 [27] we get that $CV_{(0)}(X)$ is a Q-algebra. □

5. Locally uniformly A-convexity

This section is devoted to giving some conditions to ensure that if A is a locally uniformly A-convex algebra, then $CV_{(0)}(X, A)$ is a locally uniformly A-convex algebra too. In order to do that, we need some preliminary results.

Recall that A is a *locally uniformly A-convex algebra*, if its topology can be given by a family of seminorms $\{\| \|_\alpha\}_{\alpha \in I}$ such that, for each $x \in A$, there exists $M(x) > 0$ such that

$$\max\{\|xy\|_\alpha, \|yx\|_\alpha\} \leq M(x)\|y\|_\alpha \tag{1}$$

for every $y \in A$ and $\alpha \in I$. For a locally uniformly A-convex algebra $(A, \{\| \|_\alpha\}_{\alpha \in I})$ with unit e, we can always choose the family of seminorms satisfying that $\|e\|_\alpha = 1$ for all $\alpha \in I$ (see Proposition 1 in [26]). Also we can define a norm on it as follows:

$$\|x\|_{op} = \inf\{M(x) : M(x) \text{ satisfies relation (1)}\}$$

for every $x \in A$. We know (see [23, 24, 25]) that this norm satisfies the following properties:

(1) $\|xy\|_{op} \leq \|x\|_{op}\|y\|_{op}$ for every $x, y \in A$,
(2) for each $x \in A$ and $\alpha \in I$, $\|xy\|_\alpha \leq \|x\|_{op}\|y\|_\alpha$ for every $y \in A$,
(3) if A has a unit e then, for each $\alpha \in I$, $\|x\|_\alpha \leq \|x\|_{op}$ for every $x \in A$.

If A has a unit, we can define from (3) another norm in A in the following way:

$$\|x\|_A = \sup_{\alpha \in I}\{\|x\|_\alpha\}.$$

This norm is called the *absorbing norm* in A. It follows that $\|x\|_A \leq \|x\|_{op}$ for all $x \in A$.

Proposition 5. *If $(A, \{\| \|_\alpha\}_{\alpha \in I})$ is a sequentially complete locally uniformly A-convex algebra with unit e, then the norms $\| \|_{op}$ and $\| \|_A$ are equivalent. A subset B of A is bounded with respect to the topology generated by the family of seminorms if and only if it is bounded in the norm $\| \|_{op}$.*

Proof. Oudadess [23, 25] proved that under these conditions on A, the topology induced by the absorbing norm $\| \|_A$ is finer than the topology induced by the family of seminorms. Moreover, $(A, \| \|_A)$ is a Banach algebra and a subset B of A is bounded if and only if B is bounded with respect to $\| \|_A$. Also, using arguments similar to those given in [23, 25], we obtain that $(A, \| \|_{op})$ is a Banach algebra.

Now, consider the identity function $I : (A, \| \|_{op}) \to (A, \| \|_A)$. As we mentioned before, $\|x\|_A \leq \|x\|_{op}$ for every $x \in A$. It follows from the Open Mapping Theorem that I is an isomorphism between Banach spaces. Then the conclusion follows. □

Now we apply the above results to $CV_{(0)}(X, A)$.

Proposition 6. *Let X be a completely regular Hausdorff space and let $(A, \{\| \|_\alpha\}_{\alpha \in I})$ a sequentially complete locally uniformly A-convex algebra with unit e. If $CV_{(0)}(X)$ is a locally uniformly A-convex algebra, then $CV_{(0)}(X, A)$ is a locally uniformly A-convex algebra.*

Proof. Oubbi has proved in [17] that $CV_{(0)}(X)$ is a locally uniformly A-convex algebra if and only if $CV_{(0)}(X) \subseteq C_b(X)$.

Take $\alpha \in I$, $v \in V$ and $f \in CV_{(0)}(X, A)$. If $g \in CV_{(0)}(X, A)$, since

$$\|f(x)g(x)\|_\alpha \leq \|f(x)\|_{op}\|g(x)\|_\alpha$$

for every $x \in X$, we have that

$$\|fg\|_{\alpha,v} = \sup_{x \in X} v(x)\|f(x)g(x)\|_\alpha \leq \sup_{x \in X} v(x)\|f(x)\|_{op}\|g(x)\|_\alpha. \tag{2}$$

The function $\| \|_\beta \circ f \in CV_{(0)}(X)$ for all $\beta \in I$, hence it is a bounded real-valued function. So, there is $M_\beta > 0$ such that $\sup_{x \in X} \|f(x)\|_\beta \leq M_\beta$, that is, $f(X)$ is a bounded set in A, and therefore, by Proposition 5, is bounded in the norm $\| \|_{op}$. Therefore there is $M(f) > 0$ such that

$$\sup_{x \in X} \|f(x)\|_{op} \leq M(f).$$

From relation (2) we obtain that

$$\|fg\|_{\alpha,v} \leq M(f)\|g\|_{\alpha,v}.$$

It can be proved, in a similar way, that there exists $N(f) > 0$ such that $\|gf\|_{\alpha,v} \leq N(f)\|g\|_{\alpha,v}$. □

6. Examples

Finally, we provide some examples. These examples can be found in [18]. Let X be a completely regular Hausdorff space, $(A, \{\| \ \|_\alpha\}_{\alpha\in I})$ a locally convex algebra and let the algebras of continuous functions

$$C_b(X, A) = \left\{ f : X \to A : \sup_{x\in X} \|f(x)\|_\alpha < \infty, \text{ for each } \alpha \in I \right\}$$

and

$$C_0(X, A) = \{f : X \to A : \| \ \|_\alpha \circ f \text{ vanishes at infinity, for each } \alpha \in I\}$$

both be equipped with the topology τ_∞ determined by the family of seminorms

$$\|f\|_{\alpha,\infty} = \sup_{x\in X} \|f(x)\|_\alpha$$

for each $f \in C_b(X, A)$ (resp. $f \in C_0(X, A)$) and each $\alpha \in I$. In the following examples X will denote a completely regular Hausdorff space and $(A, \{\| \ \|_\alpha\}_{\alpha\in I})$ a locally convex algebra.

Example 1. Let V be the family of all positive constant functions defined on X. Clearly V is a Nachbin family. If $v = k$ is in V and $\alpha \in I$, then we have that

$$\|f\|_{\alpha,v} = \sup_{x\in X} v(x)\|f(x)\|_\alpha = k \sup_{x\in X} \|f(x)\|_\alpha$$

for all $f \in CV_{(0)}(X, A)$. From this it follows that $CV(X, A) = C_b(X, A)$ and $CV_0(X, A) = C_0(X, A)$ where the induced topology is the uniform topology.

Example 2. Let us consider

$$V = \{\lambda\chi_K : \lambda > 0,\ K \subseteq X \text{ compact}\}$$

– the family of all positive multiples of characterirstic functions of compact sets in X. For $v \in V$ with $v = \lambda\chi_K$, we have that

$$\|f\|_{\alpha,v} = \sup_{x\in X} \lambda\chi_K(x)\|f(x)\|_\alpha = \lambda \sup_{x\in K} \|f(x)\|_\alpha.$$

We note that $CV_{(0)}(X, A) = C_c(X, A)$, the algebra of all vector-valued continuous functions with the uniform convergence topology on compact subsets of X.

Note that, since $\mathbf{1} \in CV_{(0)}(X)$ and by Proposition 2.3 of [17], we have that for some spaces X, $CV_{(0)}(X)$ is not a Q-algebra.

Example 3. Let X be a locally compact space, and V the family $C^+(X)$ of all real positive continuous functions defined on X. Then $CV_{(0)}(X) = \mathcal{K}(X)$, the algebra of all complex-valued continuous functions on X with compact support. The topology coincides with the inductive limit locally convex topology given by the Banach algebras $(C_K(X), \| \|_\infty)$ of real continuous functions with support in K, where K is a compact subset of X ([18], p. 106, Example 4).

In this case, not every element of V is bounded, and not every element of V vanishes at infinity, so $\mathbf{1} \notin CV_{(0)}(X)$.

Example 4. Let V be the family of all positive upper-semicontinuous functions that vanish at infinity. We have that $CV_{(0)}(X) = C_b(X)$ and the topology coincides with the strict topology β ([4, 30]). In this case, every element of V is bounded, so $\mathbf{1} \in CV_{(0)}(X)$. Therefore, by Proposition 2.3 of [17], $CV_{(0)}(X)$ is not a Q-algebra.

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