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Rate of convergence for rational and conjugate rational Fourier series of functions of generalized bounded variation

Hardeepbhai J. Khachar and Rajendra G. Vyas

Abstract. In this article, extending the results of Fourier and conjugate Fourier series, the rates of convergence for rational and conjugate rational Fourier series for functions of generalized bounded variation are estimated.

1. Introduction

Bojanić [1] gave a quantitative version of the Dirichlet-Jordan test on the convergence of Fourier series. Later on, this result was further generalized for functions of generalized bounded variations [3, 11]. The work of Bojanić led the groundwork for estimations related to the rate of convergence for Fourier series and conjugate Fourier series for different orthogonal systems [2, 8, 9]. Recently, some properties of Fourier series are extended for rational Fourier series, introduced by Džrbašyan [5].

The rational orthogonal system is defined as

$$\phi_0(e^{ix}) = 1, \phi_n(e^{ix}) = \frac{\sqrt{1 - \mid \alpha_n \mid^2} e^{ix}}{1 - \overline{\alpha_n} e^{ix}} \prod_{k=1}^{n-1} \frac{e^{ix} - \alpha_k}{1 - \overline{\alpha_k} e^{ix}}, \tag{1}$$

and $\phi_{-n}(e^{ix}) = \overline{\phi_n(e^{ix})}$, $\forall n \in \mathbb{N}$. Here, $\{\alpha_n\}_{n\in\mathbb{N}}$ is a complex sequence such that α_k's are in open unit disk $\mathbb{D}$. In the sequel, the following condition is assumed to be satisfied and with r as mentioned below,

$$\sup |\alpha_k| = r < 1, \ \forall k \in \mathbb{N}. \tag{2}$$

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Corresponding author: H. J. Khachar

Since $\sum_{n=1}^{\infty}(1-|\alpha_k|)=\infty$, the system in (1) is complete in $L^2[-\pi,\pi]$ [4].

For a 2π-periodic and integrable function f, the rational Fourier series of f is defined as

$$f(x)\sim\sum_{n=-\infty}^{\infty}\hat{f}(n)\phi_n(e^{ix}),$$

where $\hat{f}(n)$ is the n^{th} rational Fourier coefficient of f, given by

$$\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\overline{\phi_n(e^{ix})}dx,$$

and the conjugate rational Fourier series is given by

$$\sum_{n=-\infty}^{\infty}(-i)\text{sgn}(n)\hat{f}(n)\phi_n(e^{ix}).$$

Note that, if $\alpha_k=0,$ for all $k\in\mathbb{N}$ in (1) then $\{\phi_n(e^{ix})\}$ reduces to $\{e^{inx}\}$ and therefore the rational Fourier series (the conjugate rational Fourier series) reduces to the classical Fourier series (the conjugate Fourier series).

Some of the work related to theoretical aspects of rational Fourier series can be found in [6, 7, 10].

2. Preliminaries and notations

Throughout this paper, the notations mentioned in this section will be used.

For $n\in\mathbb{N}$ and $x\in[-\pi,\pi]$, the partial sum of the Fourier series of f is given by

$$S_nf(x)=\sum_{k=-n}^{n}\hat{f}(k)\phi_k(e^{ix})$$

and the partial sum of the conjugate Fourier series of f is given by

$$\tilde{S}_nf(x)=\sum_{k=-n}^{n}(-i)\text{sgn}(k)\hat{f}(k)\phi_k(e^{ix}).$$

Waterman [12] defined the concept of Λ-bounded variation as follows. Let $\{\lambda_n\}_{n=0}^{\infty}$ be a non decreasing sequence of positive numbers such that $\sum\frac{1}{\lambda_n}$ diverges. Then a real valued function f is said to be of Λ-bounded variation on $[a,b]$ (i.e., $f\in\Lambda BV[a,b]$) if

$$\sum_{k=1}^{n}\frac{|\ f(b_k)-f(a_k)\ |}{\lambda_k}<\infty,$$

for every sequence of non-overlapping intervals $[a_k, b_k], k = 1, ..., n$, which is contained in $[a, b]$, and the Λ-variation is defined by

$$V_\Lambda(f, [a, b]) = \sup \sum_{k=1}^{n} \left\{ \frac{| f(b_k) - f(a_k) |}{\lambda_k} \right\}.$$

Note that, if in the above definition $\lambda_n = n^\beta;\ \beta \in (0, 1)$, then f is said to be of n^β-bounded variation (i.e. $f \in \{n\}^\beta - BV[a, b]$).

As mentioned in [11], we suppose that $\frac{\lambda_{|k|}}{|k|}$ is non increasing and for fixed n, $H(t)$ is a continuously non increasing function on $[-\pi, 0)$ and $(0, \pi]$ such that

$$H(t) = \frac{\lambda_{|k|}}{t},$$

where $t = \frac{k\pi}{n+1}$ and $k = \pm 1, \pm 2, ..., \pm (n + 1)$.

The following notations will be used in the rest of the discussion:

i) $\psi_x(t) = f(x) - f(x - t),\ t \in \mathbb{R}$,

ii) $\tilde{f}(x) = \lim_{\epsilon \to 0^+} \tilde{f}(x; \epsilon) = \frac{1}{\pi} \int_{\epsilon \leq \pi} \frac{f(x - t)}{2 \tan(t/2)} dt$,

iii) $osc(\psi_x, [a, b]) = \sup_{t, y \in [a, b]} |\psi_x(t) - \psi_x(y)|$,

iv) $\eta_{km} = \frac{k\pi}{m + 1},\ \forall k = 0, 1, 2, ..., m;\ m \in \mathbb{N} \cup \{0\}$,

v) $I^+_{km} = [\eta_{km}, \eta_{(k+1)m}]$,

vi) $I^-_{km} = [-\eta_{(k+1)m}, -\eta_{km}]$.

3. Results

First, we state the results related to the rate of convergence of rational Fourier series.

Theorem 1. *If f is a bounded, measurable function in $[-\pi, \pi]$ and is regulated, i.e., $f(x) = 1/2\{f(x + 0) + f(x - 0)\}$, then*

$$|S_n f(x) - f(x)| \leq \frac{2(1 + r)}{1 - r} \sum_{k=0}^{n} \frac{1}{k + 1} \left\{ osc(\psi_x, I^+_{kn}) + osc(\psi_x, I^-_{kn}) \right\}.$$

Corollary 1. *If $f \in \Lambda BV([-\pi, \pi])$, $\frac{\pi}{n+1} = a_n < a_{n-1} < ... < a_0 = \pi$ and $-\pi = b_0 < b_1, ..., < b_n = \frac{-\pi}{n+1}$, then*

$$\left(\frac{1 - r}{1 + r}\right) |S_n f(x) - f(x)| \leq \frac{2\lambda_{n+1}}{n + 1} (V_\Lambda(\psi, [0, \pi]) + V_\Lambda(\psi, [-\pi, 0])) + \frac{2\pi}{n + 1} \sum_{i=0}^{n-1} V_\Lambda(\psi, [0, a_i])(H(a_{i+1}) - H(a_i))$$

$$+\frac{2\pi}{n+1}\sum_{i=0}^{n-1} V_\Lambda(\psi,[b_i,0])(H(b_i)-H(b_{i+1})).$$

Proof. The result can be easily obtained from Theorem 1 and then following the method used in [11]. □

Corollary 2. *If* $f \in \{n^\beta\} - BV([-\pi,\pi]),\ 0<\beta<1,$ *then*

$$|S_nf(x)-f(x)| \leq \frac{2(2-\beta)}{(n+1)^{1-\beta}}\cdot\frac{1+r}{1-r}\sum_{k=1}^{n}\frac{1}{k^\beta}\Big\{V_{\{n^\beta\}}(\psi,[0,\pi/k]) + V_{\{n^\beta\}}(\psi,[-\pi/k,0])\Big\}.$$

Theorem 1, Corollary 1 and Corollary 2 are analogues results of [11, 3] for rational Fourier series as, for $r=0$, the estimations for classical Fourier series are obtained. These results generalize the estimation of rational Fourier series for functions of bounded variation given by Tan and Zhou [10, Lemma 2.4].

The similar results can be obtained for conjugate rational Fourier series. These are stated below.

Theorem 2. *If f is bounded, measurable and regulated function in* $[-\pi,\pi]$ *then*

$$|\tilde{S}_nf(x)-\tilde{f}\left(x,\frac{\pi}{n+1}\right)| \leq \frac{2(1+r)}{1-r}\sum_{k=0}^{n}\frac{1}{k+1}\left\{osc(\psi_x,I_{kn}^+)+osc(\psi_x,I_{kn}^-)\right\}.$$

Corollary 3. *If* $f\in \Lambda BV([-\pi,\pi]),\ \frac{\pi}{n+1}=a_n<a_{n-1}<\ldots<a_0=\pi$ *and* $-\pi=b_0<b_1,\ldots,<b_n=\frac{-\pi}{n+1},$ *then*

$$\begin{aligned}\left(\frac{1-r}{1+r}\right)\left|\tilde{S}_nf(x)-\tilde{f}\left(x,\frac{\pi}{n+1}\right)\right| \leq{}& \frac{2\lambda_{n+1}}{n+1}(V_\Lambda(\psi,[0,\pi])+V_\Lambda(\psi,[-\pi,0]))\\ &+\frac{2\pi}{n+1}\sum_{i=0}^{n-1}V_\Lambda(\psi,[0,a_i])(H(a_{i+1})-H(a_i))\\ &+\frac{2\pi}{n+1}\sum_{i=0}^{n-1}V_\Lambda(\psi,[b_i,0])(H(b_i)-H(b_{i+1})).\end{aligned}$$

Corollary 4. *If* $f \in \{n^\beta\} - BV([-\pi,\pi]),\ 0<\beta<1,$ *then*

$$\left|\tilde{S}_nf(x)-\tilde{f}\left(x,\frac{\pi}{n+1}\right)\right| \leq \frac{2(2-\beta)}{(n+1)^{1-\beta}}\cdot\frac{1+r}{1-r}\sum_{k=1}^{n}\frac{1}{k^\beta}\Big\{V_{\{n^\beta\}}(\psi,[0,\pi/k]) + V_{\{n^\beta\}}(\psi,[-\pi/k,0])\Big\}.$$

Corollary 4 generalizes the result in [10, Lemma 2.4]. If $r=0$, then Theorem 2 gives estimation for conjugate Fourier series.

4. Proofs

Proof of Theorem 1. In view of [10, Lemma 2.1], for $\alpha_k = |\alpha_k|e^{ia_k}$, $n \in \mathbb{N}$ and $x \in [-\pi, \pi]$, the partial sums of the rational Fourier series are given by

$$S_n f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)D_n(x-t,x)dt,$$

where

$$D_n(t,x) = \frac{1}{2}\sum_{k=-n}^{n} \overline{\phi_k(e^{it})}\phi_k(e^{ix}) = \frac{\sin\left[\frac{x-t}{2} + \theta_n(t,x)\right]}{2\sin\left(\frac{x-t}{2}\right)}$$

and

$$\theta_n(t,x) = \int_t^x \sum_{k=1}^{n} \frac{1-|\alpha_k|^2}{1-2|\alpha_k|\cos(y-a_k)+|\alpha_k|^2}dy.$$

Note that, for $n \in \mathbb{Z}\setminus\{0\}$ and by (2), we get

$$|\phi_n(e^{ix})| = \sqrt{\frac{1-|\alpha_{|n|}|^2}{1-2|\alpha_{|n|}|\cos(x-a_{|n|})+|\alpha_{|n|}|^2}} \le \sqrt{\frac{1+r}{1-r}}. \tag{3}$$

Therefore, for $n \in \mathbb{N}$,

$$|D_n(t,x)| \le (n+1)\frac{1+r}{1-r}. \tag{4}$$

Thus,

$$\begin{aligned} f(x) - S_n f(x) &= \frac{1}{\pi}\int_0^{\pi} \psi_x(t)D_n(x-t,x)dt + \frac{1}{\pi}\int_{-\pi}^{0} \psi_x(t)D_n(x-t,x)dt \\ &:= A + B. \end{aligned}$$

Therefore,

$$\begin{aligned} A =& \frac{1}{\pi}\int_{I_{0n}^+} \psi_x(t)D_n(x-t,x)dt + \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^+}(\psi_x(t)-\psi(\eta_{kn}))D_n(x-t,x)dt \\ &+ \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^+}\psi_x(\eta_{kn})D_n(x-t,x)dt \\ =&: A_1 + A_2 + A_3. \end{aligned}$$

Similarly,

$$\begin{aligned} B =& \frac{1}{\pi}\int_{I_{0n}^-} \psi_x(t)D_n(x-t,x)dt + \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^-}(\psi_x(t)-\psi(-\eta_{kn}))D_n(x-t,x)dt \\ &+ \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^-}\psi_x(-\eta_{kn})D_n(x-t,x)dt \\ =&: B_1 + B_2 + B_3. \end{aligned}$$

By (4), we have

$$\begin{aligned}|A_1| &\leq \frac{1}{\pi}\int_{I_{0n}^+} |\psi_x(t)-\psi(0)||D_n(x-t,x)|dt\\ &\leq \frac{osc(\psi_x, I_{0n}^+)}{\pi}\int_{I_{0n}^+}(n+1)\frac{1+r}{1-r}dt\\ &\leq \frac{1+r}{1-r}osc(\psi_x, I_{0n}^+),\end{aligned}$$

and similarly

$$|B_1| \leq \frac{1+r}{1-r}osc(\psi_x, I_{0n}^-).$$

Since $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$ for $0<t<\pi$, we get

$$|A_2| \leq \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^+}|\psi_x(t)-\psi(\eta_{kn})||D_n(x-t,x)|dt \leq \sum_{k=1}^{n}\frac{1}{k+1}osc(\psi_x, I_{kn}^+),$$

and similarly

$$|B_2| \leq \sum_{k=1}^{n}\frac{1}{k+1}osc(\psi_x, I_{kn}^-).$$

Let

$$R_{km}^+ = \int_{\eta_{kn}}^{\pi} D_n(x-t,x)dt \text{ and } R_{km}^- = \int_{\eta_{kn}}^{\pi} D_n(x+t,x)dt.$$

In view of [10, Lemma 2.3], we have for $0<u<\pi$,

$$\left|\int_u^{\pi} D_n(x-t,x)dt\right| \leq \frac{\pi^2(1+r)}{2n(1-r)u} \text{ and } \left|\int_u^{\pi} D_n(x+t,x)dt\right| \leq \frac{\pi^2(1+r)}{2n(1-r)u}.$$

Thus, we get

$$|R_{kn}^+| \leq \frac{\pi(1+r)}{k(1-r)} \text{ and } |R_{kn}^-| \leq \frac{\pi(1+r)}{k(1-r)}. \tag{5}$$

We have

$$A_3 = \frac{1}{\pi}\sum_{k=1}^{n}\left\{\psi_x(\eta_{kn})-\psi_x(\eta_{(k-1)n})\right\}(R_{kn}^+)$$

and

$$B_3 = \frac{1}{\pi}\sum_{k=1}^{n}\left\{\psi_x(-\eta_{kn})-\psi_x(-\eta_{(k-1)n})\right\}(R_{kn}^-).$$

Therefore, by using (5), we get

$$|A_3| \leq \frac{1+r}{1-r}\sum_{k=1}^{n}\frac{1}{k}osc(\psi_x, I_{(k-1)n}^+) \text{ and } |B_3| \leq \frac{1+r}{1-r}\sum_{k=1}^{n}\frac{1}{k}osc(\psi_x, I_{(k-1)n}^-).$$

Thus, we have

$$|A| \leq \frac{2(1+r)}{1-r}\sum_{k=0}^{n}\frac{1}{k+1}osc(\psi_x, I_{kn}^{+}),\ |B| \leq \frac{2(1+r)}{1-r}\sum_{k=0}^{n}\frac{1}{k+1}osc(\psi_x, I_{kn}^{-}).$$

Hence, the result is proved.

□

Proof of Theorem 2. In view of [10, Lemma 2.1], for $\alpha_k = |\alpha_k|e^{ia_k},\ n \in \mathbb{N}$ and $x \in [-\pi, \pi]$, the partial sums of the conjugate rational Fourier series are given by

$$\tilde{S}_n f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)\tilde{D}_n(x-t, x)dt,$$

where

$$\tilde{D}_n(t,x) = \frac{1}{2}\sum_{k=-n}^{n}(-i)\mathrm{sgn}(k)\overline{\phi_k(e^{it})}\phi_k(e^{ix}) = \frac{\cos\left(\frac{x-t}{2}\right) - \cos\left[\frac{x-t}{2} + \theta_n(t,x)\right]}{2\sin\left(\frac{x-t}{2}\right)}$$

and

$$\theta_n(t,x) = \int_t^x \sum_{k=1}^{n}\frac{1-|\alpha_k|^2}{1-2|\alpha_k|\cos(y-a_k)+|\alpha_k|^2}dy.$$

Therefore by using (3) and the simple fact that $\mathrm{sgn}(0) = 0$, for $n \in \mathbb{N}$,

$$|\tilde{D}_n(t,x)| \leq n\frac{1+r}{1-r}. \tag{6}$$

Using the fact that

$$\int_{-\pi}^{\pi}\tilde{D}_n(x-t,x)dt = 0 \text{ and } \int_{\frac{\pi}{n+1}\leq|t|\leq\pi}\frac{1}{2\tan(t/2)}dt = 0,$$

we get

$$\begin{aligned}\tilde{S}_n f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) &= \frac{1}{\pi}\left\{-\int_0^{\pi}\psi_x(t)\tilde{D}_n(x-t,x)dt + \int_{\frac{\pi}{n+1}}^{\pi}\frac{\psi_x(t)}{2\tan(t/2)}dt\right\}\\ &+ \frac{1}{\pi}\left\{-\int_{-\pi}^{0}\psi_x(t)\tilde{D}_n(x-t,x)dt + \int_{-\pi}^{\frac{-\pi}{n+1}}\frac{\psi_x(t)}{2\tan(t/2)}dt\right\}\\ &=: \tilde{A} + \tilde{B}.\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{A} = &-\frac{1}{\pi}\int_{I_{0n}^{+}}\psi_x(t)\tilde{D}_n(x-t,x)dt\\ &+\frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^{+}}(\psi_x(t)-\psi(\eta_{kn}))\frac{\cos\left[\frac{t}{2}+\theta_n(x-t,x)\right]}{2\sin\left(\frac{t}{2}\right)}dt\end{aligned}$$

$$+\frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^{+}}\psi_x(\eta_{kn})\frac{\cos\left[\frac{t}{2}+\theta_n(x-t,x)\right]}{2\sin\left(\frac{t}{2}\right)}dt$$
$$=: \tilde{A}_1+\tilde{A}_2+\tilde{A}_3.$$

By (6), we have

$$|\tilde{A}_1| \le \frac{1}{\pi}\int_{I_{0n}^{+}}|\psi_x(t)-\psi(0)||\tilde{D}_n(x-t,x)|dt \le \frac{1+r}{1-r}osc(\psi_x, I_{0n}^{+}).$$

Let $D_n^*(x-t,x)=\frac{\cos\left[\frac{t}{2}+\theta_n(x-t,x)\right]}{2\sin\left(\frac{t}{2}\right)}$. Since $\sin\left(\frac{t}{2}\right)\ge\frac{t}{\pi}$, for $0<t<\pi$, we get

$$|\tilde{A}_2| \le \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^{+}}|\psi_x(t)-\psi(\eta_{kn})||D_n^*(x-t,x)|dt \le \sum_{k=1}^{n}\frac{1}{k+1}osc(\psi_x, I_{kn}^{+}).$$

Let $T_{km}^{+}=\int_{\eta_{kn}}^{\pi}D_n^*(x-t,x)dt$. In view of [10, Lemma 2.3], we have, for $0<u<\pi$,

$$\left|\int_{u}^{\pi}D_n^*(x-t,x)dt\right| \le \frac{\pi^2(1+r)}{2n(1-r)u}.$$

Thus, we obtain

$$|T_{kn}^{+}| \le \frac{\pi(1+r)}{k(1-r)}. \tag{7}$$

Hence, we have

$$\tilde{A}_3=\frac{1}{\pi}\sum_{k=1}^{n}\left\{\psi_x(\eta_{kn})-\psi_x(\eta_{(k-1)n})\right\}(T_{kn}^{+}).$$

Therefore, by using (7), we get

$$|\tilde{A}_3| \le \frac{1+r}{1-r}\sum_{k=1}^{n}\frac{1}{k}osc(\psi_x, I_{(k-1)n}^{+}).$$

Thus, we have

$$|\tilde{A}| \le \frac{2(1+r)}{1-r}\sum_{k=0}^{n}\frac{1}{k+1}osc(\psi_x, I_{kn}^{+})$$

and similarly

$$|\tilde{B}| \le \frac{2(1+r)}{1-r}\sum_{k=0}^{n}\frac{1}{k+1}osc(\psi_x, I_{kn}^{-}).$$

Hence, we get the result. □

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Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara - 390002, Gujarat, India

E-mail address: `hardeep1996k@gmail.com`

E-mail address: `drrgvyas@yahoo.com`