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Unique common fixed points through a unified condition

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Abstract. Fixed point theory is a crucial branch in mathematics with a colossal number of applications in countless subjects. It furnishes preeminent tools and resources for elucidating varied problems which at first glance do not look like a fixed point problem. Since and even before 1912 till now several authors launched the existence and uniqueness of common fixed points for pairs of single and set-valued maps satisfying certain compatibilities on different spaces. This paper proves existence and uniqueness of a common fixed point for pairs of occasionally weakly biased maps. This unique common fixed point is guaranteed under the concept of modified contractive modulus function. Our theorems improve some results existing in the fixed point literature.

1. Introduction and preliminaries

In 1986, Jungck [8] introduced the concept of compatible maps as follows:

Definition 1 ([8])**.** Two self-maps $\mathfrak{f}$ and $\mathfrak{g}$ of a metric space $(\mathcal{X}, d)$ are called compatible if and only if

$$\lim_{n\to+\infty} d(\mathfrak{f}\mathfrak{g}x_n, \mathfrak{g}\mathfrak{f}x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to+\infty} \mathfrak{f}x_n = \lim_{n\to+\infty} \mathfrak{g}x_n = t$ for some $t \in \mathcal{X}$.

After ten years, in [9], the same author weakened the compatibility condition by giving the weak compatibility concept.

Definition 2 ([9])**.** Two self-maps $\mathfrak{f}$ and $\mathfrak{g}$ of a metric space $(\mathcal{X}, d)$ are called weakly compatible if $\mathfrak{f}\mathfrak{t} = \mathfrak{g}\mathfrak{t}$ for some $\mathfrak{t} \in \mathcal{X}$ implies that $\mathfrak{f}\mathfrak{g}\mathfrak{t} = \mathfrak{g}\mathfrak{f}\mathfrak{t}$.

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In 2008, Al-Thagafi and Shahzad [2] generalized the weak compatibility condition by the notion of occasionally weakly compatible maps.

Definition 3 ([2])**.** Let $\mathfrak{f}$ and $\mathfrak{g}$ be self-maps of a subset D of a metric space $(\mathcal{X}, d)$. Then $\mathfrak{f}$ and $\mathfrak{g}$ are called occasionally weakly compatible if $\mathfrak{f}\mathfrak{g}x = \mathfrak{g}\mathfrak{f}x$ for some $x \in C(\mathfrak{f}, \mathfrak{g})$ where $C(\mathfrak{f}, \mathfrak{g})$ is the set of coincidence points of $\mathfrak{f}$ and $\mathfrak{g}$.

On the other hand, in 1995, Jungck and Pathak [10] introduced the concept of biased maps, a very general notion of compatible maps. In the same paper, they also gave the concept of weakly biased maps which generalizes the notion of biased maps.

Definition 4 ([10])**.** The pair $(\mathfrak{f}, \mathfrak{g})$ is $\mathfrak{g}$-biased and $\mathfrak{f}$-biased, respectively, if and only if whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ and $\mathfrak{f}x_n$, $\mathfrak{g}x_n \to t \in \mathcal{X}$, then

$$\zeta d(\mathfrak{g}\mathfrak{f}x_n, \mathfrak{g}x_n) \leq \zeta d(\mathfrak{f}\mathfrak{g}x_n, \mathfrak{f}x_n),$$
$$\zeta d(\mathfrak{f}\mathfrak{g}x_n, \mathfrak{f}x_n) \leq \zeta d(\mathfrak{g}\mathfrak{f}x_n, \mathfrak{g}x_n),$$

respectively, if $\zeta = \liminf$ and if $\zeta = \limsup$.

If the pair $(\mathfrak{f}, \mathfrak{g})$ is compatible, then, it is both $\mathfrak{f}$ and $\mathfrak{g}$-biased (see [10]). However, the converse is not true in general.

Definition 5 ([10])**.** The pair $(\mathfrak{f}, \mathfrak{g})$ is weakly $\mathfrak{g}$-biased and $\mathfrak{f}$-biased, respectively, if and only if $\mathfrak{f}\mathfrak{p} = \mathfrak{g}\mathfrak{p}$ implies

$$d(\mathfrak{g}\mathfrak{f}\mathfrak{p}, \mathfrak{g}\mathfrak{p}) \leq d(\mathfrak{f}\mathfrak{g}\mathfrak{p}, \mathfrak{f}\mathfrak{p}),$$
$$d(\mathfrak{f}\mathfrak{g}\mathfrak{p}, \mathfrak{f}\mathfrak{p}) \leq d(\mathfrak{g}\mathfrak{f}\mathfrak{p}, \mathfrak{g}\mathfrak{p}),$$

respectively.

Clearly, all biased maps are weakly biased maps (see Proposition 1.1 in [10]) but the converse is false in general.
In the paper [6] submitted in October 2009 and published in 2012, we introduced the concept of occasionally weakly biased maps which represented a generalization of weakly biased maps. Further, we used this concept to show the existence and uniqueness of common fixed points for maps satisfying different contractive conditions in a normed as well as a metric space.

Definition 6 ([6])**.** Let $\mathfrak{f}$ and $\mathfrak{g}$ be self-maps of a set $\mathcal{X}$. The pair $(\mathfrak{f}, \mathfrak{g})$ is said to be occasionally weakly $\mathfrak{f}$-biased and $\mathfrak{g}$-biased, respectively, if and only if there exists a point $\mathfrak{p}$ in $\mathcal{X}$ such that $\mathfrak{f}\mathfrak{p} = \mathfrak{g}\mathfrak{p}$ implies

$$d(\mathfrak{f}\mathfrak{g}\mathfrak{p}, \mathfrak{f}\mathfrak{p}) \leq d(\mathfrak{g}\mathfrak{f}\mathfrak{p}, \mathfrak{g}\mathfrak{p}),$$
$$d(\mathfrak{g}\mathfrak{f}\mathfrak{p}, \mathfrak{g}\mathfrak{p}) \leq d(\mathfrak{f}\mathfrak{g}\mathfrak{p}, \mathfrak{f}\mathfrak{p}),$$

respectively.

Of course, weakly $\mathfrak{f}$-biased maps and $\mathfrak{g}$-biased, respectively, are occasionally weakly $\mathfrak{f}$-biased and $\mathfrak{g}$-biased, respectively. However, the converses are not true in general. Also, occasionally weakly compatible maps are both occasionally weakly $\mathfrak{f}$-biased and $\mathfrak{g}$-biased but the converses are false in general. To this end, consider the following example.

Example 1. Let $\mathcal{X} = [0, +\infty)$ with the usual metric $d(x, y) = |x - y|$. Define $\mathfrak{f}$, $\mathfrak{g} : \mathcal{X} \to \mathcal{X}$ by

$$\mathfrak{f}x = \begin{cases} 3x^2, & \text{if } x \in [0,1], \\ 8/x, & \text{if } x \in (1, +\infty), \end{cases} \quad \mathfrak{g}x = \begin{cases} 1, & \text{if } x \in [0,1], \\ 4x, & \text{if } x \in (1, +\infty). \end{cases}$$

We have $\mathfrak{f}x = \mathfrak{g}x$ if and only if $x = 1/\sqrt{3}$ or $x = \sqrt{2}$ and

$$3\sqrt{2} = d\left(\mathfrak{fg}\left(\sqrt{2}\right), \mathfrak{f}\left(\sqrt{2}\right)\right) \leq d\left(\mathfrak{gf}\left(\sqrt{2}\right), \mathfrak{g}\left(\sqrt{2}\right)\right) = 12\sqrt{2};$$

that is, the pair $(\mathfrak{f}, \mathfrak{g})$ is occasionally weakly $\mathfrak{f}$-biased. But

$$2 = d\left(\mathfrak{fg}\left(\frac{1}{\sqrt{3}}\right), \mathfrak{f}\left(\frac{1}{\sqrt{3}}\right)\right) \nleq d\left(\mathfrak{gf}\left(\frac{1}{\sqrt{3}}\right), \mathfrak{g}\left(\frac{1}{\sqrt{3}}\right)\right) = 0;$$

i.e., the pair $(\mathfrak{f}, \mathfrak{g})$ is not weakly $\mathfrak{f}$-biased.

On the other hand we have

$$0 = d\left(\mathfrak{gf}\left(\frac{1}{\sqrt{3}}\right), \mathfrak{g}\left(\frac{1}{\sqrt{3}}\right)\right) \leq d\left(\mathfrak{fg}\left(\frac{1}{\sqrt{3}}\right), \mathfrak{f}\left(\frac{1}{\sqrt{3}}\right)\right) = 2;$$

i.e., the pair $(\mathfrak{f}, \mathfrak{g})$ is occasionally weakly $\mathfrak{g}$-biased. But, as

$$12\sqrt{2} = d\left(\mathfrak{gf}\left(\sqrt{2}\right), \mathfrak{g}\left(\sqrt{2}\right)\right) \nleq d\left(\mathfrak{fg}\left(\sqrt{2}\right), \mathfrak{f}\left(\sqrt{2}\right)\right) = 3\sqrt{2},$$

then $\mathfrak{f}$ and $\mathfrak{g}$ are not weakly $\mathfrak{g}$-biased.

Remark 1. Note that from the preceding example we have

$$\mathfrak{fg}\left(\frac{1}{\sqrt{3}}\right) = 3 \neq 1 = \mathfrak{gf}\left(\frac{1}{\sqrt{3}}\right)$$

and

$$\mathfrak{fg}\left(\sqrt{2}\right) = \sqrt{2} \neq 16\sqrt{2} = \mathfrak{gf}\left(\sqrt{2}\right).$$

That is, $\mathfrak{f}$ and $\mathfrak{g}$ are not occasionally weakly compatible maps.

In 2017, Krishnakumar and Mani [12] proved the existence of unique common fixed point of contractive maps on a complete metric space through a weakly compatible maps and contractive modulus.

Recently in 2021, Kumari and Kumar [13] proved common fixed point theorems for four weakly compatible maps using contractive modulus.

Our objective here is to improve the results of [12] and [13] using the occasionally weakly biased notion and the concept of modified contractive modulus function via an implicit relation.

Before giving our main results, recall that a function $\mathfrak{M} : [0, +\infty) \to [0, +\infty)$ is said to be a contractive modulus if $\mathfrak{M}(0) = 0$ and $\mathfrak{M}(t) < t$ for $t > 0$.

2. Main results

Now, we are ready to present our main results.

2.1. A modified contractive modulus function.

Definition 7. A function $\mathfrak{M} : [0, +\infty) \to [0, +\infty)$ is said to be a modified contractive modulus if $\mathfrak{M}$ is non-decreasing, $\mathfrak{M}(0) = 0$ and $\mathfrak{M}(t) < t$ for $t > 0$.

2.2. Implicit relations. According to [5], in his papers [18] and [19] Popa unified several explicit contractive conditions by initiating the implicit contraction type condition. Several authors used this direction to prove the existence and uniqueness of common fixed points in the settings of single and set-valued maps, in different spaces (see for example [2, 3, 4, 7, 16, 17, 20, 22, 23, 24]). Motivated by Popa, we will introduce a new type of implicit relations.

Let Ξ be a family of all functions $\xi : \mathbb{R}_+^5 \to \mathbb{R}$ satisfying the following conditions:

- (ξ_1): ξ is non-increasing in variables $\mathfrak{t}_2$, $\mathfrak{t}_3$ and $\mathfrak{t}_5$,
- (ξ_2): $\xi(\mathfrak{t}, 0, 0, 0, \mathfrak{t}) > 0 \ \forall \mathfrak{t} > 0$,
- (ξ_3): $\xi(\mathfrak{t}, 2\mathfrak{t}, 0, 0, \mathfrak{t}) > 0 \ \forall \mathfrak{t} > 0$,
- (ξ_4): $\xi(\mathfrak{t}, 0, 2\mathfrak{t}, 0, \mathfrak{t}) > 0 \ \forall \mathfrak{t} > 0$.

Example 2. $\xi(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5) = \mathfrak{t}_1 - \nu(\mathfrak{t}_2 + \mathfrak{t}_3 + \mathfrak{t}_4 + \mathfrak{t}_5)$, where $\nu \in \left(0, \frac{1}{3}\right)$.

- (ξ_1): It is clear that ξ is non-increasing in variables $\mathfrak{t}_2$ and $\mathfrak{t}_5$,
- (ξ_2): $\xi(\mathfrak{t}, 0, 0, 0, \mathfrak{t}) = \mathfrak{t}(1 - \nu) > 0 \ \forall \mathfrak{t} > 0$,
- (ξ_3): $\xi(\mathfrak{t}, 2\mathfrak{t}, 0, 0, \mathfrak{t}) = \mathfrak{t}(1 - 3\nu) > 0 \ \forall \mathfrak{t} > 0$,
- (ξ_4): $\xi(\mathfrak{t}, 0, 2\mathfrak{t}, 0, \mathfrak{t}) = \mathfrak{t}(1 - 3\nu) > 0 \ \forall \mathfrak{t} > 0$.

Example 3. $\xi(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5) = \mathfrak{t}_1 - \theta \max\{\mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5\}$, where $\theta \in \left(0, \frac{1}{2}\right)$.

- (ξ_1): Obviously,
- (ξ_2): $\xi(\mathfrak{t}, 0, 0, 0, \mathfrak{t}) = \mathfrak{t}(1 - \theta) > 0 \ \forall \mathfrak{t} > 0$,
- (ξ_3): $\xi(\mathfrak{t}, 2\mathfrak{t}, 0, 0, \mathfrak{t}) = \mathfrak{t}(1 - 2\theta) > 0 \ \forall \mathfrak{t} > 0$,
- (ξ_4): $\xi(\mathfrak{t}, 0, 2\mathfrak{t}, 0, \mathfrak{t}) = \mathfrak{t}(1 - 2\theta) > 0 \ \forall \mathfrak{t} > 0$.

Example 4. $\xi(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5) = \mathfrak{t}_1 - \zeta\mathfrak{t}_2 - \vartheta\mathfrak{t}_3 - \rho\mathfrak{t}_4 - \upsilon\mathfrak{t}_5$, where $\zeta, \vartheta, \rho, \upsilon > 0$ and $2\zeta + 2\vartheta + \rho + \upsilon < 1$.

- (ξ_1): Clearly,
- (ξ_2): $\xi(\mathfrak{t}, 0, 0, 0, \mathfrak{t}) = \mathfrak{t}(1 - \upsilon) > 0 \ \forall \mathfrak{t} > 0$,

- (ξ_3): $\xi(\mathfrak{t}, 2\mathfrak{t}, 0, 0, \mathfrak{t}) = \mathfrak{t}(1 - 2\zeta - \upsilon) > 0\ \forall \mathfrak{t} > 0$,
- (ξ_4): $\xi(\mathfrak{t}, 0, 2\mathfrak{t}, 0, \mathfrak{t}) = \mathfrak{t}(1 - 2\vartheta - \upsilon) > 0\ \forall \mathfrak{t} > 0$.

2.3. A unique common fixed point theorem for four maps. We start by formulating and proving our first result.

Theorem 1. *Let $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$ be four self-maps of a metric space $(\mathcal{X}, d)$ satisfying the following condition:*

$$\begin{aligned}&\xi(d^2(\mathfrak{f}x, \mathfrak{g}y), \mathfrak{M}(d(\mathfrak{h}x, \mathfrak{k}y))\mathfrak{M}(d(\mathfrak{f}x, \mathfrak{h}x)), \mathfrak{M}(d(\mathfrak{h}x, \mathfrak{k}y))\mathfrak{M}(d(\mathfrak{g}y, \mathfrak{k}y)),\\ &\mathfrak{M}(d(\mathfrak{f}x, \mathfrak{h}x))\mathfrak{M}(d(\mathfrak{g}y, \mathfrak{k}y)), \mathfrak{M}(d(\mathfrak{h}x, \mathfrak{g}y))\mathfrak{M}(d(\mathfrak{f}x, \mathfrak{k}y))) \le 0, \qquad (1)\end{aligned}$$

for all x, y in $\mathcal{X}$, where $\mathfrak{M}$ is a modified contractive modulus function and $\xi \in \Xi$. If $\mathfrak{f}$ and $\mathfrak{h}$ are occasionally weakly $\mathfrak{h}$-biased, and $\mathfrak{g}$ and $\mathfrak{k}$ are occasionally weakly $\mathfrak{k}$-biased, then $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$ have a unique common fixed point.

Proof. By hypotheses, there are two points $\mathfrak{u}$ and $\mathfrak{v}$ in $\mathcal{X}$ such that $\mathfrak{f}\mathfrak{u} = \mathfrak{h}\mathfrak{u}$ implies $d(\mathfrak{h}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{u}) \le d(\mathfrak{f}\mathfrak{h}\mathfrak{u}, \mathfrak{f}\mathfrak{u})$ and $\mathfrak{g}\mathfrak{v} = \mathfrak{k}\mathfrak{v}$ implies $d(\mathfrak{k}\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{v}) \le d(\mathfrak{g}\mathfrak{k}\mathfrak{v}, \mathfrak{g}\mathfrak{v})$.

First, we are going to prove that $\mathfrak{f}\mathfrak{u} = \mathfrak{g}\mathfrak{v}$. Suppose that $\mathfrak{f}\mathfrak{u} \ne \mathfrak{g}\mathfrak{v}$. From inequality (1) we have

$$\begin{aligned}&\xi(d^2(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{v}), \mathfrak{M}(d(\mathfrak{h}\mathfrak{u}, \mathfrak{k}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{u})), \mathfrak{M}(d(\mathfrak{h}\mathfrak{u}, \mathfrak{k}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{v})),\\ &\mathfrak{M}(d(\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{u}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{v})), \mathfrak{M}(d(\mathfrak{h}\mathfrak{u}, \mathfrak{g}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{u}, \mathfrak{k}\mathfrak{v})))\\ &= \xi(d^2(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{v}), 0, 0, 0, \mathfrak{M}^2(d(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{v}))) \le 0,\end{aligned}$$

since ξ is non-increasing in $\mathfrak{t}_5$, we get

$$0 \ge \xi(d^2(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{v}), 0, 0, 0, \mathfrak{M}^2(d(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{v}))) \ge \xi(d^2(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{v}), 0, 0, 0, d^2(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{v}))$$

which contradicts (ξ_2). Thus $\mathfrak{f}\mathfrak{u} = \mathfrak{g}\mathfrak{v}$.

Now, we assert that $\mathfrak{f}\mathfrak{f}\mathfrak{u} = \mathfrak{f}\mathfrak{u}$. If not, then the use of condition (1) gives

$$\begin{aligned}&\xi(d^2(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{v}), \mathfrak{M}(d(\mathfrak{h}\mathfrak{f}\mathfrak{u}, \mathfrak{k}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{f}\mathfrak{u})), \mathfrak{M}(d(\mathfrak{h}\mathfrak{f}\mathfrak{u}, \mathfrak{k}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{v})),\\ &\mathfrak{M}(d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{f}\mathfrak{u}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{v})), \mathfrak{M}(d(\mathfrak{h}\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{k}\mathfrak{v}))) \le 0;\end{aligned}$$

i.e.,

$$\xi(d^2(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u}), \mathfrak{M}(d(\mathfrak{h}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{u}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{f}\mathfrak{u})), 0, 0, \mathfrak{M}(d(\mathfrak{h}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{u}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u}))) \le 0.$$

By the triangle inequality we have $d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{f}\mathfrak{u}) \le d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u}) + d(\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{f}\mathfrak{u})$. Since $\mathfrak{f}$ and $\mathfrak{h}$ are occasionally weakly $\mathfrak{h}$-biased we get $d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{f}\mathfrak{u}) \le 2d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u})$ and by the definition of $\mathfrak{M}$ we obtain $\mathfrak{M}(d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{f}\mathfrak{u})) \le \mathfrak{M}(2d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u})) < 2d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u})$. As ξ is non-increasing in variables $\mathfrak{t}_2$ ant $\mathfrak{t}_5$, we find

$$\begin{aligned}0 &\ge \xi(d^2(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u}), \mathfrak{M}(d(\mathfrak{h}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{u}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{f}\mathfrak{u})), 0, 0, \mathfrak{M}(d(\mathfrak{h}\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{u}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u})))\\ &\ge \xi(d^2(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u}), 2d^2(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u}), 0, 0, d^2(\mathfrak{f}\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u}))\end{aligned}$$

which contradicts (ξ_3). Hence $\mathfrak{f}\mathfrak{f}\mathfrak{u} = \mathfrak{f}\mathfrak{u}$ and consequently $\mathfrak{h}\mathfrak{f}\mathfrak{u} = \mathfrak{f}\mathfrak{u}$.

Now, suppose that $\mathfrak{g}\mathfrak{g}\mathfrak{v} \neq \mathfrak{g}\mathfrak{v}$. Using inequality (1) we obtain

$$\xi(d^2(\mathfrak{f}\mathfrak{u}, \mathfrak{g}\mathfrak{g}\mathfrak{v}), \mathfrak{M}(d(\mathfrak{h}\mathfrak{u}, \mathfrak{k}\mathfrak{g}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{u})), \mathfrak{M}(d(\mathfrak{h}\mathfrak{u}, \mathfrak{k}\mathfrak{g}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v})),$$
$$\mathfrak{M}(d(\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{u}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v})), \mathfrak{M}(d(\mathfrak{h}\mathfrak{u}, \mathfrak{g}\mathfrak{g}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{u}, \mathfrak{k}\mathfrak{g}\mathfrak{v}))) \leq 0;$$

i.e.,

$$\xi(d^2(\mathfrak{g}\mathfrak{v}, \mathfrak{g}\mathfrak{g}\mathfrak{v}), 0, \mathfrak{M}(d(\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v})),$$
$$0, \mathfrak{M}(d(\mathfrak{g}\mathfrak{v}, \mathfrak{g}\mathfrak{g}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v}))) \leq 0.$$

Similarly, by the triangle inequality we have $d(\mathfrak{g}\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v}) \leq d(\mathfrak{g}\mathfrak{g}\mathfrak{v}, \mathfrak{g}\mathfrak{v}) + d(\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v})$. Since $\mathfrak{g}$ and $\mathfrak{k}$ are occasionally weakly $\mathfrak{k}$-biased we get $d(\mathfrak{g}\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v}) \leq 2d(\mathfrak{g}\mathfrak{g}\mathfrak{v}, \mathfrak{g}\mathfrak{v})$ and by the definition of $\mathfrak{M}$ we obtain $\mathfrak{M}(d(\mathfrak{g}\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v})) \leq \mathfrak{M}(2d(\mathfrak{g}\mathfrak{g}\mathfrak{v}, \mathfrak{g}\mathfrak{v})) < 2d(\mathfrak{g}\mathfrak{g}\mathfrak{v}, \mathfrak{g}\mathfrak{v})$. As ξ is non-increasing in variables $\mathfrak{t}_3$ ant $\mathfrak{t}_5$ we find

$$0 \geq \xi(d^2(\mathfrak{g}\mathfrak{v}, \mathfrak{g}\mathfrak{g}\mathfrak{v}), 0, \mathfrak{M}(d(\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v})),$$
$$0, \mathfrak{M}(d(\mathfrak{g}\mathfrak{v}, \mathfrak{g}\mathfrak{g}\mathfrak{v}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{v}, \mathfrak{k}\mathfrak{g}\mathfrak{v}))) \geq \xi(d^2(\mathfrak{g}\mathfrak{v}, \mathfrak{g}\mathfrak{g}\mathfrak{v}), 0, 2d^2(\mathfrak{g}\mathfrak{v}, \mathfrak{g}\mathfrak{g}\mathfrak{v}), 0, d^2(\mathfrak{g}\mathfrak{v}, \mathfrak{g}\mathfrak{g}\mathfrak{v}))$$

which contradicts (ξ_4). Hence $\mathfrak{g}\mathfrak{g}\mathfrak{v} = \mathfrak{g}\mathfrak{v}$ and consequently $\mathfrak{k}\mathfrak{g}\mathfrak{v} = \mathfrak{g}\mathfrak{v}$. Put $\mathfrak{f}\mathfrak{u} = \mathfrak{h}\mathfrak{u} = \mathfrak{g}\mathfrak{v} = \mathfrak{k}\mathfrak{v} = \mathfrak{w}$, then $\mathfrak{w}$ is a common fixed point of maps $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$.

For the uniqueness, let $\mathfrak{w}$ and $\mathfrak{z}$ be two distinct common fixed points of maps $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$. Then, $\mathfrak{w} = \mathfrak{f}\mathfrak{w} = \mathfrak{g}\mathfrak{w} = \mathfrak{h}\mathfrak{w} = \mathfrak{k}\mathfrak{w}$ and $\mathfrak{z} = \mathfrak{f}\mathfrak{z} = \mathfrak{g}\mathfrak{z} = \mathfrak{h}\mathfrak{z} = \mathfrak{k}\mathfrak{z}$. The use of (1) gives

$$\xi(d^2(\mathfrak{f}\mathfrak{w}, \mathfrak{g}\mathfrak{z}), \mathfrak{M}(d(\mathfrak{h}\mathfrak{w}, \mathfrak{k}\mathfrak{z}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{w}, \mathfrak{h}\mathfrak{w})), \mathfrak{M}(d(\mathfrak{h}\mathfrak{w}, \mathfrak{k}\mathfrak{z}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{z}, \mathfrak{k}\mathfrak{z})),$$
$$\mathfrak{M}(d(\mathfrak{f}\mathfrak{w}, \mathfrak{h}\mathfrak{w}))\mathfrak{M}(d(\mathfrak{g}\mathfrak{z}, \mathfrak{k}\mathfrak{z})), \mathfrak{M}(d(\mathfrak{h}\mathfrak{w}, \mathfrak{g}\mathfrak{z}))\mathfrak{M}(d(\mathfrak{f}\mathfrak{w}, \mathfrak{k}\mathfrak{z}))) \leq 0;$$

i.e.,

$$0 \geq \xi(d^2(\mathfrak{w}, \mathfrak{z}), 0, 0, 0, \mathfrak{M}^2(d(\mathfrak{w}, \mathfrak{z}))) \geq \xi(d^2(\mathfrak{w}, \mathfrak{z}), 0, 0, 0, d^2(\mathfrak{w}, \mathfrak{z}))$$

which contradicts (ξ_2). So $\mathfrak{z} = \mathfrak{w}$. □

2.4. Illustrative example.

Example 5. Let $\mathcal{X} = [0, 11)$ with the metric $d(x, y) = |x - y|$. Define

$$\mathfrak{f}x = \begin{cases} 0, & \text{if } x \in [0,1], \\ \frac{1}{4}, & \text{if } x \in (1,11), \end{cases} \quad \mathfrak{g}x = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,1], \\ \frac{1}{3}, & \text{if } x \in (1,11), \end{cases}$$

and

$$\mathfrak{h}x = \begin{cases} x^2/6, & \text{if } x \in [0,1], \\ 6, & \text{if } x \in (1,11), \end{cases} \quad \mathfrak{k}x = \begin{cases} 10x, & \text{if } x \in [0,1], \\ 9, & \text{if } x \in (1,11). \end{cases}$$

First it is easy to see that $\mathfrak{f}$ and $\mathfrak{h}$ are occasionally weakly $\mathfrak{h}$-biased and $\mathfrak{g}$ and $\mathfrak{k}$ are occasionally weakly $\mathfrak{k}$-biased.

Taking $\mathfrak{M}(t) = \frac{1}{2}t$ and $\xi(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5) = \mathfrak{t}_1 - \frac{1}{3}\max\{\mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5\}$ we get:

(1) for x, $y \in [0,1]$, we have $\mathfrak{f}x = 0$, $\mathfrak{g}y = \frac{y}{2}$, $\mathfrak{h}x = \frac{x^2}{6}$, $\mathfrak{k}y = 10y$ and

$$\begin{aligned}
& \xi(d^2(\mathfrak{f}x,\mathfrak{g}y), \mathfrak{M}(d(\mathfrak{h}x,\mathfrak{k}y))\mathfrak{M}(d(\mathfrak{f}x,\mathfrak{h}x)), \mathfrak{M}(d(\mathfrak{h}x,\mathfrak{k}y))\mathfrak{M}(d(\mathfrak{g}y,\mathfrak{k}y)),\\
& \mathfrak{M}(d(\mathfrak{f}x,\mathfrak{h}x))\mathfrak{M}(d(\mathfrak{g}y,\mathfrak{k}y)), \mathfrak{M}(d(\mathfrak{h}x,\mathfrak{g}y))\mathfrak{M}(d(\mathfrak{f}x,\mathfrak{k}y)))\\
= \ & \xi(\frac{y^2}{4}, \mathfrak{M}\left(\left|10y-\frac{x^2}{6}\right|\right)\mathfrak{M}\left(\frac{x^2}{6}\right), \mathfrak{M}\left(\left|10y-\frac{x^2}{6}\right|\right)\mathfrak{M}\left(\frac{19y}{2}\right),\\
& \mathfrak{M}\left(\frac{x^2}{6}\right)\mathfrak{M}\left(\frac{19y}{2}\right), \mathfrak{M}\left(\left|\frac{y}{5}-\frac{x^2}{6}\right|\right)\mathfrak{M}(10y))\\
= \ & \xi\left(\frac{y^2}{4}, \frac{x^2}{24}\left|10y-\frac{x^2}{6}\right|, \frac{19y}{8}\left|10y-\frac{x^2}{6}\right|, \frac{19x^2y}{48}, \frac{5y}{2}\left|\frac{y}{2}-\frac{x^2}{6}\right|\right)\\
= \ & \frac{y^2}{4}-\frac{1}{3}\max\left\{\frac{x^2}{24}\left|10y-\frac{x^2}{6}\right|, \frac{19y}{8}\left|10y-\frac{x^2}{6}\right|, \frac{19x^2y}{48}, \frac{5y}{2}\left|\frac{y}{5}-\frac{x^2}{6}\right|\right\}\\
\leq \ & 0,
\end{aligned}$$

(2) for x, $y \in (1,11)$, we have $\mathfrak{f}x = \frac{1}{4}$, $\mathfrak{g}y = \frac{1}{3}$, $\mathfrak{h}x = 6$, $\mathfrak{k}y = 9$ and

$$\begin{aligned}
& \xi(d^2(\mathfrak{f}x,\mathfrak{g}y), \mathfrak{M}(d(\mathfrak{h}x,\mathfrak{k}y))\mathfrak{M}(d(\mathfrak{f}x,\mathfrak{h}x)), \mathfrak{M}(d(\mathfrak{h}x,\mathfrak{k}y))\mathfrak{M}(d(\mathfrak{g}y,\mathfrak{k}y)),\\
& \mathfrak{M}(d(\mathfrak{f}x,\mathfrak{h}x))\mathfrak{M}(d(\mathfrak{g}y,\mathfrak{k}y)), \mathfrak{M}(d(\mathfrak{h}x,\mathfrak{g}y))\mathfrak{M}(d(\mathfrak{f}x,\mathfrak{k}y)))\\
= & \ \xi\left(\frac{1}{144}, \mathfrak{M}(3)\,\mathfrak{M}\left(\frac{23}{4}\right), \mathfrak{M}(3)\,\mathfrak{M}\left(\frac{26}{3}\right), \mathfrak{M}\left(\frac{23}{4}\right)\mathfrak{M}\left(\frac{26}{3}\right), \mathfrak{M}\left(\frac{17}{3}\right)\mathfrak{M}\left(\frac{35}{4}\right)\right)\\
= & \ \xi\left(\frac{1}{144}, \frac{69}{16}, \frac{13}{2}, \frac{299}{24}, \frac{595}{48}\right) = \frac{1}{144}-\frac{1}{3}\max\left\{\frac{69}{16}, \frac{13}{2}, \frac{299}{24}, \frac{595}{48}\right\} \leq 0,
\end{aligned}$$

(3) for $x \in [0,1]$, $y \in (1,11)$, we have $\mathfrak{f}x = 0$, $\mathfrak{g}y = \frac{1}{3}$, $\mathfrak{h}x = \frac{x^2}{6}$, $\mathfrak{k}y = 9$ and

$$\begin{aligned}
& \xi(d^2(\mathfrak{f}x,\mathfrak{g}y), \mathfrak{M}(d(\mathfrak{h}x,\mathfrak{k}y))\mathfrak{M}(d(\mathfrak{f}x,\mathfrak{h}x)), \mathfrak{M}(d(\mathfrak{h}x,\mathfrak{k}y))\mathfrak{M}(d(\mathfrak{g}y,\mathfrak{k}y)),\\
& \mathfrak{M}(d(\mathfrak{f}x,\mathfrak{h}x))\mathfrak{M}(d(\mathfrak{g}y,\mathfrak{k}y)), \mathfrak{M}(d(\mathfrak{h}x,\mathfrak{g}y))\mathfrak{M}(d(\mathfrak{f}x,\mathfrak{k}y)))\\
= \ & \xi(\frac{1}{9}, \mathfrak{M}\left(9-\frac{x^2}{6}\right)\mathfrak{M}\left(\frac{x^2}{6}\right), \mathfrak{M}\left(9-\frac{x^2}{6}\right)\mathfrak{M}\left(\frac{26}{3}\right),\\
& \mathfrak{M}\left(\frac{x^2}{6}\right)\mathfrak{M}\left(\frac{26}{3}\right), \mathfrak{M}\left(\left|\frac{1}{3}-\frac{x^2}{6}\right|\right)\mathfrak{M}(9))\\
= \ & \xi\left(\frac{1}{9}, \frac{x^2}{24}\left(9-\frac{x^2}{6}\right), \frac{13}{6}\left(9-\frac{x^2}{6}\right), \frac{13x^2}{36}, \frac{9}{4}\left|\frac{1}{3}-\frac{x^2}{6}\right|\right)\\
= \ & \frac{1}{9}-\frac{1}{3}\max\left\{\frac{x^2}{24}\left(9-\frac{x^2}{6}\right), \frac{13}{6}\left(9-\frac{x^2}{6}\right), \frac{13x^2}{36}, \frac{9}{4}\left|\frac{1}{3}-\frac{x^2}{6}\right|\right\}\\
\leq \ & 0,
\end{aligned}$$

(4) for $x \in (1, 11)$, $y \in [0, 1]$, we have $\mathfrak{f}x = \frac{1}{4}$, $\mathfrak{g}y = \frac{y}{2}$, $\mathfrak{h}x = 6$, $\mathfrak{k}y = 10y$ and

$$\begin{aligned}
& \xi(d^2(\mathfrak{f}x, \mathfrak{g}y), \mathfrak{M}(d(\mathfrak{h}x, \mathfrak{k}y))\mathfrak{M}(d(\mathfrak{f}x, \mathfrak{h}x)), \mathfrak{M}(d(\mathfrak{h}x, \mathfrak{k}y))\mathfrak{M}(d(\mathfrak{g}y, \mathfrak{k}y)),\\
& \mathfrak{M}(d(\mathfrak{f}x, \mathfrak{h}x))\mathfrak{M}(d(\mathfrak{g}y, \mathfrak{k}y)), \mathfrak{M}(d(\mathfrak{h}x, \mathfrak{g}y))\mathfrak{M}(d(\mathfrak{f}x, \mathfrak{k}y)))\\
= \;& \xi(\left(\frac{1}{4} - \frac{y}{2}\right)^2, \mathfrak{M}\left(|6 - 10y|\right)\mathfrak{M}\left(\frac{23}{4}\right), \mathfrak{M}\left(|6 - 10y|\right)\mathfrak{M}\left(\frac{19y}{2}\right),\\
& \mathfrak{M}\left(\frac{23}{4}\right)\mathfrak{M}\left(\frac{19y}{2}\right), \mathfrak{M}\left(6 - \frac{y}{2}\right)\mathfrak{M}\left(\left|\frac{1}{4} - 10y\right|\right))\\
= \;& \xi\left(\left(\frac{1}{4} - \frac{y}{2}\right)^2, \frac{23}{16}\left|6 - 10y\right|, \frac{19y}{8}\left|6 - 10y\right|, \frac{437y}{32}, \frac{1}{4}\left(6 - \frac{y}{2}\right)\left|\frac{1}{4} - 10y\right|\right)\\
= \;& \left(\frac{1}{4} - \frac{y}{2}\right)^2\\
& -\frac{1}{3}\max\left\{\frac{23}{16}\left|6 - 10y\right|, \frac{19y}{8}\left|6 - 10y\right|, \frac{437y}{32}, \frac{1}{4}\left(6 - \frac{y}{2}\right)\left|\frac{1}{4} - 10y\right|\right\}\\
\leq \;& 0.
\end{aligned}$$

So, all hypotheses of the above Theorem 1 are satisfied and 0 is the unique common fixed point of maps $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$.

Remark 2. Note that the main results of [12] and [13] are not applicable because the space is not complete and $\mathfrak{g}\mathcal{X} = \left[0, \frac{1}{2}\right] \nsubseteq \left[0, \frac{1}{6}\right] \cup \{6\} = \mathfrak{h}\mathcal{X}$.

2.5. A unique common fixed point for a sequence of maps. Now, we give a generalization of Theorem 1.

Theorem 2. *Let $\mathfrak{h}$, $\mathfrak{k}$ and $\{\mathfrak{f}_n\}_{n=1,2,\ldots}$ be maps from a metric space $(\mathcal{X}, d)$ into itself such that the pairs $(\mathfrak{f}_n, \mathfrak{h})$ and $(\mathfrak{f}_{n+1}, \mathfrak{k})$ are occasionally weakly $\mathfrak{h}$-biased and occasionally weakly $\mathfrak{k}$-biased, respectively. Suppose that the inequality*

$$\begin{aligned}
& \xi(d^2(\mathfrak{f}_n x, \mathfrak{f}_{n+1} y)\mathfrak{M}(d(\mathfrak{h}x, \mathfrak{k}y))\mathfrak{M}(d(\mathfrak{f}_n x, \mathfrak{h}x)),\\
& \mathfrak{M}(d(\mathfrak{h}x, \mathfrak{k}y))\mathfrak{M}(d(\mathfrak{f}_{n+1} y, \mathfrak{k}y)),\\
& \mathfrak{M}(d(\mathfrak{f}_n x, \mathfrak{h}x))\mathfrak{M}(d(\mathfrak{f}_{n+1} y, \mathfrak{k}y)),\\
& \mathfrak{M}(d(\mathfrak{h}x, \mathfrak{f}_{n+1} y))\mathfrak{M}(d(\mathfrak{f}_n x, \mathfrak{k}y))) \leq 0
\end{aligned} \tag{2}$$

holds for all x, y in $\mathcal{X}$, where $\mathfrak{M}$ is a modified contractive modulus function and $\xi \in \Xi$. Then $\mathfrak{h}$, $\mathfrak{k}$ and $\{\mathfrak{f}_n\}_{n=1,2,\ldots}$ have a unique common fixed point.

Proof. Putting $n = 1$, we get that maps $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{h}$ and $\mathfrak{k}$ satisfy the hypotheses of Theorem 1. Then they have a unique common fixed point $\mathfrak{w}$.

Now, letting $n = 2$, we get that maps $\mathfrak{f}_2$, $\mathfrak{f}_3$, $\mathfrak{h}$ and $\mathfrak{k}$ have a unique common fixed point $\mathfrak{z}$. Suppose that $\mathfrak{z} \neq \mathfrak{w}$. The use of inequality (2) gives

$$\xi(d^2(\mathfrak{f}_2\mathfrak{w}, \mathfrak{f}_3\mathfrak{z}), \mathfrak{M}(d(\mathfrak{h}\mathfrak{w}, \mathfrak{k}\mathfrak{z}))\mathfrak{M}(d(\mathfrak{f}_2\mathfrak{w}, \mathfrak{h}\mathfrak{w})), \mathfrak{M}(d(\mathfrak{h}\mathfrak{w}, \mathfrak{k}\mathfrak{z}))\mathfrak{M}(d(\mathfrak{f}_3\mathfrak{z}, \mathfrak{k}\mathfrak{z})),$$
$$\mathfrak{M}(d(\mathfrak{f}_2\mathfrak{w}, \mathfrak{h}\mathfrak{w}))\mathfrak{M}(d(\mathfrak{f}_3\mathfrak{z}, \mathfrak{k}\mathfrak{z})), \mathfrak{M}(d(\mathfrak{h}\mathfrak{w}, \mathfrak{f}_3\mathfrak{z}))\mathfrak{M}(d(\mathfrak{f}_2\mathfrak{w}, \mathfrak{k}\mathfrak{z}))) \leq 0;$$

i.e.,

$$0 \geq \xi(d^2(\mathfrak{w}, \mathfrak{z}), 0, 0, 0, \mathfrak{M}^2(d(\mathfrak{w}, \mathfrak{z}))) \geq \xi(d^2(\mathfrak{w}, \mathfrak{z}), 0, 0, 0, \mathfrak{M}^2(d(\mathfrak{w}, \mathfrak{z})))$$

which contradicts (ξ_2). Hence $\mathfrak{z} = \mathfrak{w}$.

Continuing in this way, we certify that $\mathfrak{w}$ is the required point; i.e., $\mathfrak{w}$ is the unique common fixed point of $\mathfrak{h}$, $\mathfrak{k}$ and $\{\mathfrak{f}_n\}_{n=1,2,\ldots}$. □

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