ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 26, Number 2, December 2022
Available online at https://ojs.utlib.ee/index.php/ACUTM

# Applications of degenerate $q$-Euler and $q$-Changhee polynomials with weight $\alpha$

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Abstract. In this paper, we give new identities involving degenerate $q$-Euler polynomials with weight $\alpha$ and $q$-Changhee polynomials of the second kind with weight $\alpha$, using the Faà di Bruno formula and some identities of the Bell polynomials of the second kind.

## 1. Introduction

Many famous scientists have defined special polynomials and given their applications in mathematics, science and engineering. There are recent investigations of identities for polynomials and numbers using their derivatives and the generating functions.

For $n \geq m \geq 0$, the Stirling numbers of the second kind $S_2(n, m)$ are the coefficients in the expansion of the falling factorial $(x)_n = x(x-1)\cdots(x-n+1)$ into powers of the real number $x$:

$$x^n = \sum_{m=0}^{n} S_2(n, m)(x)_m$$

and $S_2(n, 0) = \delta_{n,0}$, where $\delta_{i,j}$ is the Kronecker delta [4]. These numbers satisfy the recurrence relation

$$S_2(n+1, m) = S_2(n, m-1) + mS_2(n, m),$$

and can be generated by

$$\sum_{n=m}^{\infty} S_2(n, m)\frac{t^n}{n!} = \frac{(e^t - 1)^m}{m!}.$$

Received June 20, 2022.
2020 *Mathematics Subject Classification.* 05A15, 05A19, 11B68, 33E20, 05A30.
*Key words and phrases.* $q$-Euler polynomials, $q$-Changhee polynomials, generating functions.
https://doi.org/10.12697/ACUTM.2022.26.18
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The $q$-analogs are frequently studied in mathematical fields of combinatorics and special polynomials, physics, engineering. $q$-analogs also appear in the study of quantum groups, matrices, identities, dynamical systems, fractals, modular groups, designs, systems, oscillators etc. [1, 3, 9, 14, 16].

In various areas of science, special $q$-polynomials have been studied, such as $q$-Bernoulli polynomials, $q$-Euler polynomials, $q$-Changhee polynomials, degenerate $q$-Euler polynomials and degenerate $q$-Changhee polynomials. Further, algebraic and arithmetic properties of the polynomials can be found in [8, 10–12, 14–17, 19–24].

Let $p$ be a fixed prime number. $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The $p$-adic norm $|.|_p$ is normalized by $|p|_p = \frac{1}{p}$. Let $q$ be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{\frac{-1}{p-1}}$. The $q$-extension of a number $x$ is defined as $[x]_q = \frac{1-q^x}{1-q}$. It is clear that $\lim\limits_{q\to 1} [x]_q = x$.

It is well known that Euler and Changhee polynomials play an important role in combinatorial analysis and number theory. Euler polynomials [2, 7] are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1}$$

In a special case, when $x = 0$, $E_n(0) = E_n$ is called the $n$th Euler number.

The $q$-Euler numbers are defined as follows [11, 12, 21]:

$$E_{0,q} = 1, \quad q(qE_q+1)^n + E_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention about replacing $E_q^i$ by $E_{i,q}$. The authors also gave

$$E_{n,q} = \frac{[2]_q}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^i}{1+q^{i+1}}.$$

In [2], Carlitz obtained the degenerate polynomials and numbers which are related to Euler polynomials to be

$$\frac{2}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} E_n(x \mid \lambda)\frac{t^n}{n!},$$

where $\lambda \in \mathbb{R}\setminus\{0\}$. It is seen that

$$\lim_{\lambda\to 0}\sum_{n=0}^{\infty} E_n(x \mid \lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

since $\lim_{\lambda\to 0} E_n(x \mid \lambda) = E_n(x)$.

The modified $q$-Euler polynomials, denoted by $\mathcal{E}_{n,q}(x)$, are defined by

$$\mathcal{E}_{n,q}(x) = \frac{[2]_q}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}\frac{q^{ix}}{1+q^i}.$$

When $x=0$, $\mathcal{E}_{n,q}(0)=\mathcal{E}_{n,q}$ is called the $n$th modified $q$-Euler number.

In [22], the authors gave the modified $q$-Euler polynomials with weight $\alpha$ as follows:

$$\mathcal{E}_{n,q}^{(\alpha)} = \frac{[2]_q}{(1-q^\alpha)^n}\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^i q^{\alpha i}}{1+q^{\alpha i}}.$$

In [16], for any parameters $\alpha$ and $\beta$, the degenerate $q$-Euler polynomials with weight $\alpha$ are defined by the generating function to be

$$\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathcal{E}_{q^\alpha,n}(x\mid\beta)\frac{t^n}{n!}. \tag{2}$$

In the special case, when $x=0$, $\mathcal{E}_{q^\alpha,n}(0\mid\beta) := \mathcal{E}_{q^\alpha,n}(\beta)$ are called the degenerate $q$-Euler numbers with weight $\alpha$.

The Changhee polynomials are defined by Kim et al. [10, 13] as the generating function to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!}.$$

When $x=0$, $Ch_n(0)=Ch_n$ are called Changhee numbers.

In [15], the degenerate Changhee polynomials are defined to be

$$\frac{2\lambda}{2\lambda+\log(1+\lambda t)}(1+\log(1+\lambda t)^{1/\lambda})^x = \sum_{n=0}^{\infty}Ch_{n,\lambda}(x)\frac{t^n}{n!}.$$

When $x=0$, $Ch_{n,\lambda}(0)=Ch_{n,\lambda}$ are called the degenerate Changhee numbers. For $n\geq 0$, $\lim_{\lambda\to 0}Ch_{n,\lambda}=Ch_n$ is given by [14, 15, 25].

The $q$-Changhee polynomials $Ch_{n,q}(x)$ are defined by the generating function to be (see [10])

$$\frac{1+q}{1+q(1+t)}(1+t)^x = \sum_{n=0}^{\infty}Ch_{n,q}(x)\frac{t^n}{n!}\quad . \tag{3}$$

When $x=0$, $Ch_{n,q}=Ch_{n,q}(0)$ are called $q$-Changhee numbers and when $q=1$, $Ch_n=Ch_{n,1}(0)$. In [16], the generating function of $q$-Changhee polynomials of the second kind with weight $\alpha$, denoted by $\mathcal{C}_{q^\alpha,n}(x\mid\beta)$, is given by

$$\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}+(1+\lambda t)^{1/\lambda}}(1+\lambda t)^{(x+1)/\lambda} = \sum_{n=0}^{\infty}\mathcal{C}_{q^\alpha,n}(x\mid\beta)\frac{t^n}{n!}. \tag{4}$$

When $x=-1$, $\mathcal{C}_{q^\alpha,n}(-1\mid\beta)=\mathcal{C}_{q^\alpha,n}(\beta)$.

In [4,5], the Bell polynomials of the second kind $\mathcal{B}_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$, for $n \geq k \geq 0$, can be defined by

$$\mathcal{B}_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) = \sum_{\substack{1\leq i\leq n-k+1 \\ l_i\in\mathbb{N}_0 \\ \sum_{i=1}^{n-k+1} il_i=n \\ \sum_{i=1}^{n-k+1} l_i=k}} \frac{n!}{\sum_{i=1}^{n-k+1} l_i} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right),$$

where $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$. Also, these polynomials can be generated by

$$\frac{1}{k!}\left(\sum_{n=1}^{\infty} x_n \frac{t^n}{n!}\right)^k = \sum_{n=k}^{\infty} \mathcal{B}_{n,k}(x_1, x_2, \cdots)\frac{t^n}{n!}. \tag{5}$$

In [5], for a positive integer $n$, Faà di Bruno formula is described in terms of the Bell polynomials of the second kind $\mathcal{B}_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$ by

$$\frac{d^n}{dt^n}(f(h(t))) = \sum_{k=1}^{n} f^{(k)}(h(t))\mathcal{B}_{n,k}(h'(t), h''(t), \cdots, h^{(n-k+1)}(t)). \tag{6}$$

There are some interesting computations involving the Bell polynomials of the second kind. For example, the formula

$$\mathcal{B}_{n,k}\left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \cdots, \prod_{i=0}^{n-k}(1-i\lambda)\right) = \frac{(-1)^k}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(i)_{n,\lambda} \tag{7}$$

has been applied and reviewed in [8, 18–20]. Here

$$(x)_{0,\lambda} = 1 \text{ and } (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda) \quad \text{for } n \geq 1.$$

Let $g(z)$ be an analytic function of $z$. A special value of Bell polynomials of the second kind is (see [6])

$$\sum_{n=k}^{\infty} \mathcal{B}_{n,k}(g'(0), g''(0), \cdots, g^{(n+1-k)}(0))\frac{t^n}{n!} = \frac{(g(t)-g(0))^k}{k!}.$$

In [17], studying Grothendieck's inequality and correlation-preserving functions, Oertel obtained the following interesting identity for a positive integer $n$:

$$\sum_{k=0}^{2n}(-1)^k\frac{(2n+k)!}{k!}\mathcal{B}^{\circ}_{2n,k}\left(0, \frac{1}{6}, 0, \frac{3}{40}, \cdots, \frac{1+(-1)^{k+1}}{2}\frac{((2n-k)!!)^2}{(2n-k+2)}\right) = (-1)^n,$$

where

$$\mathcal{B}^{\circ}_{n,k}(x_1, x_2, \cdots, x_{n+1-k}) = \frac{k!}{n!}\mathcal{B}_{n,k}(1!x_1, 2!x_2, \cdots, (n+1-k)!x_{n+1-k}).$$

In [26], the authors defined a degenerate $\lambda$-array type polynomial $S(n,m;x;\lambda;\gamma)$ by

$$\frac{(\lambda(1+\gamma t)^{1/\gamma}-1)^m}{m!}(1+\gamma t)^{x/\gamma}=\sum_{n=0}^{\infty}S(n,m;x;\lambda;\gamma)\frac{t^n}{n!}.$$

They established two explicit formulas for $S(n,m;x;\lambda;\gamma)$ with the help of the Faà di Bruno formula and an identity of the Bell polynomials of the second kind. For example, for $n\in\mathbb{N}$,

$$S(n,m;0;\lambda;\gamma)=\frac{n!}{m!}\lambda^n(\lambda-1)^m\sum_{k=1}^{n}\frac{(m)_k}{k!}\frac{1}{1/\lambda-1}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\binom{l/\lambda}{n}.$$

## 2. Some Results

In this section, we will get new identities involving degenerate $q$-Euler numbers, degenerate $q$-Euler polynomials with weight $\alpha$ and $q$-Changhee polynomials with weight $\alpha$.

**Theorem 1.** *For a positive integer $n$, we have*

$$\mathcal{E}_{q^\alpha,n}(\beta)=[2]_{q^\alpha}\sum_{k=1}^{n}\sum_{i=0}^{k}(-1)^i\frac{q^{k(\alpha+2\beta)}(i)_{n,\lambda}}{(q^{\alpha+2\beta}+1)^{k+1}}\binom{k}{i}$$

*and*

$$\mathcal{C}_{q^\alpha,n}(\beta)=[2]_{q^\alpha}\sum_{k=1}^{n}\sum_{i=0}^{k}(-1)^i\frac{(i)_{n,\lambda}}{(q^{\alpha+2\beta}+1)^{k+1}}\binom{k}{i}.$$

*Proof.* Let $f(u)=\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}u+1}$ and $u=h(t)=(1+\lambda t)^{1/\lambda}$. From (6), we have

$$\frac{d^n}{dt^n}\left(\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}(1+\lambda t)^{1/\lambda}+1}\right)$$
$$=\sum_{k=1}^{n}\frac{d^k}{du^k}\left(\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}u+1}\right)\mathcal{B}_{n,k}(h'(t),h''(t),\cdots,h^{(n-k+1)}(t))$$
$$=\sum_{k=1}^{n}\frac{(-1)^k k!q^{k(\alpha+2\beta)}[2]_{q^\alpha}}{(q^{\alpha+2\beta}u+1)^{k+1}}$$
$$\times\mathcal{B}_{n,k}\left((1+\lambda t)^{\frac{1}{\lambda-1}},(1+\lambda t)^{\frac{1}{\lambda-2}}(1-\lambda),\cdots,(1+\lambda t)^{\frac{1}{\lambda-n+k-1}}\prod_{l=0}^{n-k}(1-l\lambda)\right)$$

and by (7), it equals to

$$\sum_{k=1}^{n}\frac{(-1)^k k!q^{k(\alpha+2\beta)}[2]_{q^\alpha}}{(q^{\alpha+2\beta}+1)^{k+1}}\frac{(-1)^k}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(i)_{n,\lambda}$$

as $t \to 0$. Considering the generating function in (2), we have the first identity. Similarly, using (4), we have the second identity. □

**Theorem 2.** *For a positive integer $n$, we have*

$$\mathcal{E}_{q^\alpha,n}(x \mid \beta) = [2]_{q^\alpha} \sum_{k=1}^{n} \sum_{l=0}^{k} \sum_{i=0}^{k} \frac{(-1)^{i+k+l} q^{l(\alpha+2\beta)}}{(q^{\alpha+2\beta}+1)^{l+1}} \frac{l!}{k!} \binom{k}{l} \binom{k}{i} (i)_{n,\lambda} (x)_{k-l}$$

*and*

$$\mathcal{C}_{q^\alpha,n}(x \mid \beta) = [2]_{q^\alpha} \sum_{k=1}^{n} \sum_{l=0}^{k} \sum_{i=0}^{k} \frac{(-1)^{i+k+l}}{(q^{\alpha+2\beta}+1)^{l+1}} \frac{l!}{k!} \binom{k}{l} \binom{k}{i} (i)_{n,\lambda} (x+1)_{k-l}.$$

*Proof.* Let $f(u) = \frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}u+1} u^x$ and $u = h(t) = (1+\lambda t)^{1/\lambda}$. From the generating function of $\mathcal{E}_{q^\alpha,n}(x \mid \beta)$ and (6) we have

$$\begin{aligned}
&\frac{d^n}{dt^n}\left(\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda}\right) \\
=&\sum_{k=1}^{n} \frac{d^k}{du^k}\left(\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}u+1}u^x\right) \mathcal{B}_{n,k}(h'(t), h''(t), \cdots, h^{(n-k+1)}(t)) \\
=&\sum_{k=1}^{n} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^l l! [2]_{q^\alpha} q^{l(\alpha+2\beta)}}{(q^{\alpha+2\beta}u+1)^{l+1}} u^{x-k+l} (x)_{k-l} \\
&\times \mathcal{B}_{n,k}\left((1+\lambda t)^{\frac{1}{\lambda-1}}, (1+\lambda t)^{\frac{1}{\lambda-2}}(1-\lambda), \cdots, (1+\lambda t)^{\frac{1}{\lambda-n+k-1}} \prod_{l=0}^{n-k}(1-l\lambda)\right).
\end{aligned}$$

With the help of (7), we write

$$\begin{aligned}
&\frac{d^n}{dt^n}\left(\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda}\right) \\
&\quad = \sum_{k=1}^{n} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^l l! [2]_{q^\alpha} q^{l(\alpha+2\beta)}}{(q^{\alpha+2\beta}+1)^{l+1}} (x)_{k-l} \frac{(-1)^k}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (i)_{n,\lambda}
\end{aligned}$$

as $t \to 0$. So, we have the first identity. Similarly, the second identity can be proved. The proof is complete. □

*Remark* 1. Some values of $\mathcal{E}_{q^\alpha,n}(x \mid \beta)$ and $\mathcal{C}_{q^\alpha,n}(x \mid \beta)$ are:

$$\begin{aligned}
\mathcal{E}_{q^\alpha,0}(x \mid \beta) =& \mathcal{C}_{q^\alpha,0}(x \mid \beta) = \frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}+1}, \\
\mathcal{E}_{q^\alpha,1}(x \mid \beta) =& \frac{[2]_{q^\alpha}(-q^{\alpha+2\beta} + xq^{\alpha+2\beta} + x)}{(q^{\alpha+2\beta}+1)^2}, \\
\mathcal{C}_{q^\alpha,1}(x \mid \beta) =& \frac{[2]_{q^\alpha}((x+1)q^{\alpha+2\beta} + x)}{(q^{\alpha+2\beta}+1)^2},
\end{aligned}$$

$$\begin{aligned}\mathcal{E}_{q^{\alpha},2}(x\mid\beta) =&\frac{[2]_q\left(-x-q^{\alpha+2\beta}\left(2x-1\right)-q^{2\alpha+4\beta}\left(x-1\right)\right)\lambda}{2\left(q^{\alpha+2\beta}+1\right)^2}\\&+\frac{[2]_q\left(q^{2\alpha+4\beta}\left(x-1\right)^2-q^{\alpha+2\beta}(2x-2x^2+1)+x^2\right)}{2\left(q^{\alpha+2\beta}+1\right)^2},\\ \mathcal{C}_{q^{\alpha},2}(x\mid\beta) =&\frac{[2]_q\left(-x-q^{\alpha+2\beta}\left(2x+1\right)-q^{2\alpha+4\beta}\left(x+1\right)\right)\lambda}{2\left(q^{\alpha+2\beta}+1\right)^2}\\&+\frac{[2]_q\left(q^{2\alpha+4\beta}\left(x+1\right)^2+q^{\alpha+2\beta}\left(2x^2+2x-1\right)+x^2\right)}{2\left(q^{\alpha+2\beta}+1\right)^2}.\end{aligned}$$

We will examine the graphical representations of these polynomials for different values of indices in the following figures.

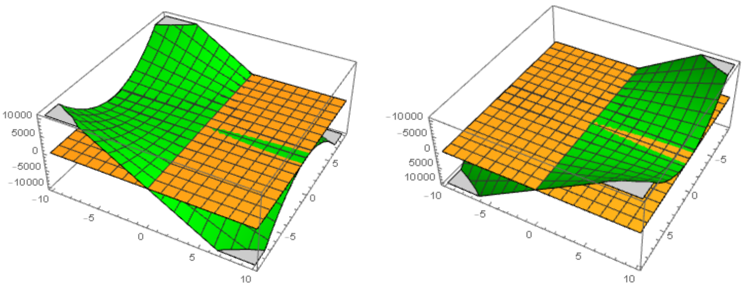


FIGURE 1. Two angles of the graph of $\mathcal{E}_{(1/3)^2,2}(1)$ for $-10<x<10$ and $-9<\lambda<9$ plotted by Wolfram Mathematica 11.2.

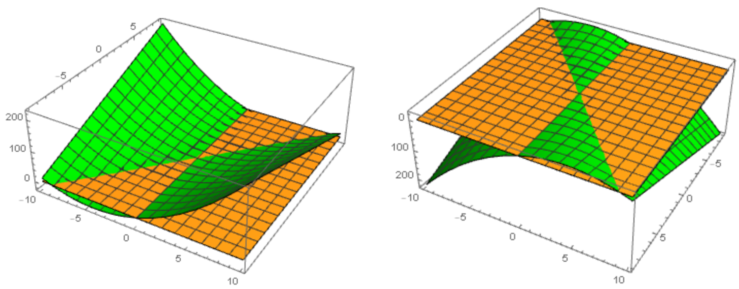


FIGURE 2. Two angles of the graph of $\mathcal{C}_{(1/3)^2,2}(1)$ for $-10<x<10$ and $-9<\lambda<9$ plotted by Wolfram Mathematica 11.2.

**Theorem 3.** *For a positive integer $n$, we have*

$$\sum_{i=1}^{n}\binom{n}{i}\mathcal{E}_{q^\alpha,n-i}(x\mid\beta)(1)_{i,\lambda}=\mathcal{E}_{q^\alpha,n}(x+1\mid\beta)-\mathcal{E}_{q^\alpha,n}(x\mid\beta)$$

*and*

$$\sum_{i=1}^{n}\binom{n}{i}\mathcal{C}_{q^\alpha,n-i}(x\mid\beta)(1)_{i,\lambda}=\mathcal{C}_{q^\alpha,n}(x+1\mid\beta)-\mathcal{C}_{q^\alpha,n}(x\mid\beta).$$

*Proof.* Consider

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\mathcal{E}_{q^\alpha,n}(x+1\mid\beta)-\mathcal{E}_{q^\alpha,n}(x\mid\beta)\right)\frac{t^n}{n!}=&\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x+1}{\lambda}}\\&-\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\\=&\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\\&\times\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right).\end{aligned}$$

By (2), we write

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\mathcal{E}_{q^\alpha,n}(x+1\mid\beta)-\mathcal{E}_{q^\alpha,n}(x\mid\beta)\right)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\mathcal{E}_{q^\alpha,n}(x\mid\beta)\frac{t^n}{n!}\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}}{n!}t^n\\&=\sum_{n=1}^{\infty}\sum_{i=1}^{n}\binom{n}{i}\mathcal{E}_{q^\alpha,n-i}(x\mid\beta)(1)_{i,\lambda}\frac{t^n}{n!}.\end{aligned}$$

Thus, comparing coefficients of the terms $t^n/n!$, we have the first identity. The second one can be verified in a similar way. □

**Theorem 4.** *For a non-negative integer $n$, we have*

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{q^\alpha,n-i}(x-\lambda\mid\beta)\mathcal{E}_{q^\alpha,i}(1\mid\beta)\\&\qquad=[2]_{q^\alpha}q^{-\alpha-2\beta}\left(x\mathcal{E}_{q^\alpha,n}(x-\lambda\mid\beta)-\mathcal{E}_{q^\alpha,n+1}(x\mid\beta)\right)\end{aligned}\tag{8}$$

*and*

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}\mathcal{C}_{q^\alpha,n-i}(x-\lambda\mid\beta)\mathcal{C}_{q^\alpha,i}(0\mid\beta)\\&\qquad=[2]_{q^\alpha}\left((x+1)\mathcal{C}_{q^\alpha,n}(x-\lambda\mid\beta)-\mathcal{C}_{q^\alpha,n+1}(x\mid\beta)\right).\end{aligned}\tag{9}$$

*Proof.* We know that

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty}\mathcal{E}_{q^\alpha,n}(x\mid\beta)\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}\mathcal{E}_{q^\alpha,n+1}(x\mid\beta)\frac{t^n}{n!}.$$

On the other hand, using the right hand side of (2) we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda}\right)\\
&\qquad= \frac{-[2]_{q^\alpha}q^{\alpha+2\beta}}{\left(q^{\alpha+2\beta}(1+\lambda t)^{\frac{1}{\lambda}}+1\right)^2}(1+\lambda t)^{\frac{x-\lambda+1}{\lambda}}\\
&\qquad\quad + x\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x-\lambda}{\lambda}}\\
&\qquad= \frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x-\lambda}{\lambda}}\\
&\qquad\quad \times\left(\frac{-q^{\alpha+2\beta}}{[2]_{q^\alpha}}\frac{[2]_{q^\alpha}}{q^{\alpha+2\beta}(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1}{\lambda}}+x\right)\\
&\qquad= \sum_{n=0}^{\infty}\mathcal{E}_{q^\alpha,n}(x-\lambda\mid\beta)\frac{t^n}{n!}\left(\frac{-q^{\alpha+2\beta}}{[2]_{q^\alpha}}\sum_{n=0}^{\infty}\mathcal{E}_{q^\alpha,n}(1\mid\beta)\frac{t^n}{n!}+x\right)\\
&\qquad= \frac{-q^{\alpha+2\beta}}{[2]_{q^\alpha}}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{\mathcal{E}_{q^\alpha,n-i}(x-\lambda\mid\beta)\mathcal{E}_{q^\alpha,i}(1\mid\beta)}{(n-i)!i!}t^n\\
&\qquad\quad + x\sum_{n=0}^{\infty}\mathcal{E}_{q^\alpha,n}(x-\lambda\mid\beta)\frac{t^n}{n!}\\
&\qquad= \sum_{n=0}^{\infty}\Bigg(x\mathcal{E}_{q^\alpha,n}(x-\lambda\mid\beta)\\
&\qquad\quad + \frac{-q^{\alpha+2\beta}}{[2]_{q^\alpha}}\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{q^\alpha,n-i}(x-\lambda\mid\beta)\mathcal{E}_{q^\alpha,i}(1\mid\beta)\Bigg)\frac{t^n}{n!}.
\end{aligned}$$

From the equality of formal power series, we have the first identity. Similarly, the second identity can be proved. The proof is complete. □

**Theorem 5.** *For a non-negative integer $n$, we have*

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{q^\alpha,n-i}(\beta)\mathcal{E}_{q^\alpha,i}(\beta)\\
&\qquad\qquad= -[2]_{q^\alpha}q^{-\alpha-2\beta}\sum_{i=0}^{n}\binom{n}{i}(1-1/\lambda)_{n-i}\lambda^{n-i}\mathcal{E}_{q^\alpha,i+1}(\beta)
\end{aligned}$$

*and*

$$\sum_{i=0}^{n}\binom{n}{i}\mathcal{C}_{q^\alpha,n-i}(\beta)\mathcal{C}_{q^\alpha,i}(\beta) = -[2]_{q^\alpha}\sum_{i=0}^{n}\binom{n}{i}(1-1/\lambda)_{n-i}\lambda^{n-i}\mathcal{C}_{q^\alpha,i+1}(\beta).$$

*Proof.* Let the function $f(t)$ be defined as

$$f(t)=\sum_{n=0}^{\infty}\mathcal{E}_{q^{\alpha},n}(\beta)\frac{t^n}{n!}=\left(\frac{q^{\alpha+2\beta}(1+\lambda t)^{1/\lambda}+1}{[2]_{q^{\alpha}}}\right)^{-1}.$$

Taking the derivative of $f(t)$ for the variable $t$, we have

$$f'(t)=\sum_{n=0}^{\infty}\mathcal{E}_{q^{\alpha},n+1}(\beta)\frac{t^n}{n!},\tag{10}$$

and thus

$$\begin{aligned}f'(t)&=-\left[2\right]_{q^{\alpha}}^{2}\left(q^{\alpha+2\beta}\left(1+\lambda t\right)^{\frac{1}{\lambda}}+1\right)^{-2}\frac{q^{\alpha+2\beta}}{\left(1+\lambda t\right)^{\frac{\lambda-1}{\lambda}}}\frac{1}{\left[2\right]_{q^{\alpha}}}\\&=-f^{2}(t)\frac{q^{\alpha+2\beta}}{\left(1+\lambda t\right)^{\frac{\lambda-1}{\lambda}}}\frac{1}{\left[2\right]_{q^{\alpha}}}.\end{aligned}$$

From here, we have

$$\begin{aligned}f^{2}(t)&=\frac{-\left[2\right]_{q^{\alpha}}}{q^{\alpha+2\beta}}\left(1+\lambda t\right)^{\frac{\lambda-1}{\lambda}}f'(t)\\&=\frac{-\left[2\right]_{q^{\alpha}}}{q^{\alpha+2\beta}}\sum_{n=0}^{\infty}\binom{\frac{\lambda-1}{\lambda}}{n}\lambda^{n}t^{n}\sum_{n=0}^{\infty}\mathcal{E}_{q^{\alpha},n+1}\left(\beta\right)\frac{t^{n}}{n!}\\&=\frac{-\left[2\right]_{q^{\alpha}}}{q^{\alpha+2\beta}}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{\frac{\lambda-1}{\lambda}}{n-i}\frac{\lambda^{n-i}\mathcal{E}_{q^{\alpha},i+1}\left(\beta\right)}{i!}t^{n}.\end{aligned}\tag{11}$$

On the other hand,

$$f^{2}(t)=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{\mathcal{E}_{q^{\alpha},n-i}(\beta)\mathcal{E}_{q^{\alpha},i}(\beta)}{(n-i)!i!}t^{n}.\tag{12}$$

Because the left hand sides of (11) and (12) are equal, the right hand sides must be equal too. From the equality of power series, we have the first identity. Similarly, the second one can be proved. So, the proof is complete. □

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