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Tensor product of partial acts

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Abstract. In this article we define the tensor product of partial acts over a semigroup and prove several properties of this tensor product. We also define the notion of a polite partial biact, which is needed to define partial actions on the tensor product of partial acts. Finally, we prove that a certain tensor functor of partial acts is a left adjoint of a certain hom-functor of partial acts.

1. Introduction

Partial structures arise very naturally in many fields of mathematics. In fact children in elementary school already encounter them, while learning about the subtraction of natural numbers or division of integers. Yet still partial structures are undeservedly not covered in almost any algebra course and are studied far too little compared to how often they appear.

In the last decades several authors have studied partial actions. For instance, Exel in [6] and Kellendok and Lawson in [8] have studied partial actions on groups. Their results have been used in several applications. Also, Megrelishvili and Schröder in [12] and Hollings in [7] have studied partial actions on monoids. So the study of partial actions has become a field of active research. A recent overview of this area is [5].

In this article we study partial actions on semigroups and so obtain so-called partial acts. In Section 2, we define partial acts, give examples and define homomorphisms of partial acts. Lastly we introduce the notion of polite partial biacts, which will become important to define partial actions on the tensor product later. In Section 3 we define the tensor product of partial acts and prove several properties of it. We also show that this construction

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induces tensor functors between categories of partial acts. Finally, in Section 4, we study a certain hom-functor of partial acts. Then we prove that there exists an adjunction between these tensor functors and hom-functors under some assumptions.

The authors assume basic familiarity with semigroup theory, for which one can reference [10], and category theory, for which one can reference [1].

2. Definition of partial acts

In this section we will give the definition of partial (right) acts and give some examples of them. The following definition of a partial act is inspired by [7], Definition 2.2.

Definition 2.1. Let $(S, *)$ be a semigroup and A a set. A partial mapping

$$\cdot\colon A \times S \to A, \quad (a, s) \mapsto a \cdot s$$

is called a **partial right S-action** if for every $a \in A$ and $s, s' \in S$ we have

$$\exists\, a \cdot s \wedge \exists\, (a \cdot s) \cdot s' \implies \exists\, a \cdot (s * s') \wedge (a \cdot s) \cdot s' = a \cdot (s * s').$$

(Here $\exists a{\cdot}s$ means that (a, s) belongs to the domain of the partial mapping $\cdot$.)

The pair $(A, \cdot)$ is called a **partial right S-act** and is denoted by A_S.

We define **partial left S-actions** and **S-acts** dually.

There are many natural examples of partial acts. For instance, every act is a partial act. Also, every set is a partial act over any semigroup, if the action is defined as the empty mapping.

Next we will give two non-trivial examples of partial acts.

Example 2.2. The set of integers $\mathbb{Z}$ is a partial right act over the semigroup $(\mathbb{N}, \cdot)$ if the action is division of integers. Indeed, if quotients $\frac{z}{n_1}$ and $(\frac{z}{n_1})/n_2$, where $z \in \mathbb{Z}$, $n_1, n_2 \in \mathbb{N}$, are integers, then $\frac{z}{n_1 n_2}$ is also an integer and

$$\frac{\left(\frac{z}{n_1}\right)}{n_2} = \frac{z}{n_1 n_2}.$$

Example 2.3. The set of rational numbers $\mathbb{Q}$ is a partial left act over the semigroup $(\mathbb{N}, \cdot)$ if the action is extracting roots. Let the roots $\sqrt[n_1]{q} \in \mathbb{Q}$ and $\sqrt[n_2]{\sqrt[n_1]{q}} \in \mathbb{Q}$ exist. This means that if $q < 0$, then n_1 and n_2 are odd numbers. In that case $n_1 n_2$ is also odd and the root $\sqrt[n_1 n_2]{q} \in \mathbb{Q}$ exists and $\sqrt[n_1 n_2]{q} = \sqrt[n_2]{\sqrt[n_1]{q}}$. On the other hand, if $q \geqslant 0$, the parity of n_1 and n_2 is not important, therefore always $\sqrt[n_1 n_2]{q} = \sqrt[n_2]{\sqrt[n_1]{q}}$, which proves that ${}_{\mathbb{N}}\mathbb{Q}$ is a partial left act.

Similarly to the case of acts, we also define partial (R, S)-biacts, where R and S are semigroups. This definition can be found in [3] (page 216).

Definition 2.4. Let S and R be semigroups. A triple $(A, \star, \cdot)$ is called a **partial** (R, S)**-biact** if $(A, \star)$ is a partial left R-act, $(A, \cdot)$ is a partial right S-act and for every $a \in A$, $r \in R$, $s \in S$ we have

$$\exists\, a \cdot s \wedge \exists\, r \star a \implies \exists\, r \star (a \cdot s) \wedge \exists\, (r \star a) \cdot s \wedge r \star (a \cdot s) = (r \star a) \cdot s.$$

We will usually omit the symbols of partial actions $\cdot$ and $\star$, if there is no threat of confusion. The partial (R, S)-biact will be denoted by ${}_RA_S$.

Next we will give two examples of non-trivial partial biacts of matrices. Let $\mathrm{Mat}(\mathbb{R})$ denote the set of all real matrices. Also, let $\mathrm{Mat}_{m,n}(\mathbb{R})$ and $\mathrm{Mat}_n(\mathbb{R})$ denote the subsets of $\mathrm{Mat}(\mathbb{R})$ of $(m \times n)$-matrices and $(n \times n)$ square matrices, respectively. We consider $\mathrm{Mat}_n(\mathbb{R})$ as a semigroup under the usual multiplication of matrices. Also, for any semigroup S, let S^{op} denote the *opposite semigroup* of S.

Example 2.5. The set $\mathrm{Mat}(\mathbb{R})$ is a partial $(\mathrm{Mat}_m(\mathbb{R}), \mathrm{Mat}_n(\mathbb{R}))$-biact, where $m, n \in \mathbb{N}$ and both of the semigroup actions are the usual matrix multiplication.

Example 2.6. The set $\mathrm{Mat}_{n,m}(\mathbb{R})$ is a partial $\left((\mathrm{Mat}_n(\mathbb{R}))^{\mathrm{op}}, (\mathrm{Mat}_m(\mathbb{R}))^{\mathrm{op}}\right)$-biact, if the action is multiplication by the inverse matrix. Namely, define two partial actions (which we denote by the same symbol $\circledast$) on $\mathrm{Mat}_{n,m}(\mathbb{R})$:

$$\circledast\colon\ (\mathrm{Mat}_n(\mathbb{R}))^{\mathrm{op}} \times \mathrm{Mat}_{n,m}(\mathbb{R}) \to \mathrm{Mat}_{n,m}(\mathbb{R}), \qquad S \circledast M := S^{-1} \cdot M,$$

$$\circledast\colon\ \mathrm{Mat}_{n,m}(\mathbb{R}) \times (\mathrm{Mat}_m(\mathbb{R}))^{\mathrm{op}} \to \mathrm{Mat}_{n,m}(\mathbb{R}), \qquad M \circledast R := M \cdot R^{-1}.$$

Let $M \in \mathrm{Mat}_{n,m}(\mathbb{R})$. If $S_1, S_2 \in \mathrm{Mat}_n(\mathbb{R})$ are invertible, then the products

$$S_1 \circledast M = S_1^{-1} \cdot M \quad \text{and} \quad S_2 \circledast (S_1 \circledast M) = S_2^{-1} \cdot (S_1^{-1} \cdot M)$$

exist and, moreover,

$$\begin{aligned} S_2 \circledast (S_1 \circledast M) = S_2^{-1} \cdot (S_1^{-1} \cdot M) = (S_2^{-1} \cdot S_1^{-1}) \cdot M = (S_1 \cdot S_2)^{-1} \cdot M = \\ = (S_2 \cdot_{\mathrm{op}} S_1)^{-1} \cdot M = (S_2 \cdot_{\mathrm{op}} S_1) \circledast M. \end{aligned}$$

Hence $\mathrm{Mat}_{n,m}(\mathbb{R})$ is a partial left act over the semigroup $((\mathrm{Mat}_n(\mathbb{R}))^{\mathrm{op}}, \cdot_{\mathrm{op}})$. Similarly we can show that $\mathrm{Mat}_{n,m}(\mathbb{R})$ is also a partial right act over the semigroup $((\mathrm{Mat}_m(\mathbb{R}))^{\mathrm{op}}, \cdot_{\mathrm{op}})$. Finally, it is easy to see, that $(S \circledast M) \circledast R = S \circledast (M \circledast R)$ holds for any M and invertible S, R. Hence $\mathrm{Mat}_{n\times m}(\mathbb{R})$ is a $((\mathrm{Mat}_n(\mathbb{R}))^{\mathrm{op}}, (\mathrm{Mat}_m(\mathbb{R}))^{\mathrm{op}})$-biact. Here we have essentially defined a division of matrices.

We note that there are many natural ways of defining homomorphisms of partial acts that generalize the notion of a homomorphism of acts, but we have chosen the following, because this works well in the sequel.

Definition 2.7. Let S be a semigroup and A_S and A'_S partial right acts. A mapping $f\colon A \to A'$ is called a **homomorphism of partial right** S**-acts** if for every $a \in A$ and $s \in S$ we have

$$\exists\, as \implies \exists\, f(a)s \,\wedge\, f(as) = f(a)s.$$

We define **homomorphims of partial left acts** dually.

Definition 2.8. Let R and S be semigroups and ${}_RA_S$ and ${}_RA'_S$ partial biacts. We call a mapping $f\colon A \to A'$ a **homomorphism of partial biacts** if $f\colon {}_RA \to {}_RA'$ is a homomorphism of partial left R-acts and $f\colon A_S \to A'_S$ is a homomorphism of partial right S-acts.

It is easy to see that the following proposition holds.

Proposition 2.9. *Partial right (left) S-acts ((R,S)-biacts) with the respective homomorphisms of partial acts form a category.*

We denote the category of partial right S-acts by PAct_S, the category of partial left S-acts by ${}_S\mathsf{PAct}$ and the category of partial (R,S)-biacts by ${}_R\mathsf{PAct}_S$.

Next we will introduce the notion of a polite partial biact. Politeness will become important for defining partial actions on the tensor product of partial acts.

Definition 2.10. Let R and S be semigroups. We call a partial (R,S)-biact $(A,\star,\cdot)$ **left polite** if, for every $a \in A$, $r \in R$, $s \in S$, we have

$$\exists\, a \cdot s \wedge \exists\, r \star (a \cdot s) \implies \exists\, r \star a \,\wedge \exists (r \star a) \cdot s. \tag{2.1}$$

If a partial (R,S)-biact ${}_RA_S$ satisfies the converse implication of (2.1), we say that ${}_RA_S$ is **right polite**. Additionally, we call an (R,S)-biact ${}_RA_S$ **polite**, if ${}_RA_S$ is both left and right polite, i.e., for every $a \in A$, $r \in R$, $s \in S$, we have

$$\exists\, as \wedge \exists\, r(as) \iff \exists\, ra \,\wedge \exists (ra)s.$$

3. Tensor product of partial acts

In this section we will define the tensor product of partial acts and prove several simple properties of this notion. Tensor product of acts over monoids was independently introduced by several authors around 1970, see [4], [9] and [13], generalizing the classical notion of the tensor product of modules over a ring with an identity. Since then it has been an important tool for study acts over monoids, in particular it has played a major role in homological classification of monoids. The monograph [10] gives a good overview of this. This section and the next section mostly generalize Chapter 2.5 in [10]. First we must introduce the notion of a tensorial mapping.

Definition 3.1. Let S be a semigroup, A_S a partial right S-act, ${}_SM$ a partial left S-act and Y a set. A mapping

$$\beta\colon A \times M \to Y$$

is called **S-tensorial** if, for every $a \in A$, $m \in M$ and $s \in S$, for which the products as and sm exist, the equality $\beta(as,m) = \beta(a,sm)$ holds.

Now we are ready to define the tensor product of partial acts.

Definition 3.2. Let S be a semigroup, A_S a partial right S-act and ${}_SM$ a partial left S-act. A set T with an S-tensorial mapping $\tau\colon A \times M \to T$ is called a **tensor product** of the partial acts A_S and ${}_SM$, if, for any set Y and any S-tensorial mapping $\beta\colon A \times M \to Y$, there exists a unique mapping $\overline{\beta}\colon T \to Y$ such that $\beta = \overline{\beta} \circ \tau$.

The definition of the tensor product of partial acts is illustrated by the following commutative diagram.

$$\begin{tikzcd} A \times M \arrow[r, "\beta"] \arrow[dr, "\tau"'] & Y \\ & T \arrow[u, dashed, "\overline{\beta}"'] \end{tikzcd}$$

Let Set denote the category of sets. Similarly to the tensor product of (usual) acts or modules, the following proposition can be easily verified (see Proposition 5.3 in [10] or Proposition 19.1 in [2]).

Proposition 3.3. *If (T, τ) and (T', τ') are tensor products of the partial acts A_S and ${}_SM$, then the sets T and T' are isomorphic in* Set.

Let A_S be a partial right S-act and ${}_SM$ a partial left S-act. Let ν be the equivalence relation on the set $A \times M$ generated by the pairs $((as, m), (a, sm))$, where $a \in A$, $m \in M$, $s \in S$ and the products as and sm exist. We call the relation ν a **partial tensor relation** and define

$$A \otimes_S M \coloneqq (A \times M)/\nu, \quad a \otimes m \coloneqq [(a, m)]_\nu.$$

Given a semigroup S, denote by S^1 the semigroup S **with an adjoined identity**, i.e. $S^1 = (S \sqcup \{1\}, *)$, where the multiplication $*$ is defined by

$$a * b = \begin{cases} ab, & \text{if } a \in S \wedge b \in S, \\ a, & \text{if } b = 1, \\ b, & \text{if } a = 1, \end{cases}$$

where $a, b \in S \sqcup \{1\}$. In the sequel, we will omit the symbol $*$. If A_S is a partial right S-act, then we can consider A also as a partial right S^1-act by defining $a1 = a$ for every $a \in A$.

Proposition 3.4. *Let A_S be a partial right S-act and ${}_SM$ a partial left S-act. Then $a \otimes m = a' \otimes m'$, $a, a' \in A$, $m, m' \in M$, if and only if there exist elements $s_1, \dots, s_k, t_1, \dots, t_k \in S^1$, $b_1, \dots, b_{k-1} \in A$ and $n_1, \dots, n_k \in M$ such that the products $s_i n_i$, $t_i n_i$, $b_i t_i$, $b_i s_{i+1}$ exist and the equations*

$$\begin{array}{rl} & s_1n_1 = m \\ as_1 = b_1t_1 & s_2n_2 = t_1n_1 \\ b_1s_2 = b_2t_2 & s_3n_3 = t_2n_2 \\ \dots & \dots \\ b_{k-1}s_k = a't_k & m' = t_kn_k \end{array} \tag{3.1}$$

hold.

Proof. Let $H = \{((as, m), (a, sm)) | a \in A, m \in M, s \in S, \exists\, as, \exists\, sm\}$ and let ν be the equivalence relation generated by the relation H. We also define the relation $\sigma \subseteq (A \times M)^2$ such that $(a, m)\sigma(a', m')$, where $a, a' \in A$ and $m, m' \in M$, if and only if there exist elements $s_1, \dots, s_k, t_1, \dots, t_k \in S^1$, $b_1, \dots, b_{k-1} \in A$ and $n_1, \dots, n_k \in M$ such that the products s_in_i, t_in_i, b_it_i, b_is_{i+1} exist and the equations (3.1) hold. It is easy to see that the relation σ is an equivalence relation.

We next show that $\nu \subseteq \sigma$. Let $((as, m), (a, sm)) \in H$. Then

$$\begin{array}{rl} & 1m = m \\ (as)1 = as & sm = sm, \end{array}$$

therefore $(as, m)\sigma(a, sm)$. Hence $H \subseteq \sigma$. Because ν is the least equivalence relation which contains the relation H and σ is an equivalence relation, we have $\nu \subseteq \sigma$.

Finally, we show that $\sigma \subseteq \nu$. Let $((a, m), (a', m')) \in \sigma$. Then there exist elements $s_1, \dots, s_k$, $t_1, \dots, t_k \in S^1$, $b_1, \dots, b_{k-1} \in A$ and $n_1, \dots, n_k \in M$ such that the products s_in_i, t_in_i, b_it_i, b_is_{i+1} exist and the equations (3.1) hold. Now note that

$$\begin{aligned}(a, m) &= (a, s_1n_1)H(as_1, n_1) = (b_1t_1, n_1)H(b_1, t_1n_1) = \dots \\ &= (a't_k, n_k)H(a', t_kn_k) = (a', m').\end{aligned}$$

Since $H \subseteq \nu$ and ν is transitive, we have $(a, m)\nu(a', m')$. Hence $\sigma \subseteq \nu$. In conclusion, $\nu = \sigma$. □

The scheme (3.1) is called an S-**tossing** between $a \otimes m$ and $a' \otimes m'$.

Proposition 3.5. *The set $A \otimes_S M$ with the mapping*

$$\tau\colon A \times M \to A \otimes_S M, \quad \tau(a, m) = a \otimes m,$$

is a tensor product of partial acts A_S and ${}_SM$.

Proof. Let Y be a set and $\beta\colon A \times M \to Y$ an S-tensorial mapping. For every $b \in A$ and $n \in M$ we define

$$\overline{\beta}\colon A \otimes_S M \to Y, \quad \overline{\beta}(b \otimes n) = \beta(b, n).$$

We show that $\overline{\beta}$ is well defined. Let $a \otimes m = a' \otimes m'$, where $a, a' \in A$, $m, m' \in M$. Then there exists an S-tossing similar to (3.1). Because β is S-tensorial, we have

$$\beta(a,m) = \beta(a,s_1n_1) = \beta(as_1,n_1) = \beta(b_1t_1,n_1) = \ldots = \beta(a',m').$$

Therefore $\overline{\beta}(a \otimes m) = \beta(a,m) = \beta(a',m') = \overline{\beta}(a' \otimes m')$, which proves that $\overline{\beta}$ is well defined. Now

$$\left(\overline{\beta} \circ \tau\right)(a,m) = \overline{\beta}(\tau(a,m)) = \overline{\beta}(a \otimes m) = \beta(a,m),$$

hence $\beta = \overline{\beta} \circ \tau$. It is straightforward to check that β is unique. □

Remark 3.6. Clearly a semigroup S can be considered as a left (or right) act over itself if the action is defined by the multiplication of S. So we may consider the tensor product $A \otimes_S S$. Tensor products of this particular form appear in Section 3.1 of [11]. They are used for globalizing the partial action of a firm strong partial act A_S.

Now we define tensor product of homomorphisms of partial acts.

Definition 3.7. Let S be a semigroup and $f\colon A_S \to A'_S$, $g\colon {}_SM \to {}_SM'$ homomorphisms of partial acts, $\tau\colon A \times M \to A \otimes_S M$ and $\tau'\colon A' \times M' \to A' \otimes_S M'$ the canonical surjections. A unique mapping $\overline{\tau' \circ (f \times g)}$, which makes the diagram

$$\begin{array}{ccc} A \times M & \xrightarrow{f \times g} & A' \times M' \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau'} \\ A \otimes_S M & \xrightarrow{\overline{\tau' \circ (f \times g)}} & A' \otimes_S M' \end{array}$$

commutative, is called the **tensor product of the homomorphisms** f **and** g. We denote the tensor product of the homomorphisms f and g by

$$f \otimes g := \overline{\tau' \circ (f \times g)}.$$

We will show that the tensor product of homomorphisms is well defined. Note that

$$(\tau' \circ (f \times g))(m,n) = f(m) \otimes g(n).$$

Let $a \in A$, $m \in M$, $s \in S$ be such that the products as and sm exist. Then the products $f(a)s$ and $sg(m)$ also exist and the equalities $f(a)s = f(as)$, $sg(m) = g(sm)$ hold. Therefore

$$\begin{aligned}(\tau' \circ (f \times g))(as,m) &= f(as) \otimes g(m) = f(a)s \otimes g(m) = f(a) \otimes sg(m) \\ &= f(a) \otimes g(sm) = (\tau' \circ (f \times g))(a,sm).\end{aligned}$$

Hence, the mapping $\tau' \circ (f \times g)$ is S-tensorial. Therefore, there exists a unique mapping $\overline{\tau' \circ (f \times g)}\colon A \otimes_S M \to A' \otimes_S M'$ such, that $\overline{\tau' \circ (f \times g)} \circ \tau = \tau' \circ (f \times g)$. Now we have

$$\overline{\tau' \circ (f \times g)}(a \otimes m) = \overline{\tau' \circ (f \times g)}(\tau(a,m)) = (\tau' \circ (f \times g))(a,m) = f(a) \otimes g(m).$$

Therefore, the tensor product of f and g exists. Note that we have proved the following lemma.

Lemma 3.8. *Let $f\colon A_S \to A'_S$ and $g\colon {}_SM \to {}_SM'$ be homomorphisms of partial acts. Then, for every $a \in A$ and $m \in M$, we have*

$$(f \otimes g)(a \otimes m) = f(a) \otimes g(m).$$

Before the next proposition recall that a morphism is called a **retraction** if it has a right inverse and a **coretraction** if it has a left inverse. It is straightforward to check that the tensor product of homomorphisms of partial acts has the following properties.

Proposition 3.9. *Let A_S be a partial right S-act and ${}_SM$ a partial left S-act. Then*

(1) $\mathrm{id}_A \otimes \mathrm{id}_M = \mathrm{id}_{A \otimes_S M}$;
(2) *if $f\colon A_S \to A'_S$ and $g\colon {}_SM \to {}_SM'$ are surjective, then $f \otimes g$ is also surjective;*
(3) *if $f\colon A_S \to A'_S$, $g\colon {}_SM \to {}_SM'$, $f'\colon A'_S \to A''_S$ and $g'\colon {}_SM' \to {}_SM''$ are homomorphisms of partial acts, then*
$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)\colon \quad A \otimes_S M \to A'' \otimes_S M'';$$
(4) *if $f\colon A_S \to A'_S$ and $g\colon {}_SM \to {}_SM'$ are retractions, coretractions or isomorphisms, then so is $f \otimes g$. In the third case one has*
$$(f \otimes g)^{-1} = f^{-1} \otimes g^{-1}.$$

Next we show that under certain assumptions we can consider the tensor product of partial acts itself as a partial biact too.

Proposition 3.10. *Let R, S be semigroups, ${}_RA_S$ a partial left polite (R, S)-biact and ${}_SM$ a partial left S-act. Then ${}_R(A \otimes_S M)$ is a partial left R-act with a partial action*

$$r(a \otimes m) := ra \otimes m \tag{3.2}$$

for all $a \in A$, $m \in M$, $r \in R$ such that the product ra exists.

Proof. Define a partial left action as in (3.2). First we show that the mapping $R \times (A \otimes_S M) \to A \otimes_S M$ is well defined. Let $a, a' \in A$, $m, m' \in M$ and $r \in R$ be such that the product ra exists and $a \otimes m = a' \otimes m'$. Due to Proposition 3.4 there exist elements $s_1, \dots, s_k, t_1, \dots, t_k \in S^1$, $b_1, \dots, b_{k-1} \in A$ and $n_1, \dots, n_k \in M$ such that the products $s_i n_i$, $t_i n_i$, $b_i t_i$, $b_i s_{i+1}$ exist and the equalities

$$\begin{aligned} & & s_1 n_1 &= m \\ a s_1 &= b_1 t_1 & s_2 n_2 &= t_1 n_1 \\ b_1 s_2 &= b_2 t_2 & s_3 n_3 &= t_2 n_2 \\ &\dots & &\dots \\ b_{k-1} s_k &= a' t_k & m' &= t_k n_k \end{aligned}$$

hold. Now

$$r(a \otimes m) = (ra) \otimes m \qquad \text{(definition of the } R\text{-action)}$$

$$
\begin{aligned}
&= (ra) \otimes (s_1 n_1) && (m = s_1 n_1 \text{ from the scheme})\\
&= ((ra)s_1) \otimes n_1 && (\text{definition of a partial biact, tensoriality})\\
&= (r(as_1)) \otimes n_1 && (\text{definition of a partial biact})\\
&= (r(b_1 t_1)) \otimes n_1 && (as_1 = b_1 t_1 \text{ from the scheme})\\
&= ((rb_1)t_1) \otimes n_1 && (\text{left politeness})\\
&= (rb_1) \otimes (t_1 n_1) && (\text{tensoriality})\\
&= (rb_1) \otimes (s_2 n_2) && (t_1 n_1 = s_2 n_2 \text{ from the scheme})\\
&= \dots\\
&= (ra') \otimes (t_k n_k) && (\text{tensoriality})\\
&= (ra') \otimes m' && (t_k n_k = m' \text{ from the scheme})\\
&= r(a' \otimes m'), && (\text{definition of the } R\text{-action})
\end{aligned}
$$

therefore, the mapping $R \times (A \otimes_S M) \to A \otimes_S M$ is well defined.

Let $r, r' \in R$, $a \in A$, $m \in M$ and assume that the products ra and $r'(ra)$ exist. Then the product $(r'r)a$ also exists and $(r'r)a = r'(ra)$ holds. Furthermore, the product $r(a \otimes m) = (ra) \otimes m$ exists and $r'((ra) \otimes m) = (r'(ra)) \otimes m$ holds. Therefore

$$r'(r(a \otimes m)) = r'((ra) \otimes m) = (r'(ra)) \otimes m = ((r'r)a) \otimes m = (r'r)(a \otimes m).$$

Hence ${}_R(A \otimes_S M)$ is a partial left R-act. □

Analogously, the following proposition can be proved.

Proposition 3.11. *Let S, T be semigroups, A_S a partial right S-act and ${}_SM_T$ a right polite partial (S,T)-biact. Then $(A \otimes_S M)_T$ is a partial right T-act with a partial action*

$$(a \otimes m)t := a \otimes mt \tag{3.3}$$

for all $a \in A$, $m \in M$ and $t \in T$ such that the product mt exists.

Next we combine the previous two propositions.

Proposition 3.12. *Let R, S, T be semigroups, ${}_RA_S$ a left polite partial (R,S)-biact and ${}_SM_T$ a right polite partial (S,T)-biact. Then ${}_R(A \otimes_S M)_T$ is a partial (R,T)-biact with the left partial action defined in* (3.2) *and right partial action defined in* (3.3). *Moreover, the biact ${}_R(A \otimes_S M)_T$ is polite.*

Proof. The set ${}_R(A \otimes_S M)_T$ is both a left partial R-act and a right partial T-act by Propositions 3.10 and 3.11, respectively.

Let $r \in R$, $t \in T$, $a \in A$, $m \in M$ be such that the products ra and mt exist. Then, there also exist the products $r(a \otimes m) = (ra) \otimes m$, $(a \otimes m)t = a \otimes (mt)$ and, hence, the products $(r(a \otimes m))t$, $r((a \otimes m)t)$ also exist. Note that

$$(r(a \otimes m))t = ((ra) \otimes m)t = (ra) \otimes (mt) = r(a \otimes (mt)) = r((a \otimes m)t).$$

Therefore ${}_R(A \otimes_S M)_T$ is a partial (R, T)-biact.

Now, let the products $(a \otimes m)t$ and $r((a \otimes m)t)$ exist. Then, the product mt also exists and $r((a \otimes m)t) = r(a \otimes (mt))$ holds. Hence, also the products ra and $r(a \otimes m)$ exist and

$$r((a \otimes m)t) = r(a \otimes (mt)) = (ra) \otimes (mt) = ((ra) \otimes m)t = (r(a \otimes m))t.$$

This means, that ${}_R(A \otimes_S M)_T$ is a left polite partial biact.

Analogously, ${}_R(A \otimes_S M)_T$ is right polite and, hence, a polite partial (R, T)-biact. $\square$

Next we will introduce tensor functors.

Proposition 3.13. *Let A_S be a right partial act. Then*

$$A \otimes_S \_\_ \colon {}_S\mathsf{PAct} \to \mathsf{Set}$$

$$\begin{array}{ccc} {}_SM & \longmapsto & A \otimes_S M \\ \Big\downarrow g & & \Big\downarrow \mathrm{id}_A \otimes g \\ {}_SM' & \longmapsto & A \otimes_S M' \end{array}$$

is a covariant functor. Similarly, for a left partial act ${}_SM$ we have a covariant functor

$$\_\_ \otimes_S M \colon \mathsf{PAct}_S \to \mathsf{Set}.$$

Moreover, if ${}_RA_S$ is a left polite partial biact, then we have a functor

$${}_RA \otimes_S \_\_ \colon {}_S\mathsf{PAct} \to {}_R\mathsf{PAct}$$

and for a right polite partial biact ${}_SM_T$ we have a functor

$$\_\_ \otimes_S M_T \colon \mathsf{PAct}_S \to \mathsf{PAct}_T.$$

Proof. It is easy to see that $A \otimes_S \_\_ \colon {}_S\mathsf{PAct} \to \mathsf{Set}$ and $\_\_ \otimes_S M \colon \mathsf{PAct}_S \to \mathsf{Set}$ are covariant functors. If ${}_RA_S$ is a polite partial biact and ${}_SM$ a partial left act, then ${}_R(A \otimes_S M)$ is a partial left act according to Proposition 3.10. Therefore we also have a functor ${}_RA \otimes_S \_\_ \colon {}_S\mathsf{PAct} \to {}_R\mathsf{PAct}$. Similarly there exists a functor $\_\_ \otimes_S M_T \colon \mathsf{PAct}_S \to \mathsf{PAct}_T$. $\square$

We call the functors defined in Proposition 3.13 **tensor functors**. Next we show that tensor product is associative up to isomorphism.

Theorem 3.14. *For every partial right act A_S, partial left act ${}_RC$ and polite partial biact ${}_SB_R$ there exists a bijection*

$$\nu_{A,B,C} \colon (A \otimes_S B) \otimes_R C \to A \otimes_S (B \otimes_R C)$$
$$\nu_{A,B,C}((a \otimes b) \otimes c) = a \otimes (b \otimes c).$$

Moreover, the family $\nu = (\nu_{A,B,C})$ is a natural transformation in each variable A, B and C.

Proof. Let A_S be a partial right act, ${}_RC$ a partial left act and ${}_SB_R$ a polite partial biact. Note that $A \otimes_S B$ is a partial right R-act and $B \otimes_R C$ is a partial left S-act with respect to the actions defined in Proposition 3.10 and Proposition 3.11.

Fix $c \in C$ and define

$$\beta_c(a, b) := a \otimes (b \otimes c)$$

for every $a \in A$ and $b \in B$. Let $a \in A$, $b \in B$, $s \in S$ be such that the products as and sb exist. Then

$$\begin{aligned}
\beta_c(a, sb) &= a \otimes ((sb) \otimes c) && \text{(the definition of } \beta_c) \\
&= a \otimes (s(b \otimes c)) && (S\text{-action of the partial biact } B \otimes_R C) \\
&= (as) \otimes (b \otimes c) && \text{(tensoriality and existence of the product } as) \\
&= \beta_c(as, b). && \text{(the definition of } \beta_c)
\end{aligned}$$

So β_c is S-tensorial and therefore there exists a unique mapping $\overline{\beta_c} \colon A \otimes_S B \to A \otimes_S (B \otimes_R C)$ such that $\beta_c = \overline{\beta_c} \circ \tau$ holds, where $\tau \colon A \times B \to A \otimes_S B$ is the canonical surjection.

Now, consider the mapping $\mu \colon (A \otimes_S B) \times C \to A \otimes_S (B \otimes_R C)$ defined by

$$\mu(a \otimes b, c) := \overline{\beta_c}(a \otimes b) = \overline{\beta_c}(\tau(a, b)) = \beta_c(a, b) = a \otimes (b \otimes c).$$

We show that μ is R-tensorial. Let $b \in B$, $c \in C$, $r \in R$ be such that the products br, rc exist and let $a \in A$. Now

$$\mu(a \otimes b, rc) = a \otimes (b \otimes rc) = a \otimes ((br) \otimes c) = \mu(a \otimes (br), c) = \mu((a \otimes b)r, c).$$

Therefore there exists a unique mapping $\overline{\mu} \colon (A \otimes_S B) \otimes_R C \to A \otimes_S (B \otimes_R C)$ such that $\mu = \overline{\mu} \circ \tau'$, where $\tau' \colon (A \otimes_S B) \times C \to (A \otimes_S B) \otimes_R C$ is the canonical surjection. Hence we have

$$\overline{\mu}((a \otimes b) \otimes c) = \overline{\mu}(\tau'(a \otimes b, c)) = \mu(a \otimes b, c) = a \otimes (b \otimes c)$$

for every $a \in A$, $b \in B$, $c \in C$. Similarly, there exists a well-defined mapping $\overline{\gamma} \colon A \otimes_S (B \otimes_R C) \to (A \otimes_S B) \otimes_R C$ such that

$$\overline{\gamma}(a \otimes (b \otimes c)) = (a \otimes b) \otimes c.$$

Clearly $\overline{\gamma} \circ \overline{\mu} = \mathrm{id}_{(A \otimes_S B) \otimes_R C}$ and $\overline{\mu} \circ \overline{\gamma} = \mathrm{id}_{A \otimes_S (B \otimes_R C)}$. Hence $\overline{\mu}$ is bijective and we take $\nu_{A,B,C} := \overline{\mu}$.

Let now $f \colon A_S \to A'_S$ be a homomorphism of partial acts. Since

$$\begin{aligned}
((f \otimes (\mathrm{id}_B \otimes \mathrm{id}_C)) \circ \nu_{A,B,C})((a \otimes b) \otimes c) &= (f \otimes (\mathrm{id}_B \otimes \mathrm{id}_C))(a \otimes (b \otimes c)) \\
&= f(a) \otimes ((\mathrm{id}_B \otimes \mathrm{id}_C)(b \otimes c)) \\
&= f(a) \otimes (\mathrm{id}_B(b) \otimes \mathrm{id}_C(c)) \\
&= f(a) \otimes (b \otimes c)
\end{aligned}$$

and $(\nu_{A',B,C} \circ ((f \otimes \mathrm{id}_B) \otimes \mathrm{id}_C))((a \otimes b) \otimes c) = f(a) \otimes (b \otimes c)$, the diagram

$$\begin{tikzcd}
(A \otimes_S B) \otimes_R C \arrow[r, "\nu_{A,B,C}"] \arrow[d, "(f\otimes \mathrm{id}_B)\otimes \mathrm{id}_C"] & A \otimes_S (B \otimes_R C) \arrow[d, "f\otimes(\mathrm{id}_B \otimes \mathrm{id}_C)"] \\
(A' \otimes_S B) \otimes_R C \arrow[r, "\nu_{A',B,C}"] & A' \otimes_S (B \otimes_R C)
\end{tikzcd}$$

is commutative. Therefore $\nu_{A,B,C}$ is natural in the first variable. Similarly it is natural in the other variables. □

We can make tensor products $(A\otimes_S B)\otimes_R C$ and $A\otimes_S(B\otimes_R C)$ isomorphic as partial acts, if A_S and ${}_RC$ are also polite biacts.

Proposition 3.15. *Let ${}_TA_S$ be left polite partial biact, ${}_SB_R$ a polite partial biact and ${}_RC$ a partial left act. Then the mapping $\nu_{A,B,C}$ from Theorem 3.14 is an isomorphism of partial left T-acts.*

Proof. Let $t \in T$, $a \in A$, $b \in B$, $c \in C$ and assume that the product $t((a \otimes b) \otimes c)$ exists. Then the products ta and $t(a \otimes b) = (ta) \otimes b$ also exist. Since

$$t((a \otimes b) \otimes c) = (t(a \otimes b)) \otimes c = ((ta) \otimes b) \otimes c \in (A \otimes_S B) \otimes_R C,$$

there exists $t(a \otimes (b \otimes c))$ and

$$\begin{aligned}\nu_{A,B,C}(t((a \otimes b) \otimes c)) &= \nu_{A,B,C}(((ta) \otimes b) \otimes c) = (ta) \otimes (b \otimes c) \\ &= t(a \otimes (b \otimes c)) = t\nu_{A,B,C}((a \otimes b) \otimes c).\end{aligned}$$

Hence $\nu_{A,B,C}$ is an isomorphism of partial left T-acts. □

The following propositions can be proven similarly.

Proposition 3.16. *Let ${}_RC_T$ be a right polite partial biact, ${}_SB_R$ a polite partial biact and A_S a partial right act. Then the mapping $\nu_{A,B,C}$ from Theorem 3.14 is an isomorphism of partial right T-acts.*

Proposition 3.17. *Let ${}_TA_S$ be left polite partial biact, ${}_RC_T$ a right polite partial biact, ${}_SB_R$ a polite partial biact. Then the mapping $\nu_{A,B,C}$ from Theorem 3.14 is an isomorphism of partial (T,P)-biacts.*

4. Hom-tensor adjunction

In this section we will study the hom-functor of partial acts and prove that there exists an adjunction between certain tensor functors and hom-functors for categories of partial acts.

There exists a hom-functor

$$\mathrm{Hom}({}_RA_S, \_)\colon {}_R\mathsf{PAct} \to \mathsf{Set},$$

which is defined for every ${}_RB, {}_RB' \in \mathrm{Ob}({}_R\mathsf{PAct})$ by the diagram

$$\begin{tikzcd}
{}_RB \arrow[r, mapsto] \arrow[d, "f"] & \mathrm{Hom}({}_RA_S, {}_RB) \arrow[d, "f\circ__"] & \ni g \arrow[d, mapsto] & {}_RA_S \arrow[r, "g"] \arrow[dr, "f\circ g"] & {}_RB \arrow[d, "f"] \\
{}_RB' \arrow[r, mapsto] & \mathrm{Hom}({}_RA_S, {}_RB') & \ni f \circ g & & {}_RB'
\end{tikzcd} .$$

Proposition 4.1. *Let R and S be semigroups and let ${}_RA_S$ be a partial biact. Then* $\mathrm{Hom}({}_RA_S, \_\_)$ *is a covariant functor* ${}_R\mathsf{PAct} \to {}_S\mathsf{PAct}$.

Proof. Let ${}_RA_S \in \mathrm{Ob}({}_R\mathsf{PAct}_S)$, ${}_RB \in \mathrm{Ob}({}_R\mathsf{PAct})$ and $f \in \mathrm{Hom}({}_RA_S, {}_RB)$. Let $s \in S$ be such that for every $a \in A$ the product as exists. We define a partial left S-action on the set $\mathrm{Hom}({}_RA_S, {}_RB)$ by

$$(sf)(a) := f(as).$$

Now, let $a \in A$ and $r \in R$ be such that ra exists. Then $f(ra)$ and $(sf)(ra)$ also exist and we can compute $(sf)(ra) = f(ras) = rf(as) = r(sf)(a)$. This means that $sf \in \mathrm{Hom}({}_RA_S, {}_RB)$. To reiterate, the product $fs \in \mathrm{Hom}({}_RA_S, {}_RB)$ exists exactly for $s \in S$ such that the product as exists for every $a \in A$.

We show that this is indeed a partial left S-action. Let $s, s' \in S$ be such that homomorphisms $s'f$ and $s'(sf)$ exist. Then products as and as' exist for every $a \in A$. Fix $a \in A$. Then there exist the products as and $(as)s'$. Therefore, $a(ss')$ also exists and $a(ss') = (as)s'$. This implies that $(ss')f$ is defined and for every $a \in A$ we have

$$((ss')f)(a) = f(a(ss')) = f((as)s') = (s'f)(as) = (s(s'f))(a),$$

which means that $(ss')f = s(s'f)$.

Hence $\mathrm{Hom}({}_RA_S, {}_RB)$ is a partial left S-act and we have a well-defined hom-functor $\mathrm{Hom}({}_RA_S, \_\_)\colon {}_R\mathsf{PAct} \to {}_S\mathsf{PAct}$. $\square$

Note that every (R, S)-biact (in the usual sense) is clearly a polite partial (R, S)-biact. The symbol ${}_R\mathsf{Act}_S$ will denote the category of (usual) (R, S)-biacts.

Proposition 4.2. *Let R and S be semigroups. For every ${}_RA_S \in \mathrm{Ob}({}_R\mathsf{Act}_S)$, ${}_SB \in \mathrm{Ob}({}_S\mathsf{PAct})$ and ${}_RC \in \mathrm{Ob}({}_R\mathsf{PAct})$ the mapping*

$$\chi\colon \mathrm{Hom}({}_SB, {}_S(\mathrm{Hom}({}_RA_S, {}_RC))) \to \mathrm{Hom}({}_R(A \otimes_S B), {}_RC),$$
$$\chi(f)(a \otimes b) = f(b)(a)$$

is a bijection.

Proof. Let ${}_RA_S \in \mathrm{Ob}({}_R\mathsf{Act}_S)$, ${}_SB \in \mathrm{Ob}({}_S\mathsf{PAct})$ and ${}_RC \in \mathrm{Ob}({}_R\mathsf{PAct})$. First, we show that the mapping χ is well defined. Fix a homomorphism of partial left S-acts $f \in \operatorname{Hom}({}_SB, {}_S(\operatorname{Hom}({}_RA_S, {}_RC)))$. Let $\beta\colon A \times B \to C$ be a mapping defined by

$$\beta(a,b) := f(b)(a)$$

where $a \in A$, $b \in B$. We show that β is S-tensorial. Let $a \in A$, $b \in B$, $s \in S$ be such that the products as and sb exist. Since $f\colon {}_SB \to {}_S(\operatorname{Hom}({}_RA_S, {}_RC))$ is a homomorphism of partial left acts and ${}_RA_S$ is a biact, there exists the product $sf(b)$ and $sf(b) = f(sb)$. Now

$$\begin{aligned}
\beta(as,b) &= f(b)(as) && \text{(the definition of } \beta)\\
&= (sf(b))(a) && (S\text{-action of the partial act } \operatorname{Hom}({}_RA_S, {}_RC))\\
&= f(sb)(a) && (\text{equality } sf(b) = f(sb))\\
&= \beta(a, sb). && \text{(the definition of } \beta)
\end{aligned}$$

Hence there exists a unique mapping $\overline{\beta}\colon A \otimes_S B \to C$ such that $\beta = \overline{\beta} \circ \tau$, where τ is the canonical surjection. Note that $\chi(f) = \overline{\beta}$ and so $\chi(f)$ is well defined. Now let $r \in R$, $a \in A$, $b \in B$ be such that the product ra is exists. Due to

$$\begin{aligned}
\chi(f)(r(a \otimes b)) &= \chi(f)((ra) \otimes b) && (\text{action of the partial act } {}_R(A \otimes_S B))\\
&= f(b)(ra) && \text{(the definition of } \chi)\\
&= r(f(b)(a)) && (f(b) \in \operatorname{Hom}({}_RA_S, {}_RC))\\
&= r(\chi(f)(a \otimes b)), && \text{(the definition of } \chi)
\end{aligned}$$

we have $\chi(f) \in \operatorname{Hom}({}_R(A \otimes_S B), {}_RC)$. In conclusion, χ is well defined.

Now let $f_1, f_2 \in \operatorname{Hom}({}_SB, {}_S\operatorname{Hom}({}_RA_S, {}_RC)))$ be such that $\chi(f_1) = \chi(f_2)$, which means that $f_1(b)(a) = f_2(b)(a)$. The last equality holds for every $a \in A$ and every $b \in B$ and therefore $f_1 = f_2$. Hence χ is injective.

Let $g \in \operatorname{Hom}({}_R(A \otimes_S B), {}_RC)$ and $f\colon {}_SB \to {}_S(\operatorname{Hom}({}_RA_S, {}_RC))$ be a mapping defined by $f(b)(a) = g(a \otimes b)$ for every $a \in A$, $b \in B$. Let $r \in R$ be such that the product ra exists. As g is a homomorphism of partial left acts, we get

$$f(b)(ra) = g((ra) \otimes b) = g(r(a \otimes m)) = rg(a \otimes m) = rf(b)(a).$$

Hence $f(b)$ is a homomorphism of partial left R-acts, as needed.

Now assume that the product sb exists. For every $s \in S$ there exists the product as, because ${}_RA_S$ is a (usual) biact, and, hence, $sf(b)$ also exists and $(sf(b))(a) = f(b)(as)$ holds for every $a \in A$. Therefore

$$f(sb)(a) = g(a \otimes sb) = g(as \otimes b) = f(b)(as) = (sf(b))(a).$$

This means that f is a homomorphism and by the definition $\chi(f) = g$. Hence χ is surjective. □

Finally, we are ready to prove that the covariant functors $\operatorname{Hom}({}_RA_S, \_\_)\colon$ ${}_R\mathsf{PAct} \to {}_S\mathsf{PAct}$ and ${}_RA \otimes_S \_\_\colon {}_S\mathsf{PAct} \to {}_R\mathsf{PAct}$ are adjoint, where ${}_RA_S$ is a biact.

Theorem 4.3. *Let R and S be semigroups and ${}_RA_S$ a biact. Then the tensor functor ${}_RA\otimes_S\_\_\colon\ {}_S\mathsf{PAct}\to{}_R\mathsf{PAct}$ is a left adjoint of the hom-functor* $\mathrm{Hom}({}_RA_S,\_\_):{}_R\mathsf{PAct}\to{}_S\mathsf{PAct}$.

Proof. Let ${}_RA_S$ be a biact. We denote a family of homomorphisms

$$\xi\coloneqq(\xi_{B,C})_{\substack{B\in\mathrm{Ob}({}_S\mathsf{PAct})\\ C\in\mathrm{Ob}({}_R\mathsf{PAct})}}\colon\mathrm{Hom}({}_S\_\_,{}_S(\mathrm{Hom}({}_RA_S,\_\_)))\Rightarrow\mathrm{Hom}({}_R(A\otimes_S\_\_),{}_R\_\_),$$

where for every ${}_SB\in\mathrm{Ob}({}_S\mathsf{PAct})$ and ${}_RC\in\mathrm{Ob}({}_R\mathsf{PAct})$ we have $\xi_{B,C}\coloneqq\chi$ defined in Proposition 4.2. We know that every component of ξ is a bijection. We fix ${}_SB,{}_SD\in\mathrm{Ob}({}_S\mathsf{PAct})$, ${}_RC\in\mathrm{Ob}({}_R\mathsf{PAct})$, $f\colon{}_SB\to{}_SD$ and show that the diagram

$$\begin{tikzcd}
{}_SB \arrow[d, "f"] & \mathrm{Hom}({}_SB,{}_S(\mathrm{Hom}({}_RA_S,{}_RC))) \arrow[r, "\xi_{B,C}"] & \mathrm{Hom}({}_R(A\otimes_S B),{}_RC) \\
{}_SD & \mathrm{Hom}({}_SD,{}_S(\mathrm{Hom}({}_RA_S,{}_RC))) \arrow[u, "__\circ f"] \arrow[r, "\xi_{D,C}"] & \mathrm{Hom}({}_R(A\otimes_S D),{}_RC) \arrow[u, "__\circ(\mathrm{id}_A\otimes f)"']
\end{tikzcd}$$

commutes. Let $\varphi\in\mathrm{Hom}({}_SD,{}_S(\mathrm{Hom}({}_RA_S,{}_RC)))$ and $a\otimes b\in A\otimes_S B$. Now

$$\begin{aligned}(\xi_{B,C}\circ(\_\_\circ f))(\varphi)(a\otimes b)&=(\xi_{B,C}(\varphi\circ f))(a\otimes b)=(\chi(\varphi\circ f))(a\otimes b)\\&=(\varphi\circ f)(b)(a)=\varphi(f(b))(a)\end{aligned}$$

and

$$\begin{aligned}((\_\_\circ(\mathrm{id}_A\otimes f))\circ\xi_{D,C})(\varphi)(a\otimes b)&=(\_\_\circ(\mathrm{id}_A\otimes f))(\xi_{D,C}(\varphi))(a\otimes b)\\&=(\_\_\circ(\mathrm{id}_A\otimes f))(\chi'(\varphi))(a\otimes b)\\&=(\chi'(\varphi)\circ(\mathrm{id}_A\otimes f))(a\otimes b)\\&=\chi'(\varphi)((\mathrm{id}_A\otimes f)(a\otimes b))\\&=\chi'(\varphi)(a\otimes f(b))\\&=\varphi(f(b))(a),\end{aligned}$$

where χ' is the bijection from Proposition 4.2 for the partial acts ${}_RA_S$, ${}_SD$ and ${}_RC$. Therefore ξ is natural in the first variable.

Similarly, ξ is also natural in the second variable. In conclusion, ξ is a natural isomorphism, which proves the adjunction ${}_RA\otimes_S\_\_\dashv\mathrm{Hom}({}_RA_S,\_\_)$. □

Every semigroup can be viewed as a biact over itself. Therefore we can make the following corollary.

Corollary 4.4. *Let R be a semigroup. The tensor functor ${}_RR\otimes_R\_\_\colon$ ${}_R\mathsf{PAct}\to{}_R\mathsf{PAct}$ is a left adjoint of the hom-functor* $\mathrm{Hom}({}_RR_R,\_\_)\colon\ {}_R\mathsf{PAct}\to{}_R\mathsf{PAct}$.

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References

[1] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and Concrete Categories – The Joy of Cats*, University of Bremen, 2004, `http://katmat.math.uni-bremen.de/acc/acc.pdf`.

[2] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics **13**, Springer-Verlag, New York-Heidelberg, 1974, DOI: 10.1007/978-1-4612-4418-9

[3] I. M. Araújo, M. J. J. Branco, V. H. Fernandes, and G. M. S. Gomes, *Semigroups and Languages*, World Scientific Publishing Co Pte Ltd., Singapur, 2004. DOI: 10.1142/5584

[4] M. Delorme, *Sur la platitude des demi-groupes de fractions*, C. R. Acad. Sci., 1969.

[5] M. Dokuchaev, *Recent developments around partial actions*, São Paulo J. Math. Sci. **13** (2019), 195–247, DOI: 10.1007/s40863-018-0087-y.

[6] R. Exel, *Partial actions of groups and actions of inverse semigroups*, Proc. Amer. Math. Soc. **126** (1998), 3481–3494.

[7] C. Hollings, *Partial actions of monoids*, Semigroup Forum. **75** (2007), 293–316. DOI: 10.1007/s00233-006-0665-7

[8] J. Kellendonk and M. V. Lawson, *Partial actions of groups*, Internat. J. Algebra Comput. **14** (2004), 87–114, DOI: 10.1142/S0218196704001657.

[9] M. Kilp, *Flat polygons*, Tartu Riikl. Ül. Toimetised **253** (1970), 66–72. (Russian)

[10] M. Kilp, U. Knauer, and A. V. Mikhalev, *Monoids, Acts and Categories: With Applications to Wreath Products and Graphs. A Handbook for Students and Researchers*, De Gruyter, Berlin, New York, 2000.

[11] G. Kudryavtseva and V. Laan, *Globalization of partial actions of semigroups*, 2022, 15 pp., `https://arxiv.org/pdf/2206.06808.pdf`.

[12] M. Megrelishvili and L. Schröder, *Globalisation of confluent partial actions on topological and metric spaces*, Topology Appl. **145** (2004), 119–145, DOI: 10.1016/j.topol.2004.06.006.

[13] Bo Stenström, *Flatness and localization over monoids*, Math. Nachr. **48** (1971), 315–334, DOI: 10.1002/mana.19710480124.

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