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On a generalized class of analytic functions related to Bazilevič functions

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Abstract. Using operator $L_p(a,c)$ introduced by Saitoh (Math. Japon. 44 (1996), 31–38) we define the subclass $H^{\nu;\mu}_{p,n}(a,c;\phi)$ of the class $\mathcal{A}(p,n)$ and establish containment, subordination and coefficient inequalities of this subclass. We indicate the connections of our results with earlier results obtained by other researchers.

1. Introduction

Let $\mathcal{A}(p,n)$ be the class of analytic functions in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} \alpha_{p+k} z^{p+k},\ z \in \mathbb{U}, \quad p, n \in \mathbb{N} := \{1, 2, \dots\}. \tag{1}$$

If f and g are two analytic functions in $\mathbb{U}$, we say that f is subordinate to g, written symbolically $f(z) \prec g(z)$, if there exists a *Schwarz function* w which is (by definition) analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. If g is univalent, $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (cf. [3, p. 90]; see also [11]).
For two functions $f(z) = z^p + \sum\limits_{n=1}^{\infty} a_n z^{n+p}$ and $g(z) = z^p + \sum\limits_{n=1}^{\infty} b_n z^{n+p}$, the *Hardamard* (or convolution) product of f and g is defined by

$$(f * g)(z) := z^p + \sum_{n=1}^{\infty} a_n b_n z^{n+p},\ z \in \mathbb{U}.$$

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Let the function θ_p be the *incomplete Beta function* defined by

$$\theta_p(a,c;z) := z^p + \sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k},\ z \in \mathbb{U}, \tag{2}$$

where $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, where $\mathbb{Z}_0^- := \{0, -1, -2, \dots\}$, and the symbol $(x)_m$ represents the *shifted factorial* given by

$$(x)_m = \begin{cases} 1, & \text{if} \quad m = 0,\ x \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}, \\ x(x+1)\dots(x+m-1), & \text{if} \quad m \in \mathbb{N},\ x \in \mathbb{C}. \end{cases}$$

Using the function θ_p given by (2) and the convolution product, let us recall that the convolution operator $L_p(a,c) : \mathcal{A}(p,n) \to \mathcal{A}(p,n)$ is defined by

$$L_p(a,c)f(z) := \theta_p(a,c;z) * f(z),\ z \in \mathbb{U}. \tag{3}$$

The operator $L_p(a,c)$ defined on the class $\mathcal{A}(p)$ was studied by Saitoh in [17], which generalized the Carlson and Shaffer [1] to certain subclasses of starlike, convex and prestarlike hypergeometric functions.

Remark 1. Note that for $f \in \mathcal{A}(p,1)$ we have $L_p(a,a)f(z) = f(z)$. Some other results concerning this operator are given in [4, 16, 14, 9, 13].

If the function f is given by (1), then from (3) we get

$$G(z) := L_p(a,c)f(z) = z^p + \sum_{k=n}^{\infty} \frac{(a)_k}{(c)_k} \alpha_{p+k} z^{p+k},\ z \in \mathbb{U}, \quad a, c > 0, \tag{4}$$

and this function will be used to define the following subclass of $\mathcal{A}(p,n)$.

Definition 1. For $\nu,\ \mu \in \mathbb{C}$, with $\operatorname{Re}\nu > 0$, we define the classes

$$H_{p,n}^{\nu;\mu}(a,c;\phi) := \left\{ f \in \mathcal{A}(p,n) \colon \left(\frac{G(z)}{z^p}\right)^{\nu} \left[(1-\mu p) + \mu\, \frac{zG'(z)}{G(z)}\right] \prec \phi(z) \right\}, \tag{5}$$

where ϕ is a convex (univalent) function in $\mathbb{U}$, with $\phi(0) = 1$, and G is given by (4). (All the powers are understood as principal values, that is $\log 1 = 0$.)

Remark 2. Now we will prove that there exist values of parameters such that $H_{p,n}^{\nu;\mu}(a,c;\phi) \neq \emptyset$.

For example, if $p = 2$, $n = 1$, and $f(z) = z^2 + \lambda z^3 \in \mathcal{A}(2,1)$, then it is necessary to find a "small enough" $\lambda \in \mathbb{C}$, i.e. $|\lambda|$ is close to zero, such that the subordination

$$\left(\frac{G(z)}{z^p}\right)^{\nu} \left[(1 - \mu p) + \mu\, \frac{zG'(z)}{G(z)}\right] \prec \frac{1+z}{1-z} = \phi(z)$$

holds for some values of the parameters, where ϕ is a convex (univalent) function in $\mathbb{U}$, with $\phi(0) = 1$ and $\operatorname{Re}\phi(z) > 0$, $z \in \mathbb{U}$. Using the definition of subordination (see also [11], p. 21), as $\operatorname{Re}\phi(z) > 0$, the above subordination is equivalent to

$$\varphi(z) := \operatorname{Re}\left\{\left(\frac{G(z)}{z^p}\right)^{\nu}\left[(1-\mu p)+\mu\,\frac{zG'(z)}{G(z)}\right]\right\} > 0,\ z \in \mathbb{U},$$

and for the special values

$$\nu = 3, \quad \mu = 0.2, \quad a = 2, \quad c = 8, \quad \lambda = 0.3, \tag{6}$$

the image of the unit circle $\partial\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$ under the function φ is nonnegative, because

$$\varphi(z) > 0.7 > 0,\ z \in \mathbb{U},$$

(see Figure 1 made with MAPLE™ computer software). Thus, for the values

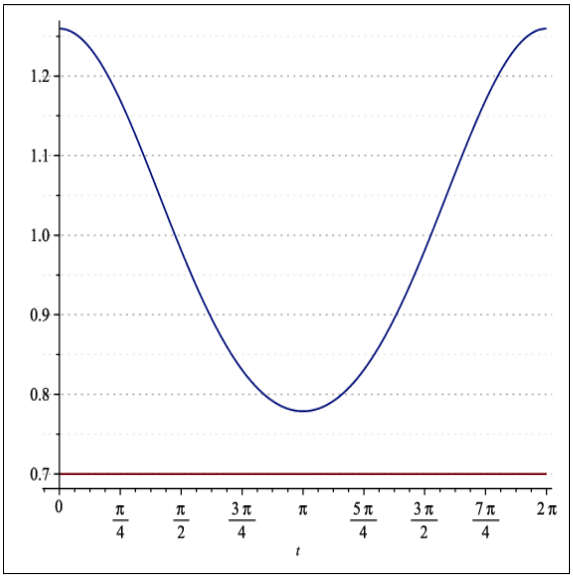

FIGURE 1. The graphics of $\varphi\left(e^{it}\right)$ for $t \in [0, 2\pi]$.

given by (6) we have $f(z) = z^2 + \lambda z^3 \in H_{p,n}^{\nu,\mu}(a, c; \phi)$, hence there exist values for the parameters such that $H_{p,n}^{\nu,\mu}(a, c; \phi) \neq \emptyset$.

Remark 3. Upon specifying different parameters and function ϕ we get different subclasses as follows.

1. Let us denote $H_{p,n}^{\nu,\mu}(a, c; \phi) =: H_{p,n}^{\nu,\mu}(a, c; A, B)$ for $\phi(z) = \dfrac{1 + Az}{1 + Bz}$ and $-1 \leq B < A \leq 1$. As a special case we mention that the subclass $H_{1,n}^{\nu,\mu}(a, a; 1 - 2\rho, -1)$ $(0 \leq \rho < 1)$ has been studied by Liu [8].

2. For $\mu p = 1$ we get the class

$$H_{p,n}^{\nu,1/p}(a, c; A, B) = \left\{f \in \mathcal{A}(p, n) : \frac{1}{p}\,\frac{zG'(z)}{G(z)}\left(\frac{G(z)}{z^p}\right)^{\nu} \prec \frac{1 + Az}{1 + Bz}\right\}.$$

(i) For the special case $A = 1 - 2\rho$, $B = -1$, $0 \leq \rho < 1$, the above class is denoted by $B^{\nu}_{p,n}(a, c; \rho)$ and consists of the functions $f \in \mathcal{A}(p, n)$ that satisfy the inequality

$$\operatorname{Re}\left\{\frac{1}{p}\frac{zG'(z)}{G(z)}\left(\frac{G(z)}{z^p}\right)^{\nu}\right\} > \rho,\ z \in \mathbb{U}.$$

(ii) The subclass

$$B^{\nu}(\rho) := B^{\nu}_{1,1}(a, a; \rho) = \left\{f \in \mathcal{A}(1, 1) : \operatorname{Re}\frac{z^{1-\nu} f'(z)}{f^{1-\nu}(z)} > \rho,\ z \in \mathbb{U}\right\},$$

with $\nu > 0$, $0 \leq \rho < 1$, was investigated by Owa [12] and also in [2] (for $\lambda = 0$). The class $B^{\nu}(0)$ is the subclass of Bazilevič function of type ν, studied by Singh [18] with respect to starlike function $g(z) = z$.

(iii) The subclass $B^{\nu}_{1,n}(a, a; \rho)$ with $0 \leq \rho < 1$ has been studied by Ling et al. [7].

Assume that a, b and c are complex numbers such that $c \notin \mathbb{Z}^{-}_{0}$. The function

$$\mathcal{F}(a, b; c; z) := {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},\ z \in \mathbb{U},$$

is called the *Gaussian hypergeometric function.* It is analytic in $\mathbb{U}$ and satisfies the hypergeometric differential equation

$$z(1 - z)w''(z) + [c - (a + b + 1)z]\, w'(z) - abw(z) = 0,\ z \in \mathbb{C}.$$

The Gaussian hypergeometric function has several remarkable properties, for example (see [5]):

$$\mathcal{F}(a, b; c; z) = (1 - z)^{c-a-b}\mathcal{F}(c - a, c - b, c; z), \tag{7}$$

$$\mathcal{F}(a, b; c; z) = (1 - z)^{-a}\mathcal{F}\left(a, c - b, c; \frac{z}{z - 1}\right). \tag{8}$$

If $\operatorname{Re} c > \operatorname{Re} b > 0$, then there is a probability measure on $[0, 1]$ given by

$$d\eta(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)}\, t^{b-1}(1 - t)^{c-b-1}\, dt, \tag{9}$$

such that

$$\mathcal{F}(a, b; c; z) = \int_0^1 (1 - tz)^{-a}\, d\eta(t). \tag{10}$$

In the ongoing work, subset relations, the upper and lower bound of $\operatorname{Re}\left(\frac{G(z)}{z^p}\right)^{\nu}$ for $f \in H^{\nu;\mu}_{p,n}(a, c; A, B)$ are obtained using some subordinations connected methods. The results we established are more general than those of the works of some earlier researchers and, moreover, we get some new outcomes.

2. Preliminary results

The following results of this section are essential to prove our main results.

Lemma 1 ([6], Theorem 1; [11], Theorem 3.1b)**.** *Let h be convex in $\mathbb{U}$, with $h(0) = a$, and $\gamma \in \mathbb{C}$, with $\gamma \neq 0$ and $\operatorname{Re}\gamma \geq 0$. If the function $F(z) = a + d_n z^n + d_{n+1} z^{n+1} + \ldots$ is analytic in $\mathbb{U}$ and*

$$F(z) + \frac{1}{\gamma} zF'(z) \prec h(z), \tag{11}$$

then

$$F(z) \prec q(z) := \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z t^{\frac{\gamma}{n}-1} h(t)\, dt \prec h(z).$$

The function q is convex and is the best dominant of the differential subordination (11)*.*

For the above lemma see also [10].

Lemma 2 ([3], Theorem 6.4(i); [15], Rogosinski's Theorem X)**.** *Let $f(z) = \sum\limits_{k=1}^{\infty} d_k z^k$ be analytic in $\mathbb{U}$, and $g(z) = \sum\limits_{k=1}^{\infty} b_k z^k$ be a convex function in $\mathbb{U}$, such that $f(z) \prec g(z)$.*

(*i*) *If g is convex, then $|d_k| \leq |g'(0)| = |b_1|$, for $k = 1, 2, \ldots$.*

(*ii*) *If g is starlike (starlike with respect to 0), then $|d_k| \leq k\,|g'(0)| = k|b_1|$, for $k = 2, 3, \ldots$.*

3. Main results

Our first result deals with subordination properties of the functions that belong to the class $H_{p,n}^{\nu;\mu}(a, c; \phi)$.

Theorem 1. *Suppose that $\operatorname{Re}\dfrac{\nu}{\mu} \geq 0$, where $\mu \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. If $f \in H_{p,n}^{\nu;\mu}(a, c; \phi)$, then*

$$\left(\frac{G(z)}{z^p}\right)^{\nu} \prec Q(z) := \frac{\nu}{n\mu\, z^{\frac{\nu}{n\mu}}} \int_0^z \phi(t) t^{\frac{\nu}{n\mu}-1}\, dt \prec \phi(z). \tag{12}$$

The function Q is convex and is the best dominant of the above differential subordination.

Proof. If $f \in H_{p,n}^{\nu;\mu}(a, c; \phi)$, then the left-hand side function from the subordination that appeared in (5) is an analytic function in $\mathbb{U}$. Therefore, the function L defined by

$$L(z) = \left(\frac{G(z)}{z^p}\right)^{\nu},\ z \in \mathbb{U}, \tag{13}$$

is analytic in $\mathbb{U}$, and from (4) it follows that

$$L(z) = \left(\frac{G(z)}{z^p}\right)^{\nu} = 1 + d_n z^n + \ldots,\ z \in \mathbb{U}.$$

Logarithmically differentiating (13), since $f \in H^{\nu,\mu}_{n,p}(a, c : \phi)$ we get

$$\left(\frac{G(z)}{z^p}\right)^{\nu} \left[(1 - \mu p) + \mu \frac{zG'(z)}{G(z)}\right] = L(z) + \frac{\mu}{\nu} zL'(z) \prec \phi(z),$$

and from this subordination, using Lemma 1 we get the desired result (12). □

For the case $\phi(z) = \dfrac{1 + Az}{1 + Bz}$ and $-1 \le B < A \le 1$ we could give the next result, where the middle term function becomes the Gaussian hypergeometric function.

Theorem 2. *Suppose that*

$$\operatorname{Re}\left(\frac{\nu}{\mu}\right) > 0,\ \textit{with}\ \nu,\ \mu \in \mathbb{C}, \tag{14}$$

and $f \in H^{\nu;\mu}_{p,n}(a, c; A, B)$.

(i) *The subordination*

$$\left(\frac{G(z)}{z^p}\right)^{\nu} \prec q(z) := 1 + \frac{\nu}{\nu + n\mu} \frac{(A-B)z}{1+Bz} \mathcal{F}\left(1, 1; \frac{\nu}{n\mu} + 2; \frac{Bz}{Bz + 1}\right) \prec \frac{1+Az}{1+Bz}$$

holds. The function q is convex and is the best dominant of the above differential subordination.

(ii) *The image of the unit disk $\mathbb{U}$ under the function $\left(\frac{G(z)}{z^p}\right)^{\nu}$ lies within the lines $\operatorname{Re} w = \rho_1$ and $\operatorname{Re} w = \rho_2$, where*

$$\rho_1 := 1 + \frac{\nu}{\nu + n\mu} \frac{B - A}{1 - B} \mathcal{F}\left(1, 1; \frac{\nu}{n\mu} + 2; \frac{B}{B - 1}\right),\ \textit{if} \quad -1 \le B < A \le 1,$$

and

$$\rho_2 := \begin{cases} 1 + \dfrac{\nu}{\nu + n\mu} \dfrac{A - B}{1 + B} \mathcal{F}\left(1, 1; \dfrac{\nu}{n\mu} + 2; \dfrac{B}{1 + B}\right), & \textit{if}\ \ -1 < B < A \le 1, \\ +\infty, & \textit{if}\ \ -1 = B < A \le 1. \end{cases}$$

The bounds ρ_1 and ρ_2 are the best possible.

Proof. If we take in Theorem 1 the function $\phi(z) = \dfrac{1 + Az}{1 + Bz}$, with $-1 \le B < A \le 1$, if $f \in H^{\nu;\mu}_{p,n}(a, c; A, B)$, then from (12) it follows that

$$\left(\frac{G(z)}{z^p}\right)^{\nu} \prec q(z) := \frac{\nu}{n\mu\, z^{\frac{\nu}{n\mu}}} \int_0^z t^{\frac{\nu}{n\mu} - 1} \frac{1 + At}{1 + Bt}\, dt \prec \frac{1 + Az}{1 + Bz}.$$

Using the variable change $t = zu$ in the above integral, we get

$$\begin{aligned} q(z) =& \frac{\nu}{n\mu\, z^{\frac{\nu}{n\mu}}} \int_0^z t^{\frac{\nu}{n\mu}-1} \frac{1+At}{1+Bt}\, dt \\ =& \frac{\nu}{n\mu} \int_0^1 u^{\frac{\nu}{n\mu}-1} \frac{1+Azu}{1+Bzu}\, du \qquad (15) \\ =& \frac{\nu}{n\mu} \int_0^1 u^{\frac{\nu}{n\mu}-1} \left[1 + \frac{(A-B)zu}{1+Bzu}\right] du \\ =& 1 + \frac{\nu}{\nu+n\mu}(A-B)z \int_0^1 (1-u(-Bz))^{-1}\, d\eta(u), \qquad (16) \end{aligned}$$

where $d\eta(u)$ is given by (9) and is a probability measure on $[0,1]$ for $a := 1$, $b = (\nu/n\mu) + 1$ and $c = b + 1$.

Since $\operatorname{Re} \dfrac{\nu}{\mu} > 0$, we have $\operatorname{Re} c > \operatorname{Re} b > 1 > 0$, and using the relations (7) and (8), according to (10) the above formula (16) leads to

$$q(z) = 1 + \frac{\nu}{\nu+n\mu}\,(A-B)z\,\mathcal{F}\left(1, \frac{\nu}{n\mu}+1; \frac{\nu}{n\mu}+2; -Bz\right).$$

Then, from (7) and (8) it follows that

$$\begin{aligned} q(z) =& 1 + \frac{\nu}{\nu+n\mu}\,\frac{(A-B)z}{1+Bz}\,\mathcal{F}\left(1, 1; \frac{\nu}{n\mu}+2; \frac{Bz}{1+Bz}\right) \qquad (17) \\ =& 1 + \frac{\frac{\nu}{n\mu}}{1+\frac{\nu}{n\mu}}\,\frac{(A-B)z}{1+Bz}\,\mathcal{F}\left(1, 1; \frac{\nu}{n\mu}+2; \frac{Bz}{1+Bz}\right). \end{aligned}$$

If we denote

$$Q_1(z) := \frac{\frac{\nu}{n\mu}}{1+\frac{\nu}{n\mu}}\,\frac{(A-B)z}{1+Bz}, \quad Q_2(z) := \mathcal{F}\left(1, 1; \frac{\nu}{n\mu}+2; \frac{Bz}{1+Bz}\right),$$

we have $q(z) = 1 + Q_1(z)Q_2(z)$, $z \in \mathbb{U}$. From (14) the functions Q_1 and Q_2 have real coefficients, thus the function q will have real coefficients. Therefore, $q(\overline{z}) = \overline{q(z)}$ for all $z \in \mathbb{U}$, hence the image of $\mathbb{U}$ under q, i.e. $q(\mathbb{U})$, is symmetric with respect to the real axis. Using the fact that q is a convex function in $\mathbb{U}$ and $q(\mathbb{U})$ is symmetric with respect to the real axis, it follows that

$$\begin{aligned} \mathrm{I} &:= \inf\left\{\operatorname{Re} q(z) : z \in \mathbb{U}\right\} = \min\left\{q(-1); q(1)\right\}, \\ \mathrm{S} &:= \sup\left\{\operatorname{Re} q(z) : z \in \mathbb{U}\right\} = \max\left\{q(-1); q(1)\right\}. \end{aligned}$$

To determine I and S, first we see that under the assumption (14) and using (15) we have

$$\operatorname{Re} q(z) = \frac{\nu}{n\mu} \int_0^1 u^{\frac{\nu}{n\mu}-1} \operatorname{Re} \frac{1+Azu}{1+Bzu}\, du.$$

Denoting

$$\Phi(\zeta) := \frac{1+A\zeta}{1+B\zeta},\ \zeta = zu \in \mathbb{U},$$

it could be easily checked that $\Phi(-1) < \Phi(1)$, whenever $-1 \leq B < A \leq 1$. Remark that in the case $B = -1$, the right hand-side of this inequality becomes $\Phi(1) = +\infty$.

From the above two formulas for I, S, and the inequality $\Phi(-1) < \Phi(1)$, using (17) it is easy to deduce the following facts:

If $-1 < B < A \leq 1$, then

$$\mathrm{I} = q(-1) = 1 + \frac{\nu}{\nu + n\mu}\,\frac{B-A}{1-B}\,\mathcal{F}\left(1, 1; \frac{\nu}{n\mu} + 2; \frac{B}{B-1}\right),$$

$$\mathrm{S} = q(-1) = 1 + \frac{\nu}{\nu + n\mu}\,\frac{A-B}{1+B}\,\mathcal{F}\left(1, 1; \frac{\nu}{n\mu} + 2; \frac{B}{1+B}\right).$$

If $-1 = B < A \leq 1$, then the only difference from the above relations will be that

$$\mathrm{S} = q(-1) = +\infty.$$

Therefore, assuming that assumption (14) holds, we finally obtain part (ii) of our theorem. $\square$

If we set $\mu = n = p = 1$, $a = c$, $A = 1 - 2\rho$ with $0 \leq \rho < 1$, and $B = -1$ in Theorem 2, then we get the following special case.

Corollary 1. *If $f \in B^{\nu}(\rho)$ with $\nu > 0$, and $0 \leq \rho < 1$, (see Remark 3, 2(ii)) then the following are true.*

(i) The subordination

$$\left(\frac{f(z)}{z}\right)^{\nu} \prec R(z) := 1 + \frac{\nu}{\nu+1}\,\frac{2(1-\rho)z}{1-z}\,\mathcal{F}\left(1, 1; \nu+2; \frac{z}{z-1}\right) \prec \frac{1+(1-2\rho)z}{1-z}$$

holds. The function R is convex and is the best dominant of the above differential subordination.

(ii) Moreover,

$$\mathrm{Re}\left(\frac{f(z)}{z}\right)^{\nu} > \varrho := 1 - \frac{\nu(1-\rho)}{\nu+1}\,\mathcal{F}\left(1, 1; \nu+2 : \frac{1}{2}\right) > \rho,\ z \in \mathbb{U}.$$

The bound ϱ is the best possible.

Remark 4. (i) If ψ is the logarithmical derivative of the function Γ, that is

$$\psi(z) := \frac{d}{dz}\big(\log \Gamma(z)\big) = \frac{\Gamma'(z)}{\Gamma(z)},$$

then it is well-known that (see [5])

$$\mathcal{F}\left(1, 1; \nu+2; \frac{1}{2}\right) = (\nu+1)\left[\psi\left(\frac{\nu+2}{2}\right) - \psi\left(\frac{\nu+1}{2}\right)\right],$$

$$\text{if} \quad \nu + 1 \notin \{-1, -2, -3, \dots\}.$$

Hence, part (ii) of the above corollary could be reformulated as follows.

If $f \in B^{\nu}(\rho)$ with $\nu > 0$, and $0 \le \rho < 1$, then

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\nu} > \widehat{\varrho} := 1 - \nu(1-\rho)\left[\psi\left(\frac{\nu+2}{2}\right) - \psi\left(\frac{\nu+1}{2}\right)\right] > \rho, \ z \in \mathbb{U},$$

and the bound $\widehat{\varrho}$ is the best possible.

(ii) Upon taking $\rho = 0$ in Corollary 1 we obtain Lemma 4 of [18].

Theorem 3. *Let* $\nu,\ \mu > 0$, *and* $f \in H^{\nu,\mu}_{p,n}(a,c;A,B)$ *with* $-1 \le B < A \le 1$.

(i) *If* $\mu p \ge 1$, *then*

$$\operatorname{Re}\left[\mu\frac{zG'(z)}{G(z)}\left(\frac{G(z)}{z^p}\right)^{\nu}\right] > \widehat{\rho}, \ z \in \mathbb{U},$$

where

$$\widehat{\rho} := \widehat{\rho}(\mu, p; A, B) = \mu p\,\frac{1-A}{1-B}.$$

(ii) *If* $\mu p < 1$ *and* $B \neq -1$, *then*

$$\operatorname{Re}\left[\mu\frac{zG'(z)}{G(z)}\left(\frac{G(z)}{z^p}\right)^{\nu}\right] > \widetilde{\rho}, \ z \in \mathbb{U},$$

where

$$\widetilde{\rho} := \widetilde{\rho}(\mu, p; A, B) = (\mu p - 1)\,\frac{1+A}{1+B} + \frac{1-A}{1-B}.$$

The bounds $\widehat{\rho}$ *and* $\widetilde{\rho}$ *are the best possible.*

Proof. If $f \in H^{\nu,\mu}_{n,p}(a,c;A,B)$, then from Theorem 2 (i) it follows that the sharp subordination

$$\left(\frac{G(z)}{z^p}\right)^{\nu} \prec \frac{1+Az}{1+Bz}$$

holds, hence

$$\operatorname{Re}\left(\frac{G(z)}{z^p}\right)^{\nu} > \frac{1-A}{1-B}, \ z \in \mathbb{U}. \tag{18}$$

According to the subordination from (5) we have

$$\operatorname{Re}\left[\mu\frac{zG'(z)}{G(z)}\left(\frac{G(z)}{z^p}\right)^{\nu}\right] > (\mu p - 1)\operatorname{Re}\left(\frac{G(z)}{z^p}\right)^{\nu} + \frac{1-A}{1-B}, \ z \in \mathbb{U}. \tag{19}$$

(i) Supposing that $\mu p \ge 1$, from (18) and the above inequality we obtain

$$\operatorname{Re}\left[\mu\frac{zG'(z)}{G(z)}\left(\frac{G(z)}{z^p}\right)^{\nu}\right] > (\mu p - 1)\frac{1-A}{1-B} + \frac{1-A}{1-B} = \mu p\,\frac{1-A}{1-B} = \widehat{\rho}, \ z \in \mathbb{U}.$$

(ii) Assuming that $\mu p < 1$ and $B \neq -1$, from Theorem 2 (i) we similarly get the sharp inequality

$$\operatorname{Re}\left(\frac{G(z)}{z^p}\right)^{\nu} < \frac{1+A}{1+B}, \ z \in \mathbb{U},$$

and form (19) it follows that

$$\operatorname{Re}\left[\mu\frac{zG'(z)}{G(z)}\left(\frac{G(z)}{z^p}\right)^{\nu}\right] > (\mu p-1)\frac{1+A}{1+B}+\frac{1-A}{1-B}=\widetilde{\rho},\ z\in\mathbb{U}.$$

□

Theorem 4. *Let* $\nu,\ \mu\in\mathbb{C}$, *with* $\operatorname{Re}\nu>0$, *and* $-1\leq B<A\leq 1$. *If* $f\in H_{p,n}^{\nu;\mu}(a,c;A,B)$ *is of the form* (1), *then*

$$|\alpha_{p+n}|\leq\frac{A-B}{|n\mu+\nu|}\,\frac{(c)_n}{(a)_n}. \tag{20}$$

Proof. Since $f\in H_{n,p}^{\mu,\nu}(a,c;A,B)$, we have

$$(1-\mu p)\left(\frac{G(z)}{z^p}\right)^{\nu}+\mu\frac{zG'(z)}{G(z)}\left(\frac{G(z)}{z^p}\right)^{\nu}=1+\frac{(a)_n}{(c)_n}\alpha_{p+n}z^n(n\mu+\nu)+\dots$$
$$\prec\frac{1+Az}{1+Bz}=:\phi(z). \tag{21}$$

Since the function ϕ of the subordination (21) is convex in $\mathbb{U}$, from Lemma 2 it follows that

$$\left|\frac{(a)_n}{(c)_n}\alpha_{p+n}(n\mu+\nu)\right|\leq|A-B|,$$

and taking into the account that $a,c>0$ the above inequality proves that the conclusion (20) holds. □

Remark 5. (i) For the particular case $\nu\geq 1$, $c=a$, and $n=\mu=p=1=A=-B$, the above theorem agrees with the result of Theorem 6 of [18] for $\alpha\geq 1$.
(ii) Upon taking $p=n=1, a=c, A=1-2\rho, B=-1$ in the above result we get an agreement with the result obtained in Theorem 3.9 of [2] (for $\lambda=0$).

Conclusions

The new subclass $H_{p,n}^{\nu;\mu}(a,c;\phi)$ of the class $\mathcal{A}(p,n)$ that we introduced using the $L_p(a,c)$ convolution operator of Saitoh [17] and the concept of subordination generalize many well-known classes defined and studied by several authors. The functions that belong to these classes satisfy the sharp subordination property given by Theorem 1. For the special choice of ϕ we got the sharp subordination and inequalities of Theorem 2, and the fact that the convolution product of these functions by the Saitoh operator is a Bazilevič function (see Theorem 3). A coefficient inequality for these function is given in the last result. We emphasize that for some particular cases of the parameters these results reduce to some previous ones found earlier in different papers.

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