ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 26, Number 2, December 2022
Available online at https://ojs.utlib.ee/index.php/ACUTM

Asymmetric response of inelastic circular plates to blast loading

Jaan Lellep and Annely Mürk

Abstract. The dynamic behaviour of clamped circular plates subjected to the concentrated blast loading is studied. The load is applied at a non-central point of the plate, non-axisymmetric deflections are taken into account. An approximate theoretical procedure developed earlier is applied for the evaluation of residual maximal deflections. The solution technique is based on the idea of equality of the power of the internal and external work, respectively. As it was shown earlier this concept leads to results which are close to exact ones in the case of axisymmetric loading of circular plates, also in the case of circular cylindrical shells.

1. Introduction

The response of beams and axisymmetric plates to quasi-static and dynamic loadings has been studied by many researchers making use of the concept of a rigid plastic body. The first papers on this topic were published by Hopkins and Prager [6]. In these studies the motion of ideal rigid-perfectly plastic plates simply supported at the edge is prescribed in the case of the material obeying the Tresca yield hexagon. Wang and Hopkins [27] extended the solution by Hopkins and Prager to the case of plates clamped at the edge and subjected to the impulsive loading. The dynamic plastic response of circular and annular plates due to the central pressure loading was examined by Florence [2] – [5]. Evidently, the distributions of stresses and strains in the plates subjected to the transverse pressure loading depend on the shape of the dynamic pressure pulse. The influence of the pressure pulse on the residual deflections of circular and annular plates was investigated by Youngdahl [30] and Krajcinovic [10, 11], also by Shen and Jones [25], Wen et al. [29], Ma [18], Mazalov and Nemirovski [19], Micallef et al. [20], Salupere [23].

Received August 31, 2022.
2020 *Mathematics Subject Classification.* 74K20.
Key words and phrases. Circular plate, plasticity, blast loading.
https://doi.org/10.12697/ACUTM.2022.26.21

The solutions to the problems of dynamic loading of axisymmetric plates accounting for the transverse shear effects are developed by Li and Jones [16], Chung Kim Yuen et al. [1], Nurick and Martin [21], Jones [6, 7], Lellep and Torn [15], Wierzbicki and Jones [28], Oliviera and Jones [22]. The case of a rectangular impulse distributed over a central region of the plate was investigated by Liu and Stronge [16]. In the mentioned papers circular and annular plates subjected to axisymmetric loadings are considered. In this case it is reasonable to expect that the plastic response to loading is also axisymmetric. Asymmetric loading of rigid-plastic circular plates is studied by Lellep and Mürk [12] – [14] in the cases of rectangular and impulsive pressure loadings. In the present paper the case of asymmetrical loading of inelastic circular plates is studied under the assumption that the load intensity is exponentially decaying in time. An approximate method suggested by Jones [7] is used for evaluation of maximal residual deflections.

2. Formulation of the problem

Let us consider the plastic response of an eccentrically loaded circular plate to the blast loading. It is assumed that the intensity of the external loading is defined as

$$P(r, \theta, t) = p(t) f(r, \theta), \tag{1}$$

where t stands for the time whereas r and θ are the polar coordinates (the polar radius and the polar angle, respectively). Here the function $f(r, \theta)$ is assumed to be a piecewise continuous function of polar coordinates and $p(t)$ is a decaying function of time. In the case of blast loading one can take

$$p(t) = \begin{cases} p_0 e^{-\beta t}, & t \in [0, t_1], \\ 0, & t > t_1, \end{cases} \tag{2}$$

where p_0 and β, also t_1 are given constants. The approximate method of mode form motions will be employed in the current paper. Thus, one can state that the deflection rate at each point of the plate is defined as

$$\dot{W}(r, \theta, t) = \dot{W}_0(t)\, \varphi(r, \theta), \tag{3}$$

where (see Lellep and Mürk [12] – [14])

$$\varphi(r, \theta) = 1 - \frac{r}{r_*(\theta)}. \tag{4}$$

In (4) the function $r_*(\theta)$ presents the right-hand part of the equation of the boundary of the plate $r = r_*(\theta)$. The plate under consideration has piecewise constant thickness

$$h = h_j,\ j = 1, \ldots, n \tag{5}$$

for $r \in (r_j, r_{j+1})$. In principle, $r_j = r_j(\theta)$ are given smooth functions of the angle θ. For the sake of simplicity, in the present study we confine our attention to the case where $r_j(\theta) = const$; $j = 1, \ldots, n$. The contour of the plate is denoted by $r_{n+1} = r_*$. As the contour of the plate is a circle of radius R one has

$$r_{n+1} = a\cos\theta + \sqrt{R^2 - a^2\sin^2\theta}, \tag{6}$$

where a stands for the distance between the centre of the plate O and the origin of the polar coordinates O_1. Evidently, in the cartesian coordinates the points lying at the boundary of the plate have the form:

$$\begin{aligned} x &= r_*\cos\theta, \\ y &= r_*\sin\theta. \end{aligned} \tag{7}$$

It is assumed that the plate is fully clamped at the contour, e.g. at $r = r_*$. The aim of the study is the determination of residual deflections of the circular plate of stepped thickness (5) subjected to the blast loading (2). In this paper we confine our attention to the case of the concentrated load applied asymmetrically.

3. Determination of residual deflections

It is assumed that the continuous yield line fan is formed by internal forces. The internal energy dissipation corresponding to (3), (4) can be calculated as (see Save et al. [24], Skrzypek and Hetnarski [26], Lellep and Mürk [12]–[14])

$$\dot{D}_i = \dot{W}_0 \sum_{j=0}^{n} M_{0j} \int_{r_j}^{r_{j+1}} \int_{0}^{2\pi} \frac{1}{r_*}\left(1 + \frac{r_*'^{\,2}}{r_*^2}\right) d\theta dr, \tag{8}$$

where M_{0j} is the yield moment for $h = h_{0j}$,

$$M_{0j} = \frac{\sigma_0 h_{0j}^2}{4}, \tag{9}$$

and σ_0 stands for the yield stress of the material. In (8), the following notation is used:

$$\begin{aligned} \dot{W}_0 &= \frac{\partial W_0}{\partial t}, \\ r_*' &= \frac{\partial r_*}{\partial \theta}. \end{aligned} \tag{10}$$

Making use of (6) one can define

$$r_*' = -a\sin\theta - \frac{a^2 \sin 2\theta}{2\sqrt{R^2 - a^2\sin^2\theta}}. \tag{11}$$

The external power (the power of inertial forces and of the loading (1)) is

$$\dot{D}_e = P\dot{W}_0 - \iint\limits_{(S)} \mu \dot{W}\ddot{W}\, dS, \tag{12}$$

where $-\mu\,\ddot{W}$ stands for the inertial force of the element dS, μ being the density of the material.

Following the approach presented by Jones [7] we assume that $\dot{D}_i = \dot{D}_e$. This equality together with (1)–(4) yields the result

$$\ddot{W}_0(t) = const, \tag{13}$$

provided $P(t) = const$. However, if $P = P(t)$, then $\ddot{W}_0 = \ddot{W}_0(t)$. The equation (13) can be easily integrated to give:

$$\dot{W}_0 = \ddot{W}_0 \cdot t + C_1 \tag{14}$$

and

$$W_0 = \frac{1}{2}\ddot{W}_0\, t^2 + C_1 t + C_2, \tag{15}$$

where C_1 and C_2 are arbitrary constants of integration. Making use of the initial conditions

$$\begin{aligned} \dot{W}_0(0) &= v_0, \\ W_0(0) &= w_{00}, \end{aligned} \tag{16}$$

in the particular case when $v_0 = 0$, $w_{00} = 0$ one obtains $C_1 = C_2 = 0$. Thus, according to (14)–(16) one has

$$\dot{W}_0(t) = \ddot{W}_0 \cdot t + v_0 \tag{17}$$

and

$$W_0(t) = \frac{1}{2}\ddot{W}_0\, t^2 + v_0 t + w_{00}. \tag{18}$$

Equalizing the internal power and the power of external forces according to (8), (12) one obtains

$$
\begin{aligned}
&\sum_{j=0}^{n} M_{0j} \int\limits_{r_j}^{r_{j+1}} \int\limits_{0}^{2\pi} \left(\frac{1}{r_*} + \frac{{r_*'}^2}{r_*^3} \right) d\theta dr = \\
&= P - \mu \ddot{W}_0 \sum_{j=0}^{n} \int\limits_{0}^{2\pi} \int\limits_{r_j}^{r_{j+1}} h_j \left(1 - \frac{r}{r_*} \right)^2 r dr d\theta - \int\limits_{0}^{2\pi} \frac{M_{0n}}{r_*} d\theta.
\end{aligned}
\tag{19}
$$

It easily follows from (19) that

$$
\ddot{W}_0 = \frac{P - \int\limits_0^{2\pi} \frac{M_{0n}}{r_*} d\theta - \sum\limits_{j=0}^{n} M_{0j} \int\limits_{r_j}^{r_{j+1}} \int\limits_{0}^{2\pi} \left(\frac{1}{r_*} + \frac{{r_*'}^2}{r_*^3} \right) d\theta dr}{\mu \sum\limits_{j=0}^{n} \int\limits_{r_j}^{r_{j+1}} \int\limits_{0}^{2\pi} h_j \left(1 - \frac{r}{r_*} \right)^2 r dr d\theta}. \tag{20}
$$

It is worthwhile to mention that the acceleration of the plate is defined by (20) during the active stage of loading. In the unloading phase the acceleration can be obtained from (20), taking $P = 0$. In this case according to (20), $\ddot{W}_0 < 0$, as might be expected. In the unloading phase $\ddot{W}_0(t) = const$. In the case of the exponential loading (1), (2) it is reasonable to present (20) as

$$
\ddot{W}_0 = \frac{1}{A} \left(\dot{P}_0 e^{-\beta t} - B \right), \tag{21}
$$

where

$$
A = \mu \sum_{j=0}^{n} \int\limits_{0}^{2\pi} \int\limits_{r_j}^{r_{j+1}} h_j \left(1 - \frac{r}{r_*} \right)^2 r dr d\theta \tag{22}
$$

and

$$
B = \int\limits_{0}^{2\pi} \frac{M_{0n}}{r_*} d\theta - \sum_{j=0}^{n} M_{0j} \int\limits_{r_j}^{r_{j+1}} \int\limits_{0}^{2\pi} \left(\frac{1}{r_*} + \frac{{r_*'}^2}{r_*^3} \right) d\theta dr. \tag{23}
$$

Integration with respect to time in (21) yields

$$\dot{W}_0 = -\frac{P_0 e^{-\beta t}}{A\beta} - \frac{B}{A}t + C_1,$$
$$W_0 = \frac{P_0 e^{-\beta t}}{A\beta^2} - \frac{B}{2A}t^2 + C_1 t + C_2, \tag{24}$$

where C_1, C_2 stand for arbitrary constants. The relations (24) corresponding to the initial conditions (16) with $v_0 = 0$, $w_{00} = 0$ have the forms

$$\dot{W}_0 = \frac{P_0}{A\beta}\left(1 - e^{-\beta t}\right) - \frac{B}{A}t \tag{25}$$

and

$$W_0 = \frac{P_0}{A\beta^2}\left(e^{-\beta t} - 1 + \beta t\right) - \frac{B}{2A}t^2. \tag{26}$$

Evidently, (25) and (26) hold for the first phase for $t \in [0, t_0]$, provided $t_0 \leq t_1$. The case $t_1 < t_0$ will be studied later. During the second stage of motion $p(t) = 0$; $t > t_1$. For $t > t_1$, the relations (25), (26) are not valid. The acceleration for this stage of motion can be obtained from (20) taking $P = 0$. Thus, the subsequent motion takes place with the acceleration

$$\ddot{W}_0 = \frac{-\int\limits_0^{2\pi} \frac{M_{0n}}{r_*} d\theta - \sum\limits_{j=0}^{n} M_{0j} \int\limits_{r_j}^{r_{j+1}} \int\limits_0^{2\pi} \frac{1}{r_*}\left(1 + \frac{r_*'^{\,2}}{r_*^2}\right) d\theta dr}{\mu \sum\limits_{j=0}^{n} \int\limits_{r_j}^{r_{j+1}} \int\limits_0^{2\pi} h_j \left(1 - \frac{r}{r_*}\right)^2 r dr d\theta}. \tag{27}$$

The motion ceases at the time instant t_2 when the velocity vanishes. Thus

$$\dot{W}_0(t_2) = 0. \tag{28}$$

Evidently

$$\dot{W}_0 = \ddot{W}_0\left(t - t_1\right) + \dot{W}_0\left(t_1\right), \tag{29}$$

where $\dot{W}_0(t_1)$ is calculated with the help of (25) taking $t = t_1$. In the similar way one can define

$$W_0 = \frac{\ddot{W}_0}{2}\left(t - t_1\right)^2 + \dot{W}_0(t_1)\left(t - t_1\right) + W_0(t_1), \tag{30}$$

where $\dot{W}_0(t_1)$ is defined by (26) at $t = t_1$. Making use of (28), (29) one can define the terminal time of motion

$$t_2 = t_1 - \frac{1}{A\ddot{W}_0}\left(\frac{P_0}{\beta}\left(1 - e^{-\beta t_1}\right) - Bt_1\right) \tag{31}$$

where the acceleration $\ddot{W}_0$ is defined by (27). The maximal permanent deflection can be calculated from (30), (31) as

$$\begin{aligned} W_1 = & \frac{P_0}{A\beta^2}\left(e^{-\beta t_1} - 1 + \beta t_1\right) - \frac{B}{2A}t_1^2 \\ & + \frac{3\left(\frac{P_0}{\beta}\left(1 - e^{-\beta t_1}\right) - Bt_1\right)^2}{A\left(P_0 e^{-\beta t_1} - B\right)}. \end{aligned} \tag{32}$$

4. Numerical results

The residual deflections at the final instant t_2 are calculated numerically after the determination of quantities A and B. The results of calculations are presented in Figures 1 − 4 for the case of the plate subjected to the concentrated force applied at the distance $a = 0.5R$ and $a = 0.1R$ from the centre of the plate.

The permanent maximal deflections in Figures 1 − 4 are depicted for $\beta = 0.2$. It is assumed that

$$h = \begin{cases} h_0, & r \in [0, r_1], \\ h_1, & r \in [r_1, r_*], \end{cases}$$

where $r_1 = \mathit{const}$ and r_* is defined by (6).

In Figure 1 the permanent deflections of circular plates of constant thickness are presented.

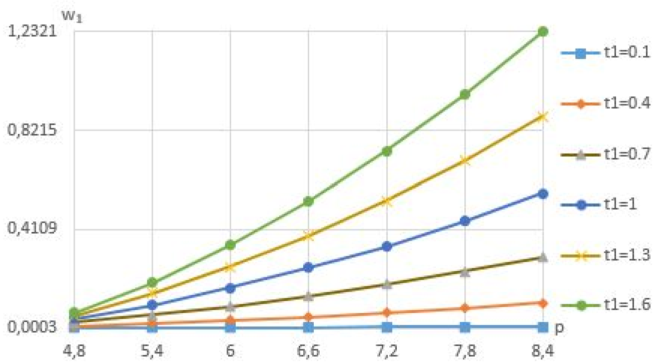

FIGURE 1. Maximal permanent deflection of the plate of constant thickness.

The curves depicted in Figure 1 correspond to the loading times $t_1 = 0.1, \ldots, t_1 = 1.6$. Here $\beta = 0.2$, $a = 0.1R$. It can be seen from Figure 1 that the larger is the load intensity the larger is the residual permanent deflection as might be expected. The maximal permanent deflections of the plate versus the duration of the loading phase are depicted in Figure 2 for different values of the duration of the external loading.

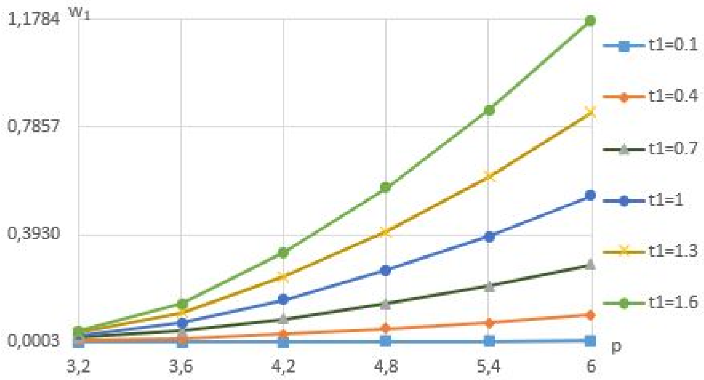

FIGURE 2. Maximal permanent deflection of the stepped plate.

The curves presented in Figure 2 correspond to the loading duration $t_1 = 0.1, \ldots, t_1 = 1.6$. Here $a = 0.1R$ and $h_1 = 0.8h_0$. It can be seen from Figure 2 that the longer is the loading phase the larger are residual deflections. In Figure 3 and Figure 4 the maximal residual deflections are presented in the cases where $a = 0.5R$, $r = 0.2R$ and $a = 0.1R$, $r = 0.5R$, respectively. Here, as previously $\beta = 0.2$ and $h_1 = 0.8h_0$.

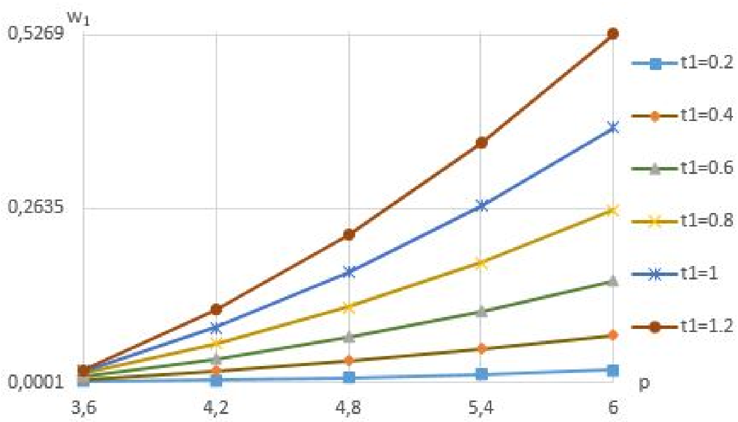

FIGURE 3. Maximal permanent deflection of the stepped plate.

Comparing the residual deflections depicted in Figure 2 and Figure 4 one can see that the larger is the mass of the plate the smaller is the residual deflection. Maximal permanent deflections of simply supported and fully

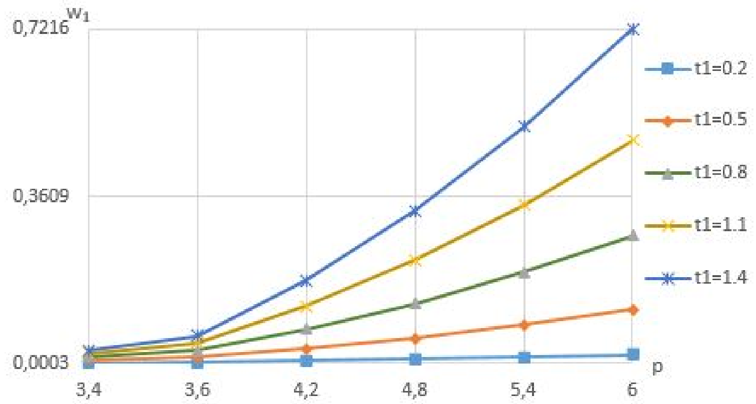

FIGURE 4. Maximal permanent deflection of the stepped plate.

clamped circular plates are shown in Figure 5 for different values of the length of the loading period.

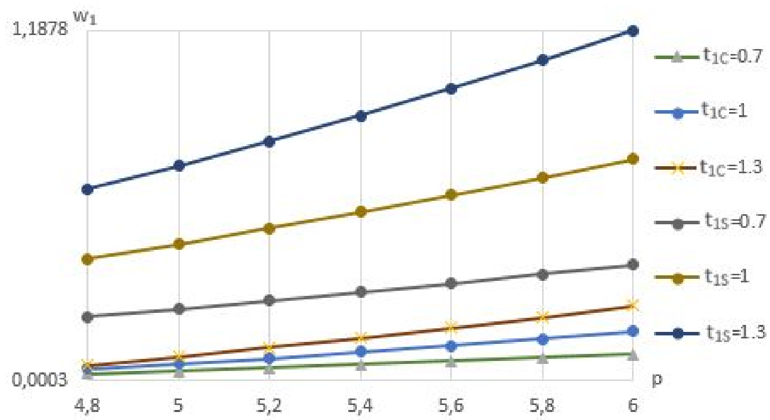

FIGURE 5. Maximal permanent deflections of simply supported (t_{1S}) and clamped plates (t_{1C}).

It can be seen from Figure 5 that the permanent deflections of clamped plates are smaller than those corresponding to simply supported plates calculated by the authors in [13].

5. Concluding remarks

An approximate theoretical method developed earlier is applied to the clamped circular plate subjected to the asymmetrically loaded circular plate clamped at the edge. Numerical results are obtained for fully clamped circular plates of piecewise constant thickness subjected to the concentrated loading whose intensity is exponentially decaying. It is shown that the maximal residual displacements depend on the time of the active loading and directly on the intensity of the transverse loading. In this sense the similarity of the current case and the case of rectangular impulse can be observed. Naturally, the relationship between the length of the loading stage and the maximal permanent deflection is not a linear one. Nevertheless, the higher is the load level the larger is the maximal permanent deflection, as might be expected.

References

[1] S. Chung Kim Yuen, G. N. Nurick, G. S. Langdon, and Y. Iyer, *Deformation of thin plates subjected to impulsive load: Part III – an update 25 years on*, Int. J. Impact Eng. **107** (2017), 108–117.

[2] A. L. Florence, *Clamped circular rigid-plastic plates under blast loading*, ASME, J. Appl. Mech. **33** (1966), 256–260.

[3] A. L. Florence, *Clamped circular rigid-plastic plates under central blast loading*, Int. J. Solids Struct. **2** (1966), 319–335.

[4] A. L. Florence, *Response of circular plates to central pulse loading*, Int. J. Solids Struct. **13** (1977), 1091–1102.

[5] A. L. Florence, *Annular plate under a transverse line impulse*, AIAA J. **3** (1965), 1729–1732.

[6] H. G. Hopkins and W. Prager, *On the dynamics of plastic circular plates*, Z. Angew. Math. Phys. **5** (1954), 317–330.

[7] N. Jones, *Structural Impact*, Cambridge Uniersity Press, Cambridge, 1989.

[8] N. Jones and J. G. Oliveira, *Dynamic plastic response of circular plates with transverse shear and rotatory inertia*, ASME, J. Appl. Mech. **47** (1980), 27–34.

[9] K. L. Komarov and Y. V. Nemirovski, *Dynamics of Rigid-Plastic Elements of Structures*, Nauka, Novosibirsk, 1984. (Russian)

[10] D. Krajcinovic, *Dynamic analysis of clamped plastic circular plates*, Int. J. Mech. Sci. **14** (1972), 225–234.

[11] D. Krajcinovic, *On approximate solutions for rigid-plastic structures subjected to dynamic loeading*, Int. J. Non-Linear Mech. **7** (1972), 571–575.

[12] J. Lellep and A. Mürk, *Asymmetric dynamic plastic behaviour of circular plates*, Proc. OAS 2013, Tartu, (2013), 70–75.

[13] J. Lellep and A. Mürk, *Asymmetric blast loading of inelastic circular plates*, Proc. OAS 2015, Tartu, (2015), 59–64.

[14] J. Lellep and A. Mürk, *Inelastic response of axisymmetric plates with cracks*, Int. Mech. Mater. Design **3** (2006), 237–251.

[15] J. Lellep and K. Torn, *Dynamic plastic behaviour of annular plates with shear effects*, Int. J. Impact Eng. **34** (2007), 1061–1080.

[16] Q. M. Li and N. Jones, *Blast loading of fully clamped circular plates with transverse shear effects*, Int. J. Solids Struct. **31** (1994), 1861–1876.

[17] D. Liu and W. Stronge, *Deformation of simply supported circular plate by central pressure pulse*, Int. J. Solids Struct. **33** (1996), 283–299.

[18] G. Ma, S. Iwaski, Y. Miyamoto, and H. Deto, *Dynamic plastic behaviour of circular plates using unified yield criterion*, Int. J. Solids Struct. **36** (1999), 3257–3275.

[19] V. N. Mazalov and Y. V. Nemirovski, *Dynamical loading of rigid-plastic annular plates*, Int. J. Non-Linear Mech. **11** (1976), 25–39.

[20] K. Micallef, A. S. Fallah, D. J. Pope, and L. A. Louca, The dynamic performance of simply-supported rigid-plastic circular steel plates subjected to localised blast loading, Int. J. Mech. Sci. **65** (2012), 177–191.

[21] G. N. Nurick and J. B. Martin, *Deformation of thin plates subjected to impulsive loading – a review*, Int. J. Impact Eng. **8** (1989), 159–170.

[22] J. G. Oliveira and N. Jones, *Some remarks on the influence of transverse shear on the plastic yielding of structures*, Int. J. Mech. Sci. **20** (1978), 759–765.

[23] A. Salupere, *Optimal design of dynamically loaded rigid-plastic stepped circular plates.*, Acta Comment. Univ. Tartu, **939** (1992), 20–35.

[24] M. A. Save, C. E. Massonnet, and G. D. Saxce, *Plastic Limit Analysis of Plates, Shells and Disks*, Elsevier, Amsterdam, 1997.

[25] W. Q. Shen and N. Jones, *Dynamic response and failure of fully clamped circular plates under impulsive loading*, Int. J. Impact Eng. **13** (1993), 259–278.

[26] J. Skrzypek and R. B. Hetnarski, *Plasticity and Creep: Theory, Examples and Problems*, CRC Press, Boca Raton, 1993.

[27] A. J. Wang and H. G. Hopkins, *On the plastic deformation of built-in circular plates under impulsive load*, J. Mech. Phys. Solids **13** (1954), 22–37.

[28] T. Wierzbicki and N. Jones, *Structural Failure*, Wiley-Interscieance, Hoboken, 1989.

[29] H. M. Wen, T. X. Yu, and J. N. Reddy, *A note on clamped circular plates under impulsive loading*, Mech. Stuct. Mach. **23** (1995), 331–342.

[30] C. K. Youngdahl, *Influence of pulse shape on the final plastic deformation of a circular plate*, Int. J. Solids Stuct. **7** (1971), 1127–1142.

Institute of Mathematics and Statistics, University of Tartu, Estonia
E-mail address: jaan.lellep@ut.ee
E-mail address: annely.murk@ut.ee