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Specht's ratio and logarithmic mean in time scale dynamic inequalities and their retrospective variants

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Abstract. In this research article, we investigate reverse Radon's inequality, reverse Bergström's inequality, the reverse weighted power mean inequality, reverse Schlömilch's inequality, reverse Bernoulli's inequality, and reverse Lyapunov's inequality with Specht's ratio on time scales. We also present reverse Rogers–Hölder's inequality with logarithmic mean and Specht's ratio on time scales. The time scale dynamic inequalities unify and extend some continuous inequalities and their corresponding discrete and quantum versions.

1. Introduction

Specht's ratio [7, 17] is defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}},$$

where $h > 0$, $h \neq 1$.

Here, we present some properties of Specht's ratio (see [7, 17, 18] for the proofs and details):

(i) $S(1) = 1$ and $S(h) = S\left(\frac{1}{h}\right) > 1$ for all $h > 0$;
(ii) $S(h)$ is a monotone increasing function on $(1, \infty)$ and monotone decreasing function on $(0, 1)$.

The logarithmic mean $L(m, M)$ (cf. [10]) is defined by

$$L(m, M) = \frac{M - m}{\log M - \log m},$$

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where m and M are real numbers with $0 < m < M$.

We state here some reverse integral inequalities (cf. [19]) with Specht's ratio.

Let $f(x)$ and $g(x)$ be positive and continuous functions. If $\beta > 0$, then

$$\frac{\left(\int_a^b S\left(\frac{\Omega f^{\beta+1}(x)}{\Lambda g^{\beta+1}(x)}\right) f(x)dx\right)^{\beta+1}}{\left(\int_a^b g(x)dx\right)^{\beta}} \geq \int_a^b \frac{f^{\beta+1}(x)}{g^{\beta}(x)}dx, \tag{1}$$

where

$$\Lambda = \int_a^b \frac{f^{\beta+1}(x)}{g^{\beta}(x)}dx \text{ and } \Omega = \int_a^b g(x)dx.$$

Inequality (1) is called reverse Radon's integral inequality.

Let $f(x)$ and $w(x)$ be positive continuous functions with $\int_a^b w(x)dx = 1$. If $0 < \delta_1 < \delta_2$, then

$$\left(\int_a^b S\left(\frac{f^{\delta_2}(x)}{\Lambda}\right) w(x)f^{\delta_1}(x)dx\right)^{\frac{1}{\delta_1}} \geq \left(\int_a^b w(x)f^{\delta_2}(x)dx\right)^{\frac{1}{\delta_2}}, \tag{2}$$

where

$$\Lambda = \int_a^b w(x)f^{\delta_2}(x)dx.$$

We will prove the results given in (1) and (2) on time scales. The calculus of time scales was initiated by Stefan Hilger, as given in [9]. A time scale is an arbitrary nonempty closed subset of the real numbers. The theory of time scales is applied to combine results in one comprehensive form. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The time scales calculus is studied as delta calculus, nabla calculus, and diamond-α calculus. This hybrid theory is also widely applied on dynamic inequalities, see [1, 6, 12, 13, 14, 15]. The basic work on dynamic inequalities is done by Ravi Agarwal, George Anastassiou, Martin Bohner, Allan Peterson, Donal O'Regan, Samir Saker, and many other authors.

In this paper, it is assumed that all integrals considered exist and are finite, $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$, and $[a, b]_{\mathbb{T}}$ denotes the intersection of the real interval $[a, b]$ with the time scale $\mathbb{T}$.

2. Preliminaries

We need the basic concepts of delta calculus here. The results of delta calculus are adopted from monographs [4, 5].

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the forward graininess function. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the backward graininess function. If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$, then we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^Δ is defined as follows.

Let $t \in \mathbb{T}^k$. If there exists $f^\Delta(t) \in \mathbb{R}$ such that, for all $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|$$

for all $s \in U$, then f is said to be delta differentiable at t, and $f^\Delta(t)$ is called the delta derivative of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous), if it is continuous at each right-dense point and there exists a finite left-sided limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [4, 5].

Definition 1. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [2, 4, 5].

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Further, $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$.

A function $f : \mathbb{T}_k \to \mathbb{R}$ is called nabla differentiable at $t \in \mathbb{T}_k$, with nabla derivative $f^\nabla(t)$, if there exists $f^\nabla(t) \in \mathbb{R}$ such that, given any $\epsilon > 0$, there is a neighborhood V of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \le \epsilon|\rho(t) - s|$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be left-dense continuous (ld-continuous), provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [2, 4, 5].

Definition 2. A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Now we present a short introduction to the diamond-α derivative as given in [1, 16].

Definition 3. Let $\mathbb{T}$ be a time scale and let $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}$, the diamond-α dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \quad 0 \le \alpha \le 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

The diamond-α derivative reduces to the standard Δ-derivative for $\alpha = 1$, or the standard ∇-derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0, 1)$.

Theorem 1 (see [16]). *Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$. Write $f^{\sigma}(t) = f(\sigma(t))$, $g^{\sigma}(t) = g(\sigma(t))$, $f^{\rho}(t) = f(\rho(t))$, and $g^{\rho}(t) = g(\rho(t))$. Then*

(i) *$f \pm g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with*
$$(f \pm g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) \pm g^{\diamond_\alpha}(t);$$
(ii) *$fg : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with*
$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1-\alpha) f^{\rho}(t) g^{\nabla}(t);$$
(iii) *if $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, then $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with*

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{f^{\diamond_\alpha}(t)g^{\sigma}(t)g^{\rho}(t) - \alpha f^{\sigma}(t)g^{\rho}(t)g^{\Delta}(t) - (1-\alpha) f^{\rho}(t)g^{\sigma}(t)g^{\nabla}(t)}{g(t)g^{\sigma}(t)g^{\rho}(t)}.$$

Definition 4 (see [16]). Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha) \int_a^t h(s)\nabla s, \quad 0 \le \alpha \le 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 2 (see [16]). *Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\diamond_\alpha$-integrable functions on $[a, b]_{\mathbb{T}}$. Then*

(i) $\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s$;
(ii) $\int_a^t c f(s) \diamond_\alpha s = c \int_a^t f(s) \diamond_\alpha s$;
(iii) $\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s$;

(iv) $\int_a^t f(s) \diamond_\alpha s = \int_a^b f(s) \diamond_\alpha s + \int_b^t f(s) \diamond_\alpha s$;
(v) $\int_a^a f(s) \diamond_\alpha s = 0$.

We need the following results.

The famous reverse Young's inequality (cf. [18]) with Specht's ratio can be written as

$$S\left(\frac{a}{b}\right) a^{\frac{1}{p}} b^{\frac{1}{q}} \geq \frac{a}{p} + \frac{b}{q} \tag{3}$$

for positive numbers a, b, and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$.

Now we present reverse Rogers–Hölder's inequality (cf. [6]) with Specht's ratio on time scales in the following result.

Theorem 3. *Let $a, b \in \mathbb{T}_k^k$ and $w, f, g \in C\left([a,b]_{\mathbb{T}}, \mathbb{R}_0^+ - \{0\}\right)$ such that f^p and g^q are $\diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$. If $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then*

$$\int_a^b S\left(\frac{\Omega f^p(x)}{\Lambda g^q(x)}\right) w(x) f(x) g(x) \diamond_\alpha x \geq \left(\int_a^b w(x) f^p(x) \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b w(x) g^q(x) \diamond_\alpha x\right)^{\frac{1}{q}}, \tag{4}$$

where

$$\Lambda = \int_a^b w(x) f^p(x) \diamond_\alpha x \text{ and } \Omega = \int_a^b w(x) g^q(x) \diamond_\alpha x,$$

and $S(\cdot)$ is Specht's ratio.

For $a, b > 0$, the logarithmic mean and Specht's ratio are used in a converse difference inequality of Young's inequality (cf. [18]) as

$$L(a,b) \log S\left(\frac{a}{b}\right) \geq \frac{1}{p} a + \frac{1}{q} b - a^{\frac{1}{p}} b^{\frac{1}{q}} \tag{5}$$

holds for $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$.

3. Main results

In order to present our main results, we first give the following extension of reverse Radon's inequality with Specht's ratio on time scales.

Theorem 4. *Let $w, f, g \in C\left([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\}\right)$ be $\diamond_\alpha$-integrable functions. If $\beta > 0$ and $\gamma \geq 1$, then*

$$\frac{\left(\int_a^b S\left(\frac{\Omega |f(x)|^{\beta+\gamma}}{\Lambda |g(x)|^{\beta+\gamma}}\right) |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x\right)^{\beta+\gamma-1}} \geq \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x, \tag{6}$$

where $\Lambda = \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x$ *and* $\Omega = \int_a^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x$*.*

Proof. Let $p = \beta + \gamma$ and $q = \frac{\beta+\gamma}{\beta+\gamma-1}$. Then, similar to inequality (4), we have that

$$\int_a^b S\left(\frac{\Omega_1|f(x)|^{\beta+\gamma}}{\Lambda_1|g(x)|^{\frac{\beta+\gamma}{\beta+\gamma-1}}}\right)|w(x)||f(x)g(x)| \diamond_\alpha x$$
$$\geq \left(\int_a^b |w(x)||f(x)|^{\beta+\gamma} \diamond_\alpha x\right)^{\frac{1}{\beta+\gamma}} \left(\int_a^b |w(x)||g(x)|^{\frac{\beta+\gamma}{\beta+\gamma-1}} \diamond_\alpha x\right)^{\frac{\beta+\gamma-1}{\beta+\gamma}}, \quad (7)$$

where $\Lambda_1 = \int_a^b |w(x)||f(x)|^{\beta+\gamma} \diamond_\alpha x$ and $\Omega_1 = \int_a^b |w(x)||g(x)|^{\frac{\beta+\gamma}{\beta+\gamma-1}} \diamond_\alpha x$. Replacing $|f(x)|$ and $|g(x)|$ by $F(x)$ and $G(x)$, respectively, and letting $F(x) = \frac{|f(x)|}{|g(x)|^{\frac{\beta+\gamma-1}{\beta+\gamma}}}$ and $G(x) = |g(x)|^{\frac{\beta+\gamma-1}{\beta+\gamma}}$ in (7), we have

$$\int_a^b S\left(\frac{\Omega_2|f(x)|^{\beta+\gamma}}{\Lambda_2|g(x)|^{\beta+\gamma}}\right)|w(x)||f(x)| \diamond_\alpha x$$
$$\geq \left(\int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta+\gamma-1}} \diamond_\alpha x\right)^{\frac{1}{\beta+\gamma}} \left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\frac{\beta+\gamma-1}{\beta+\gamma}}, \quad (8)$$

where $\Lambda_2 = \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta+\gamma-1}} \diamond_\alpha x$ and $\Omega_2 = \int_a^b |w(x)||g(x)| \diamond_\alpha x$. Taking both sides of inequality (8) to the power $\beta + \gamma$, we get

$$\left(\int_a^b S\left(\frac{\Omega_2|f(x)|^{\beta+\gamma}}{\Lambda_2|g(x)|^{\beta+\gamma}}\right)|w(x)||f(x)| \diamond_\alpha x\right)^{\beta+\gamma}$$
$$\geq \left(\int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta+\gamma-1}} \diamond_\alpha x\right)\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\beta+\gamma-1}. \quad (9)$$

Replacing $|w(x)|$ by $|w(x)||g(x)|^{\gamma-1}$ in inequality (9), we get

$$\left(\int_a^b S\left(\frac{\Omega|f(x)|^{\beta+\gamma}}{\Lambda|g(x)|^{\beta+\gamma}}\right)|w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x\right)^{\beta+\gamma}$$
$$\geq \left(\int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x\right)\left(\int_a^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x\right)^{\beta+\gamma-1}, \quad (10)$$

where $\Lambda = \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x$ and $\Omega = \int_a^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x$. Inequality (10) directly yields inequality (6). This completes the proof. □

Remark 1. If $\mathbb{T} = \mathbb{R}$, $\gamma = 1$, $w \equiv 1$, and $f(x)$ and $g(x)$ are positive functions, then inequality (6) reduces to inequality (1).

Remark 2. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $\gamma = 1$, $w \equiv 1$, $f(k) = x_k > 0$, and $g(k) = y_k > 0$ for $k = 1, 2, \ldots, n$. Then a discrete version of the inequality (6) reduces to

$$\frac{\left(\sum_{k=1}^{n} S\left(\frac{\Omega x_k^{\beta+1}}{\Lambda y_k^{\beta+1}}\right) x_k\right)^{\beta+1}}{\left(\sum_{k=1}^{n} y_k\right)^{\beta}} \geq \sum_{k=1}^{n} \frac{x_k^{\beta+1}}{y_k^{\beta}}, \tag{11}$$

where $\Lambda = \sum_{k=1}^{n} \frac{x_k^{\beta+1}}{y_k^{\beta}}$ and $\Omega = \sum_{k=1}^{n} y_k$.

Inequality (11) is given in [19].

Next, we give the following extension of reverse Bergström's inequality with Specht's ratio on time scales.

Corollary 1. *Let $w, f, g \in C\left([a, b]_{\mathbb{T}}, \mathbb{R} - \{0\}\right)$ be $\diamond_\alpha$-integrable functions. Then*

$$\frac{\left(\int_a^b S\left(\frac{\Omega|f(x)|^2}{\Lambda|g(x)|^2}\right) |w(x)||f(x)| \diamond_\alpha x\right)^2}{\int_a^b |w(x)||g(x)| \diamond_\alpha x} \geq \int_a^b \frac{|w(x)||f(x)|^2}{|g(x)|} \diamond_\alpha x, \tag{12}$$

where $\Lambda = \int_a^b \frac{|w(x)||f(x)|^2}{|g(x)|} \diamond_\alpha x$ and $\Omega = \int_a^b |w(x)||g(x)| \diamond_\alpha x$.

Proof. If we put $\beta = \gamma = 1$, then (12) follows by Theorem 4. □

Remark 3. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in \mathbb{R} - \{0\}$, and $g(k) = y_k > 0$ for $k = 1, 2, \ldots, n$. Then a discrete version of the inequality (12) reduces to

$$\frac{\left(\sum_{k=1}^{n} S\left(\frac{\Omega x_k^2}{\Lambda y_k^2}\right) x_k\right)^2}{\sum_{k=1}^{n} y_k} \geq \sum_{k=1}^{n} \frac{x_k^2}{y_k}, \tag{13}$$

where $\Lambda = \sum_{k=1}^{n} \frac{x_k^2}{y_k}$ and $\Omega = \sum_{k=1}^{n} y_k$.

Inequality (13) is obtained with Specht's ratio, which is a reverse discrete version of classical Bergström's inequality [3, 11]. Bergström's inequality is also called Titu Andreescu's inequality or Engel's inequality in literature.

Next, we present the following extension of the reverse weighted power mean inequality with Specht's ratio on time scales.

Corollary 2. *Let $w, f \in C\left([a, b]_{\mathbb{T}}, \mathbb{R} - \{0\}\right)$ be $\diamond_\alpha$-integrable functions. If $0 < \delta_1 < \delta_2$, then*

$$\left(\frac{\int_a^b S\left(\frac{\Omega|f(x)|^{\delta_2}}{\Lambda}\right)|w(x)||f(x)|^{\delta_1}\diamond_\alpha x}{\int_a^b |w(x)|\diamond_\alpha x}\right)^{\frac{1}{\delta_1}} \geq \left(\frac{\int_a^b |w(x)||f(x)|^{\delta_2}\diamond_\alpha x}{\int_a^b |w(x)|\diamond_\alpha x}\right)^{\frac{1}{\delta_2}}, \quad (14)$$

where $\Lambda = \int_a^b |w(x)||f(x)|^{\delta_2}\diamond_\alpha x$ *and* $\Omega = \int_a^b |w(x)|\diamond_\alpha x$.

Proof. Since $0 < \delta_1 < \delta_2$, we have $\frac{\delta_2}{\delta_1} > 1$. Putting $g \equiv 1$ in Theorem 4, we obtain

$$\frac{\left(\int_a^b S\left(\frac{\Omega|f(x)|^{\frac{\delta_2}{\delta_1}}}{\Lambda_1}\right)|w(x)||f(x)|\diamond_\alpha x\right)^{\frac{\delta_2}{\delta_1}}}{\left(\int_a^b |w(x)|\diamond_\alpha x\right)^{\frac{\delta_2}{\delta_1}-1}} \geq \int_a^b |w(x)||f(x)|^{\frac{\delta_2}{\delta_1}}\diamond_\alpha x,$$

where $\Lambda_1 = \int_a^b |w(x)||f(x)|^{\frac{\delta_2}{\delta_1}}\diamond_\alpha x$ and $\Omega = \int_a^b |w(x)|\diamond_\alpha x$. Replacing, in the preceding inequality, $|f(x)|$ by $|f(x)|^{\delta_1}$ and taking both sides of the inequality to the power $\frac{1}{\delta_2} > 0$, we deduce the desired result (14). □

Now, we state the following extension of reverse Schlömilch's inequality with Specht's ratio on time scales.

Corollary 3. *Let* $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ *be* $\diamond_\alpha$*-integrable functions. If* $0 < \delta_1 < \delta_2$*, then*

$$\left(\int_a^b S\left(\frac{|f(x)|^{\delta_2}}{\Lambda}\right)|w(x)||f(x)|^{\delta_1}\diamond_\alpha x\right)^{\frac{1}{\delta_1}} \geq \left(\int_a^b |w(x)||f(x)|^{\delta_2}\diamond_\alpha x\right)^{\frac{1}{\delta_2}}, \quad (15)$$

where $\Lambda = \int_a^b |w(x)||f(x)|^{\delta_2}\diamond_\alpha x$.

Proof. Without loss of generality, we may suppose that $\int_a^b |w(x)|\diamond_\alpha x = 1$. Therefore, by inequality (14), we have inequality (15). □

Remark 4. If $\mathbb{T} = \mathbb{R}$, and $w(x)$ and $f(x)$ are positive functions, then inequality (15) reduces to inequality (2).

Remark 5. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w(k) = p_k > 0$ and $f(k) = x_k > 0$ for $k = 1, 2, \ldots, n$. Then a discrete version of inequality (15) reduces to

$$\left(\sum_{k=1}^n S\left(\frac{x_k^{\delta_2}}{\Lambda}\right)p_k x_k^{\delta_1}\right)^{\frac{1}{\delta_1}} \geq \left(\sum_{k=1}^n p_k x_k^{\delta_2}\right)^{\frac{1}{\delta_2}}, \quad (16)$$

where $\Lambda = \sum_{k=1}^{n} p_k x_k^{\delta_2}$.

Inequality (16) is given in [19]. Inequality (16) is obtained with Specht's ratio, which is a reverse discrete version of classical Schlömilch's inequality [8, p. 26]. Further, if we put $\delta_2 = 2\delta_1$, then (16) takes the form of an inequality given in [19, Remark 3.3].

Now, we present the following reverse Bernoulli's inequality with Specht's ratio on time scales.

Theorem 5. *Let $f \in C\left([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\}\right)$. If $0 < \xi < 1$, then*

$$S\left(|f(x)|\right)|f(x)|^{\xi} \geq 1 + \xi\left(|f(x)| - 1\right). \tag{17}$$

Proof. If we put $p = \frac{1}{\xi}$ for $p > 1$ in (3), then we get the desired claim. □

Remark 6. Inequality (17) is just an inverse of the following well-known Bernoulli's inequality [8, pp. 40–41]:

$$x^{\xi} \leq 1 + \xi(x-1)$$

for $x \geq 0$ and $0 < \xi < 1$.

Next, we present the following extension of reverse Lyapunov's inequality with Specht's ratio on time scales.

Theorem 6. *Let $w, f, g \in C\left([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\}\right)$ be $\diamond_\alpha$-integrable functions. If $0 < \delta_1 < \delta_2 < \delta_3 < \infty$, then*

$$\begin{aligned}\left(\int_a^b S\left(\frac{\Omega|g(x)|^{\delta_1}}{\Lambda|g(x)|^{\delta_3}}\right)|w(x)||f(x)||g(x)|^{\delta_2} \diamond_\alpha x\right)^{\delta_3-\delta_1} \\ \geq \left(\int_a^b |w(x)||f(x)||g(x)|^{\delta_1} \diamond_\alpha x\right)^{\delta_3-\delta_2} \\ \times \left(\int_a^b |w(x)||f(x)||g(x)|^{\delta_3} \diamond_\alpha x\right)^{\delta_2-\delta_1},\end{aligned} \tag{18}$$

where

$$\Lambda = \int_a^b |w(x)||f(x)||g(x)|^{\delta_1} \diamond_\alpha x \text{ and } \Omega = \int_a^b |w(x)||f(x)||g(x)|^{\delta_3} \diamond_\alpha x,$$

and $S(\cdot)$ is Specht's ratio.

Proof. Let us consider

$$p = \tfrac{\delta_3-\delta_1}{\delta_3-\delta_2} \text{ and } q = \tfrac{\delta_3-\delta_1}{\delta_2-\delta_1}.$$

Then clearly $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{\delta_1}{p} + \frac{\delta_3}{q} = \delta_2$. Applying reverse Rogers–Hölder's inequality similar to (4) with Specht's ratio on time scales, we get

$$\left(\int_a^b S\left(\frac{\Omega|g(x)|^{\delta_1}}{\Lambda|g(x)|^{\delta_3}}\right)|w(x)||f(x)||g(x)|^{\delta_2}\diamond_\alpha x\right)^{\delta_3-\delta_1}$$
$$=\left(\int_a^b S\left(\frac{\Omega|g(x)|^{\delta_1}}{\Lambda|g(x)|^{\delta_3}}\right)|w(x)||f(x)|^{\frac{1}{p}+\frac{1}{q}}|g(x)|^{\frac{\delta_1}{p}+\frac{\delta_3}{q}}\diamond_\alpha x\right)^{\delta_3-\delta_1}$$
$$=\left(\int_a^b S\left(\frac{\Omega|g(x)|^{\delta_1}}{\Lambda|g(x)|^{\delta_3}}\right)|w(x)|\left(|f(x)||g(x)|^{\delta_1}\right)^{\frac{1}{p}}\left(|f(x)||g(x)|^{\delta_3}\right)^{\frac{1}{q}}\diamond_\alpha x\right)^{\delta_3-\delta_1}$$
$$\geq\left(\int_a^b |w(x)||f(x)||g(x)|^{\delta_1}\diamond_\alpha x\right)^{\frac{\delta_3-\delta_1}{p}}\left(\int_a^b |w(x)||f(x)||g(x)|^{\delta_3}\diamond_\alpha x\right)^{\frac{\delta_3-\delta_1}{q}}$$
$$=\left(\int_a^b |w(x)||f(x)||g(x)|^{\delta_1}\diamond_\alpha x\right)^{\delta_3-\delta_2}\left(\int_a^b |w(x)||f(x)||g(x)|^{\delta_3}\diamond_\alpha x\right)^{\delta_2-\delta_1}.$$

This completes the proof. □

Remark 7. Let $\alpha=1$, $\mathbb{T}=\mathbb{Z}$, $a=1$, $b=n+1$, $w\equiv 1$, $f(k)=x_k>0$, and $g(k)=y_k>0$ for $k=1,2,\dots,n$. Then a discrete version of inequality (18) reduces to

$$\left(\sum_{k=1}^{n} S\left(\frac{\Omega y_k^{\delta_1}}{\Lambda y_k^{\delta_3}}\right)x_k y_k^{\delta_2}\right)^{\delta_3-\delta_1}\geq\left(\sum_{k=1}^{n}x_k y_k^{\delta_1}\right)^{\delta_3-\delta_2}\left(\sum_{k=1}^{n}x_k y_k^{\delta_3}\right)^{\delta_2-\delta_1}, \quad (19)$$

where $\Lambda=\sum_{k=1}^{n}x_k y_k^{\delta_1}$ and $\Omega=\sum_{k=1}^{n}x_k y_k^{\delta_3}$.

Inequality (19) is just an inverse of the following classical Lyapunov's discrete inequality [8, p. 27]

$$\left(\sum_{k=1}^{n}x_k y_k^{\delta_2}\right)^{\delta_3-\delta_1}\leq\left(\sum_{k=1}^{n}x_k y_k^{\delta_1}\right)^{\delta_3-\delta_2}\left(\sum_{k=1}^{n}x_k y_k^{\delta_3}\right)^{\delta_2-\delta_1}.$$

Next, we give the following extension of reverse Rogers–Hölder's inequality with the logarithmic mean and Specht's ratio on time scales.

Theorem 7. *Let $w,f,g\in C([a,b]_{\mathbb{T}},\mathbb{R}-\{0\})$ be $\diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$. If $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, then*

$$\int_a^b |w(x)||f(x)g(x)|\diamond_\alpha x$$
$$\geq\left[1-\int_a^b L\left(\frac{|w(x)||f(x)|^p}{\Lambda},\frac{|w(x)||g(x)|^q}{\Omega}\right)\log S\left(\frac{\Omega|f(x)|^p}{\Lambda|g(x)|^q}\right)\diamond_\alpha x\right]$$
$$\times\left(\int_a^b |w(x)||f(x)|^p\diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b |w(x)||g(x)|^q\diamond_\alpha x\right)^{\frac{1}{q}}, \quad (20)$$

where $\Lambda=\int_a^b|w(x)||f(x)|^p\diamond_\alpha x$, $\Omega=\int_a^b|w(x)||g(x)|^q\diamond_\alpha x$, $L(\cdot,\cdot)$ is the logarithmic mean, and $S(\cdot)$ is Specht's ratio.

Proof. Let $\Phi(x) = \frac{|w(x)||f(x)|^p}{\Lambda}$ and $\Psi(x) = \frac{|w(x)||g(x)|^q}{\Omega}$ on $[a,b]_{\mathbb{T}}$. Inequality (5) takes the form

$$L\left(\frac{|w(x)||f(x)|^p}{\Lambda}, \frac{|w(x)||g(x)|^q}{\Omega}\right) \log S\left(\frac{\Omega|f(x)|^p}{\Lambda|g(x)|^q}\right) \geq \frac{1}{p}\frac{|w(x)||f(x)|^p}{\Lambda} + \frac{1}{q}\frac{|w(x)||g(x)|^q}{\Omega} - \frac{|w(x)||f(x)g(x)|}{\Lambda^{\frac{1}{p}}\Omega^{\frac{1}{q}}}. \quad (21)$$

By integrating both sides with respect to x from a to b, inequality (21) becomes

$$\int_a^b L\left(\frac{|w(x)||f(x)|^p}{\Lambda}, \frac{|w(x)||g(x)|^q}{\Omega}\right) \log S\left(\frac{\Omega|f(x)|^p}{\Lambda|g(x)|^q}\right) \diamond_\alpha x \geq 1 - \int_a^b \frac{|w(x)||f(x)g(x)|}{\Lambda^{\frac{1}{p}}\Omega^{\frac{1}{q}}} \diamond_\alpha x. \quad (22)$$

Inequality (20) follows from (22). This completes the proof. □

Remark 8. If we set $\alpha = 1$, then we get delta versions, and if we set $\alpha = 0$, then we get nabla versions of diamond-α integral operator inequalities presented in this article. Also, if we set $\mathbb{T} = \mathbb{Z}$, then we get discrete versions, and if we set $\mathbb{T} = \mathbb{R}$, then we get continuous versions of diamond-α integral operator inequalities presented in this article.

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