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On dominions of certain ample monoids

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ABSTRACT. A semigroup S is called left ample if it can be embedded in the symmetric inverse semigroup $\mathcal{I}_X$ of partial bijections of a non-empty set X such that the image of S is closed under the unary operation $\alpha \longmapsto \alpha\alpha^{-1}$, where α^{-1} is the inverse of α in $\mathcal{I}_X$. Right ample semigroups are defined dually. A semigroup is called ample if it is both left and right ample. A monoid is (left, right) ample if it is (left, right) ample as a semigroup. We observe that the dominion of an ample subsemigroup of $\mathcal{I}_X$ coincides with the inverse subsemigroup of $\mathcal{I}_X$ generated by it. We then determine the dominions of certain submonoids of $\mathcal{I}_n$, the symmetric inverse semigroup over a finite chain $1 < 2 < \cdots < n$.

1. Introduction and preliminaries

We know from the Wagner–Preston representation theorem that any inverse semigroup can be embedded in the *symmetric inverse semigroup* $\mathcal{I}_X$ of partial bijections of a non-empty set X (see, for instance, [7] Theorem 5.1.7). A semigroup S is called *left ample* if it can be embedded in some $\mathcal{I}_X$ (or in any inverse semigroup, for that matter) such that the image of S is closed under the unary operation $\alpha \longmapsto \alpha\alpha^{-1} = I_{dom\alpha}$, where we are identifying S with its isomorphic copy in $\mathcal{I}_X$, the maps are written to the right of their arguments, $\alpha^{-1} \in \mathcal{I}_X$ is the inverse of $\alpha \in S$ and $I_{dom\alpha}$ denotes the identity map on the domain of α. We shall call $\mathcal{I}_X$ a symmetric inverse semigroup *associated* with S. *Right ample* semigroups are defined dually. A semigroup is called *ample* if it is both left and right ample. A monoid is (left, right) *ample* if it is (left, right) ample as a semigroup. Let S be a left (respectively, right) ample semigroup with associated symmetric inverse semigroup $\mathcal{I}_X$ (respectively, $\mathcal{I}_Y$). Then the problem of finding a set Z such that $\mathcal{I}_Z$ (as an associated symmetric inverse semigroup) makes S into a left as well as

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right ample semigroup is, in general, undecidable [5]. A subsemigroup S of a semigroup T is called *full* if it contains all idempotents of T. In particular, every full subsemigroup of an inverse semigroup is ample. Notwithstanding, $(\mathbb{N}, \cdot)$ is an ample submonoid of $(\mathbb{R}, \cdot)$ that is not full. In this article we shall study the dominions of certain ample submonoids of $\mathcal{I}_n$, the symmetric inverse semigroup over a finite chain $1 < 2 < \cdots < n$ of natural numbers (see Figure 1).

For standard concepts in semigroup theory we refer the reader to Howie [7] or Higgins [6]. For further details about ample semigroups (monoids) the reader is referred to [4] and the references contained therein.

Recall that a morphism $f : A \longrightarrow B$ in a category $\mathcal{C}$ is called an *epimorphism*, shortly *epi*, if for all $C \in \mathsf{Ob}(\mathcal{C})$ and for all $g, h \in \mathsf{Hom}_{\mathcal{C}}(B, C)$

$$fg = fh \Longrightarrow g = h.$$

In concrete categories surjective morphisms are always epis. The converse is, however, not true. Particularly, there exist non-surjective epis in the categories of all semigroups and all monoids and their homomorphisms, see for instance [8].

2. Dominions

A semigroup S is called an *oversemigroup* of a semigroup U if the latter is a subsemigroup of the former. *Overmonoids* are defined similarly. Given an oversemigroup S of a semigroup U, an element $d \in S$ is said to be in the *dominion* of U if for all pairs of semigroup homomorphisms $f, g : S \longrightarrow T$ we have:

$$f\,|_U = g\,|_U \Longrightarrow (d)f = (d)g. \tag{1}$$

The set of all elements of S satisfying implication (1) is called the *dominion* of U in S; we denote it by $Dom_S U$. Dominions of monoids are defined similarly. A semigroup (respectively, monoid) is said to be *absolutely closed* if $Dom_S U = U$ for every oversemigroup (respectively, overmonoid) S of U.

Theorem 1 (Theorem 8.3.6 of [7]). *Inverse semigroups are absolutely closed.*

Note that a morphism $f : S \longrightarrow T$ in the category of semigroups (monoids) is an epi if and only if $Dom_T \mathrm{Im} f = T$. In this case we say that S is *epimorphically embedded* in T.

Remark 1. The following statements can be easily verified.

(1) Let U be a subsemigroup of a semigroup S. Then $Dom_S U$ is an oversemigroup of U and a subsemigroup of S.

(2) If U_1 and U_2 are subsemigroups of a semigroup S with $U_1 \subseteq U_2$ then $Dom_S U_1 \subseteq Dom_S U_2$.

(3) $Dom_S(Dom_S U) = Dom_S U$ for every subsemigroup U of S.

Conditions (1)–(3) imply that Dom_S is a 'closure operator' in the sense of universal algebra [2].

Proposition 1. *Let U be an ample semigroup with associated symmetric inverse semigroup $\mathcal{I}_X$ (that makes it both left and right ample). Then $Dom_{\mathcal{I}_X} U = \langle U\rangle_{\mathrm{INV}}$, where $\langle U\rangle_{\mathrm{INV}}$ is the inverse subsemigroup of $\mathcal{I}_X$ generated by U.*

Proof. If U is an inverse semigroup, then, by Theorem 1, there is nothing to prove. So, assume that U is an ample semigroup that is not inverse. By Remark 1 part (2) and Theorem 1, we have:

$$Dom_{\mathcal{I}_X} U \subseteq Dom_{\mathcal{I}_X} \langle U\rangle_{\mathrm{INV}} = \langle U\rangle_{\mathrm{INV}}.$$

To prove the reverse inclusion, let us make the following observations. By Remark 1 part (1), $U \subseteq Dom_{\mathcal{I}_X} U$. Also, since U is not inverse, there exists $x \in U$ such that $x^{-1} \in \langle U\rangle_{\mathrm{INV}} \setminus U$, where x^{-1} is the inverse of x in $\mathcal{I}_X$. Now, because U is left and right ample with respect to $\mathcal{I}_X$, we have $xx^{-1}, x^{-1}x \in U$.

Let $f, g : \mathcal{I}_X \longrightarrow T$ be semigroup homomorphisms with $f\,|_U = g\,|_U$. Then, we may calculate

$$\begin{aligned} (x^{-1})f &= (x^{-1}xx^{-1})f = (x^{-1})f(xx^{-1})f \\ &= (x^{-1})f(xx^{-1})g = (x^{-1})f(x)g(x^{-1})g \\ &= (x^{-1})f(x)f(x^{-1})g = (x^{-1}x)f(x^{-1})g \\ &= (x^{-1}x)g(x^{-1})g = (x^{-1}xx^{-1})g \\ &= (x^{-1})g. \end{aligned} \tag{2}$$

Hence, $x^{-1} \in Dom_{\mathcal{I}_X} U$. Now, because the generating set of $\langle U\rangle_{\mathrm{INV}}$ is contained in $Dom_{\mathcal{I}_X} U$, it follows that $\langle U\rangle_{\mathrm{INV}} \subseteq Dom_{\mathcal{I}_X} U$. □

Corollary 1. *If U is a left and right ample semigroup with respect to the same associated symmetric inverse semigroup $\mathcal{I}_X$ then U is epimorphically embedded in $\langle U\rangle_{\mathrm{INV}}$.*

Proof. It suffices to prove that $\langle U\rangle_{\mathrm{INV}} \subseteq Dom_{\langle U\rangle_{\mathrm{INV}}} U$. Let $x \in \langle U\rangle_{\mathrm{INV}}$. Then $x = u_1u_2\cdots u_n$, where $u_1, u_2, \ldots, u_n \in \mathcal{I}_X$ are such that u_i or u_i^{-1} belongs to U for all $i \in \{1, 2, \ldots, n\}$. Let $f, g : \langle U\rangle_{\mathrm{INV}} \longrightarrow T$ be semigroup homomorphisms with $f\,|_U = g\,|_U$. Then, by virtue of calculation (2), we have $(u_i)f = (u_i)g$ for all $i \in \{1, 2, \ldots, n\}$. Thus $(x)f = (x)g$, whence $x \in Dom_{\langle U\rangle_{\mathrm{INV}}} U$, as required. □

3. Subsemigroups of $\mathcal{I}_n$

Let $\mathcal{I}_n$ denote the symmetric inverse semigroup over a chain

$$1 < 2 < \cdots < n \tag{3}$$

of natural numbers. In this section we shall determine the dominions of certain submonoids of $\mathcal{I}_n$, henceforth called *special submonoids.* We shall omit parenthesis around the arguments of partial transformations, whenever they are not necessary.

(1) Let $\mathcal{S}_n$ denote the symmetric group of all bijections of chain (3). We define $\mathcal{I}'_n := (\mathcal{I}_n \smallsetminus \mathcal{S}_n) \cup \{\iota\}$, where ι denotes the identity of $\mathcal{S}_n$. It is an easy exercise to prove that $\mathcal{I}'_n$ is an inverse submonoid of $\mathcal{I}_n$. The elements of $\mathcal{I}'_n$ are called *strict partial bijections.*

(2) We call $\alpha \in \mathcal{I}_n$ *order-decreasing* if $x\alpha \leq x$ for all $x \in Dom\,\alpha$. We denote by $\mathcal{DI}_n$ the special submonoid of all order-decreasing partial bijections. The special submonoid $\mathcal{DI}^+_n$ of all *order-increasing* partial bijections is defined dually [10].

(3) A partial bijection $\alpha \in \mathcal{I}_n$ is said to be *order-preserving* or *monotone* if

$$\forall x, y \in Dom\,\alpha,\ x < y \text{ implies that } x\alpha < y\alpha.$$

We denote by $\mathcal{OI}_n$ the special submonoid of all monotone partial bijections [3].

(4) An element α of $\mathcal{I}_n$ is called a *contraction* (respectively, *expansion*) if for all $x, y \in Dom\,\alpha$ we have $|x\alpha - y\alpha| \leq |x - y|$ (respectively, $|x\alpha - y\alpha| \geq |x - y|$). The special submonoid of all contractions (respectively, expansions) is denoted by $\mathcal{CI}_n$ (respectively, $\mathcal{CI}^*_n$) [1].

(5) We say that $\alpha \in \mathcal{I}_n$ is *order-reversing* or *antitone* if

$$\forall x, y \in Dom\,\alpha,\ x < y \text{ implies that } y\alpha < x\alpha.$$

Note that the only order-reversing full transformation of (3) is

$$e : \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ n & n-1 & \ldots & 2 & 1 \end{pmatrix}.$$

Let $\mathcal{RI}_n$ denote the subset of $\mathcal{I}'_n$ comprising all monotone as well as all antitone partial bijections. Then it is an easy exercise to show that $\mathcal{RI}_n$ is a (special) submonoid of $\mathcal{I}_n$. (Indeed, $\mathcal{RI}_n = \mathcal{ORI}_n \smallsetminus \{e\}$, where $\mathcal{ORI}_n$ denotes the submonoid of $\mathcal{I}_n$ containing all monotone and all antitone partial bijections, see [3]).

The special submonoid of $\mathcal{I}_n$ comprising all partial bijections that are both order-preserving and order-decreasing is denoted by $\mathcal{ODI}_n$. The special submonoids $\mathcal{OCI}_n$, $\mathcal{RDI}_n$, $\mathcal{RCI}_n$, $\mathcal{DCI}_n$, $\mathcal{ODCI}_n$, $\mathcal{RDCI}_n$ etc. are defined analogously, see the following lattice diagram.

The study of these submonoids is not only motivated by their natural occurrence, but also by their elegant use in the enumerative combinatorial problems [1, 9, 10, 11]. The aim of this section is find the dominions of certain special submonoids, see Remark 3.

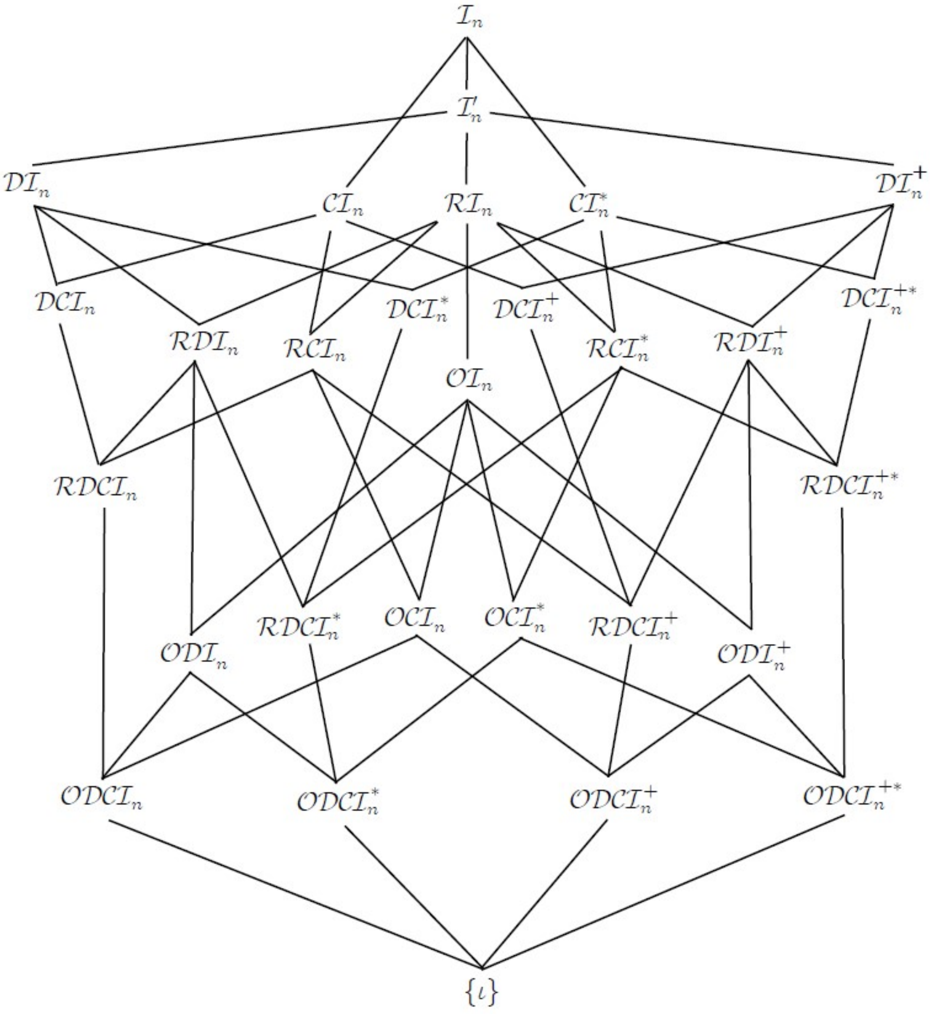

FIGURE 1. The lattice of special submonoids.

Let S be a special submonoid and let $\alpha \in S$. Then it can be easily verified that $\alpha\alpha^{-1} = I_{dom\alpha}$ and $\alpha^{-1}\alpha = I_{dom\alpha^{-1}}$ belong to S. Thus we have the following lemma.

Lemma 1. *The special submonoids of $\mathcal{I}_n$ are all ample (with $\mathcal{I}_n$ being their associated symmetric inverse semigroup).*

Lemma 2. *$\mathcal{OI}_n$, $\mathcal{DI}_n$, $\mathcal{DI}^+_n$ and $\mathcal{RI}_n$ are submonoids of $\mathcal{I}'_n$.*

Proof. Let $\alpha \in \mathcal{OI}_n$. Suppose on the contrary that $\alpha \in \mathcal{S}_n \setminus \{\iota\}$. Then $\operatorname{Dom}\alpha = \operatorname{Im}\alpha = \{1, 2, \ldots, n\}$. Because $\alpha \neq \iota$, there exists $j \in \operatorname{Dom}\alpha$ such that $j\alpha = k \neq j$. Let us consider the case when $k < j$ (the other case, wherein $k > j$, can be dealt with similarly). Since α is a monotone bijection, it must map $\{1, 2, \ldots, j-1\}$ to $\{1, 2, \ldots, k-1\}$ in a one-to-one fashion. This gives a contradiction, because $k - 1 < j - 1$. Similarly, we can show that $\mathcal{DI}_n$ and $\mathcal{DI}_n^+$ are submonoids of $\mathcal{I}'_n$.

Lastly, $\mathcal{RI}_n$ is a submonoid of $\mathcal{I}'_n$ because $\mathcal{OI}_n \subseteq \mathcal{I}'_n$ and the only order-reversing full bijection, viz. e, does not belong to $\mathcal{RI}_n$. □

Let f be a bijective mapping from a semigroup S to a semigroup T. We call f an *anti-isomorphism* if $(xy)f = (y)f(x)f$, for all $x, y \in S$.

Proposition 2. *The map $\alpha \longmapsto \alpha^{-1}$ is an anti-isomorphism from*

(1) $\mathcal{OI}_n$ to $\mathcal{OI}_n$,
(2) $\mathcal{RI}_n$ to $\mathcal{RI}_n$,
(3) $\mathcal{DI}_n$ to $\mathcal{DI}_n^+$,
(4) $\mathcal{CI}_n$ to $\mathcal{CI}_n^*$,
(5) $\mathcal{DCI}_n$ to $\mathcal{DI}_n^+ \cap \mathcal{CI}_n^*$ $(:= \mathcal{DCI}_n^{+*})$ and
(6) $\mathcal{ODCI}_n$ to $\mathcal{ODI}_n^+ \cap \mathcal{OCI}_n^*$ $(:= \mathcal{ODCI}_n^{+*})$.

Proof. Obviously, α is order-preserving (respectively, order-reversing) if and only if α^{-1} is order-preserving (respectively, order-reversing). It is also clear that α is order-increasing (respectively, expansion) if and only if α^{-1} is order-decreasing (respectively, contraction). Moreover, the map $(\alpha)\psi = \alpha^{-1}$ is a bijection with

$$(\alpha\beta)\psi = (\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1} = (\beta)\psi(\alpha)\psi. \tag{4}$$

Thus $\alpha \longmapsto \alpha^{-1}$ is an anti-isomorphism in all the above-mentioned cases. □

Remark 2. Clearly $\alpha \longmapsto \alpha^{-1}$ is also an anti-isomorphism from $\mathcal{DI}_n^+$ to $\mathcal{DI}_n$, from $\mathcal{CI}_n^*$ to $\mathcal{CI}_n$, from $\mathcal{DCI}_n^{+*}$ to $\mathcal{DCI}_n$ and from $\mathcal{ODCI}_n^{+*}$ to $\mathcal{ODCI}_n$.

The following corollary immediately follows from Proposition 2.

Corollary 2. *$\mathcal{OI}_n$ and $\mathcal{RI}_n$ are inverse submonoids of $\mathcal{I}'_n$.*

Remark 3. Because $\mathcal{I}_n$, $\mathcal{I}'_n$, $\mathcal{RI}_n$, $\mathcal{OI}_n$ and $\{\iota\}$ are inverse monoids, they coincide, by Theorem 1, with their dominions in $\mathcal{I}_n$. On the contrary, it also follows from Proposition 2 that the remaining submonoids in Figure 1 are all non-inverse. We shall use Proposition 1 to find the dominions for 12 of the remaining 24 submonoids. To avoid somewhat cumbersome notations, such as $Dom_{\mathcal{I}_n}\mathcal{ODCI}_n^{+*} = \mathcal{OI}_n$, the dominions will always be described in terms of Corollary 1, see Theorems 2, 3, 4, and Corollary 3.

Remark 4. Each $\alpha \in \mathcal{I}_n$ can be pictured as a digraph $\mathcal{G}$ whose set of vertices is $\operatorname{Dom}\alpha \cup \operatorname{Im}\alpha$. We have an edge from vertex u to vertex v if $u\alpha = v$. The in- (respectively, out-) degree of every $v \in \operatorname{Dom}\alpha \smallsetminus \operatorname{Im}\alpha$ is 0 (respectively, 1). Dually, the in- (respectively, out-) degree of every vertex $v \in \mathrm{Im}\alpha \smallsetminus \mathrm{Dom}\,\alpha$ is 1 (respectively, 0). On the other hand, both the in- and out-degrees of $v \in \mathrm{Dom}\,\alpha \cap \mathrm{Im}\alpha$ are 1. This implies that each of the connected components of $\mathcal{G}$ either consists of

- a unique *path*: $(y_1, \dots, y_k)$, where $y_i\alpha = y_{i+1}$, $i = 1, \dots, k-1$,
- or, a unique *cycle*: $(x_1, \dots, x_r)$, such that $x_i\alpha = x_{i+1}$, $i = 1, \dots, r-1$, $x_r\alpha = x_1$,
- or, a unique *fixed point* $z = z\alpha$.

Note that a path $\widehat{\pi} = (y_1, \dots, y_k)$ may be treated as an element of $\mathcal{I}'_k$ (and hence of $\mathcal{I}'_n$) because $y_1 \notin \mathrm{Im}\alpha$ (equivalently, $y_k \notin \mathrm{Dom}\,\alpha$). Similarly, a cycle $\widehat{\mu} = (x_1, \dots, x_r)$ may be viewed as an element of $\mathcal{S}_r \subseteq \mathcal{I}_n$.

Remark 5. Let the digraph $\alpha \in \mathcal{I}_n$ have component paths $\widehat{\pi}_1, \widehat{\pi}_2 \dots, \widehat{\pi}_r$ and component cycles $\widehat{\mu}_1, \widehat{\mu}_2, \dots, \widehat{\mu}_s$. Define $\pi_i, \mu_j \in \mathcal{I}_n$, $1 \le i \le r$, $1 \le j \le s$ by

$$\begin{aligned}(x)\pi_i &= \begin{cases} (x)\,\widehat{\pi}_i, & \text{if } x \in \mathrm{Dom}\ \widehat{\pi}_i, \\ x, & \text{if } x \in \mathrm{Dom}\ \alpha \smallsetminus \mathrm{Dom}\ \widehat{\pi}_i\,, \end{cases} \\ (x)\mu_j &= \begin{cases} (x)\,\widehat{\mu}_j, & \text{if } x \in \mathrm{Dom}\ \widehat{\mu}_j, \\ x, & \text{if } x \in \mathrm{Dom}\ \alpha \smallsetminus \mathrm{Dom}\ \widehat{\mu}_j. \end{cases}\end{aligned}$$

Then $\alpha = \pi_1 \circ \cdots \circ \pi_r \circ \mu_1 \circ \cdots \circ \mu_s$ (with the product on the right hand side being commutative).

Proof. Straightforward verification. □

To keep the notations simple, the component paths $\widehat{\pi}_1, \widehat{\pi}_2, \dots, \widehat{\pi}_r$ and component cycles $\widehat{\mu}_1, \widehat{\mu}_2, \dots, \widehat{\mu}_s$ for any $\alpha \in \mathcal{I}_n$ will be identified, respectively, with the partial bijections $\pi_1, \dots, \pi_r$ and $\mu_1, \dots, \mu_s$ defined in Remark 5.

Lemma 3. *Let $\pi : (y_1, \dots, y_k)$ be a path in the digraph of $\alpha \in \mathcal{I}_n$. Then, for any $1 < m < k$, $\pi = \rho_1 \circ \rho_2$, where $\rho_1, \rho_2 \in \mathcal{I}'_n$ are defined below.*

$$\begin{aligned}\rho_1 &: \begin{pmatrix} y_1 & y_2 & \cdots & y_{m-1} & y_m & y_{m+1} & \cdots & y_{k-1} \\ y_1 & y_2 & \cdots & y_{m-1} & y_{m+1} & y_{m+2} & \cdots & y_k \end{pmatrix}, \\ \rho_2 &: \begin{pmatrix} y_1 & y_2 & \cdots & y_{m-1} & y_{m+1} & y_{m+2} & \cdots & y_k \\ y_2 & y_3 & \cdots & y_m & y_{m+1} & y_{m+2} & \cdots & y_k \end{pmatrix}.\end{aligned}$$

Proof. Straightforward. □

In fact one can easily prove the following generalized version of the above lemma.

Lemma 4. *Let $\pi : (y_1, \dots, y_k)$ be a path in the digraph of $\alpha \in \mathcal{I}_n$ and let $1 = m_0 < m_1 < m_2 < \cdots < m_{r-1} < m_r = k$. Then, there exists a* *factorization* $\pi = \rho_1 \circ \rho_2 \circ \cdots \circ \rho_r$*, where* ρ_1*,* ρ_i*,* $2 < i < r-1$*, and* ρ_r *are the elements of* $\mathcal{I}'_n$ *defined below.*

$$\rho_1 : \begin{pmatrix} y_1 & y_2 & \cdots & y_{m_{r-1}-1} & y_{m_{r-1}} & y_{m_{r-1}+1} & \cdots & y_{m_r-1} \\ y_1 & y_2 & \cdots & y_{m_{r-1}-1} & y_{m_{r-1}+1} & y_{m_{r-1}+2} & \cdots & y_{m_r} \end{pmatrix},$$

$$\rho_i : \begin{pmatrix} y_1 & \cdots & y_{m_{r-i}-1} & y_{m_{r-i}} & \cdots y_{m_{r-i+1}-1} & y_{m_{r-i+1}+1} & \cdots & y_{m_r} \\ y_1 & \cdots & y_{m_{r-i}-1} & y_{m_{r-i}+1} \cdots & y_{m_{r-i+1}} & y_{m_{r-i+1}+1} & \cdots & y_{m_r} \end{pmatrix},$$

$$\rho_r : \begin{pmatrix} y_1 & y_2 & \cdots & y_{m_1-1} & y_{m_1+1} & y_{m_1+2} & \cdots & y_{m_r} \\ y_2 & y_3 & \cdots & y_{m_1} & y_{m_1+1} & y_{m_1+2} & \cdots & y_{m_r} \end{pmatrix}.$$

Proof. Recursively apply Lemma 3. □

Lemma 5. *Let* $\alpha \in \mathcal{OI}_n$*. Then the digraph of* α *does not contain any cycles. Moreover, if* $\pi = \rho_1 \circ \rho_2 \circ \cdots \circ \rho_r$ *is a factorization of a component path of* α*, as given by Lemma 4, then* $\rho_i \in \mathcal{OI}_n$ *for all* $i \in \{1, 2, \ldots, r\}$*.*

Proof. Let $\alpha \in \mathcal{OI}_n$. Suppose on the contrary that the digraph of α contains a cycle:

$$\mu : (x_1, x_2, \ldots, x_k), \;\; k \geq 2.$$

If $x_1 < x_2$ then $x_2 = (x_1)\,\alpha < (x_2)\,\alpha = x_3$, since α is monotone. Iterating the argument we get

$$x_1 < x_2 < \cdots < x_k < x_1,$$

a contradiction. Similarly, $x_1 > x_2$ gives a contradiction. Thus the digraph of α does not contain any cycles.

To prove the second part, consider a path $\pi = (y_1, \ldots, y_k)$ in the digraph of α. Let ρ_1, ρ_i, $2 < i < r-1$, and ρ_r be the components of a decomposition of α given by Lemma 4. Because α is monotone, we have either

$$y_1 < y_2 < \cdots < y_k$$

or

$$y_1 > y_2 > \cdots > y_k.$$

In the former (respectively, latter) case both the rows in ρ_i, $1 < i < r$, are written in ascending (respectively, descending) order, going from left to right. This implies that ρ_i, $1 < i < r$, are all monotone. □

Lemma 6. *Let* $\mu : (x_1, \ldots, x_k)$ *be a cycle in the digraph of* $\alpha \in \mathcal{I}'_n \setminus \{\iota\}$*. Then we have* $\mu = \sigma \circ \pi$ *such that* σ *is the path* (x_1, x') *and* π *is the path* $(x', x_2, x_3, \ldots, x_{k-1}, x_k, x_1)$ *for some* $x' \in \{1, 2, \ldots, n\} \setminus \mathrm{Im}\mu$*.*

Proof. Let $\alpha \in \mathcal{I}'_n \setminus \{\iota\}$ be a cycle. Then note that $\mathrm{Im}\mu = \mathrm{Dom}\mu$ and there exists $x' \in \{1, 2, \ldots, n\} \setminus \mathrm{Im}\mu$. Now, it is a routine to verify that

$$\sigma \;\; = \;\; \begin{pmatrix} x_1 & x_2 & \ldots & x_{k-1} & x_k \\ x' & x_2 & \ldots & x_{k-1} & x_k \end{pmatrix} \text{and}$$

$$\pi \;=\; \begin{pmatrix} x' & x_2 & \dots & x_{k-1} & x_k \\ x_2 & x_3 & \dots & x_k & x_1 \end{pmatrix}$$

satisfy the requirements of the lemma. □

Theorem 2. *$\mathcal{DI}_n$ and $\mathcal{DI}_n^+$ are epimorphically embedded in $\mathcal{I}'_n$.*

Proof. By Corollary 1, Proposition 2 and Remark 2, it suffices to show that $\mathcal{I}'_n$ is the inverse submonoid of $\mathcal{I}_n$ generated by $\mathcal{DI}_n$ (equivalently, $\mathcal{DI}_n^+$). Indeed, we need to prove that any element of $\mathcal{I}'_n$ can be expressed as a product of elements belonging to $\mathcal{DI}_n \cup \mathcal{DI}_n^+$.

Let α be an arbitrary element of $\mathcal{I}'_n$ and let $\xi = (y_1, \dots, y_k)$ be a component path in the digraph of α. If $k = 2$ then ξ is either order-decreasing or order-increasing. If $k \geq 3$, then, applying Lemma 4 with appropriate division points $m_1, \dots, m_{r-1}$, we may write $\xi = \xi_1 \circ \xi_2 \circ \cdots \circ \xi_r$ such that $\xi_i \in \mathcal{DI}_n \cup \mathcal{DI}_n^+$, $1 \leq i \leq r$.

Also, if μ is a component cycle in the digraph of α then there exists a decomposition $\mu = \sigma \circ \pi$, as described in Lemma 6. It is clear that $\sigma \in \mathcal{DI}_n \cup \mathcal{DI}_n^+$, whereas π, being a path, can be further factorized into $\rho_1 \circ \rho_2 \circ \cdots \circ \rho_k$ with $\rho_i \in \mathcal{DI}_n \cup \mathcal{DI}_n^+$, $1 \leq i \leq k$, as discussed above.

Lastly, $\iota \in \mathcal{DI}_n \cap \mathcal{DI}_n^+$. Hence, $\mathcal{I}'_n$ is the inverse subsemigroup of $\mathcal{I}_n$ generated by $\mathcal{DI}_n$ (equivalently, $\mathcal{DI}_n^+$). □

Theorem 3. *The special submonoids $\mathcal{ODCI}_n$, $\mathcal{ODCI}_n^{+*}$, $\mathcal{ODCI}_n^*$ and $\mathcal{ODCI}_n^+$ are all epimorphically embedded in $\mathcal{OI}_n$.*

Proof. Recall from Lemma 2 that $\mathcal{OI}_n$ is contained in $\mathcal{I}'_n$. Let

$$\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$$

be a factorization of $\alpha \in \mathcal{OI}_n$ as given by Theorem 2. By Lemma 5, we have $\alpha_i \in \mathcal{ODI}_n \cup \mathcal{ODI}_n^+$, $1 \leq i \leq k$. Let $\gamma \in \mathcal{ODI}_n$ be defined by

$$\gamma : \begin{pmatrix} x_1 & x_2 & \dots & x_k \\ y_1 & y_2 & \dots & y_k \end{pmatrix},\ x_i, y_i \in \{1, 2, \dots, n\},\ 1 \leq i \leq k < n.$$

We may assume, without loss of generality, that

$$x_1 < x_2 < \cdots < x_{k-1} < x_k.$$

Then

$$y_1 < y_2 < \cdots < y_{k-1} < y_k,$$

because γ is order-preserving. We also have

$$y_1 \leq x_1,\ y_2 \leq x_2,\ \dots,\ y_k \leq x_k,$$

as γ is order-decreasing. Now, define δ, $\xi \in \mathcal{I}'_n$ by

$$\delta : \begin{pmatrix} x_1 & x_2 & \dots & x_k \\ y_1 & y_1+1 & \dots & y_1+k-1 \end{pmatrix},$$

$$\xi : \begin{pmatrix} y_1 & y_1+1 & \dots & y_1+k-1 \\ y_1 & y_2 & \cdots & y_k \end{pmatrix}.$$

Then, clearly $\gamma = \delta \circ \xi$. Also, it can be easily verified that $\delta \in \mathcal{ODCI}_n$ and $\xi \in \mathcal{ODCI}_n^{+*}$. Using a similar argument, one can show that every $\beta \in \mathcal{ODI}_n^{+}$ can be factorized in $\mathcal{ODCI}_n \cup \mathcal{ODCI}_n^{+*}$. Thus the inverse subsemigroup $\mathcal{OI}_n$ is generated by $\mathcal{ODCI}_n$, as well as $\mathcal{ODCI}_n^{+*}$.

To prove that each of $\mathcal{ODCI}_n^{*}$ and $\mathcal{ODCI}_n^{+}$ also generate $\mathcal{OI}_n$ consider again $\alpha \in \mathcal{OI}_n$ with a factorization

$$\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$$

and $\gamma \in \mathcal{ODI}_n$ both as defined above. Define

$$\delta' : \begin{pmatrix} x_1 & x_2 & \dots & x_{k-1} & x_k \\ x_k - k & x_k-(k-1) & \dots & x_k - 1 & x_k \end{pmatrix},$$

$$\xi' : \begin{pmatrix} x_k - k & x_k-(k-1) & \dots & x_k - 1 & x_k \\ y_1 & y_2 & \cdots & y_{k-1} & y_k \end{pmatrix}.$$

Then $\gamma = \delta' \circ \xi'$, where $\delta' \in \mathcal{ODCI}_n^{+}$ and $\xi' \in \mathcal{ODCI}_n^{*}$. Also, any $\beta \in \mathcal{ODI}_n^{+}$ can be factorized in $\mathcal{ODCI}_n^{+} \cup \mathcal{ODCI}_n^{*}$ by using a similar argument. $\square$

Corollary 3. *$\mathcal{ODI}_n$, $\mathcal{ODI}_n^{+}$, $\mathcal{OCI}_n$ and $\mathcal{OCI}_n^{*}$ are epimorphically embedded in $\mathcal{OI}_n$.*

Proof. Let $\mathcal{ODCI}_n = U$, $\mathcal{ODI}_n = V$, $\mathcal{OI}_n = S$, $\mathcal{I}_n = T$. Then $U \subseteq V \subseteq S \subseteq T$. Now, observe that $Dom_T V = S$:

$$\begin{aligned} S &= Dom_T U, \text{ by Theorem 3 and Corollary 1,} \\ &\subseteq Dom_T V, \text{ by part(2) of Remark 1,} \\ &\subseteq S, \text{ by Proposition 1, for } S \text{ is an inverse monoid.} \end{aligned}$$

This implies by Corollary 1 that $\mathcal{ODI}_n$ is epimorphically embedded in $\mathcal{OI}_n$. That the remaining three submonoids epimorphically embed in $\mathcal{OI}_n$ can be shown similarly. $\square$

Theorem 4. *$\mathcal{RCI}_n$ and $\mathcal{RCI}_n^{*}$ are epimorphically embedded in $\mathcal{RI}_n$.*

Proof. Let $\alpha \in \mathcal{RI}_n$. If α is order-preserving, i.e. $\alpha \in \mathcal{OI}_n$, then we may write by Corollary 3

$$\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k,$$

where $\alpha_i \in \mathcal{OCI}_n \cup \mathcal{OCI}_n^* \subseteq \mathcal{RCI}_n \cup \mathcal{RCI}_n^*$, $1 \leq i \leq k$. So, assume that

$$\alpha : \begin{pmatrix} x_1 & x_2 & \dots & x_k \\ y_1 & y_2 & \dots & y_k \end{pmatrix}, \, x_i, y_i \in \{1, 2, \dots, n\}, 1 \leq i \leq k < n$$

is an order-reversing bijection. We may suppose, without loss of generality, that $x_1 < x_2 < \cdots < x_k$. This also necessitates: $y_k < y_{k-1} < \cdots < y_1$. We shall need the following chain to define the factors of α,

$$z_1 < z_1 + 1 < \cdots < z_p,$$

where $z_1 = \min\{x_1, y_k\}$, $z_p = \max\{x_k, y_1\}$. Let us define

$$\xi_1 \colon \begin{pmatrix} x_1 & x_2 & \dots & x_{k-1} & x_k \\ z_1 & z_1 + 1 & \dots & z_1 + k - 2 & z_1 + k - 1 \end{pmatrix},$$

$$\xi_2 \colon \begin{pmatrix} z_1 & z_1 + 1 & \dots & z_1 + k - 2 & z_1 + k - 1 \\ y_1 & y_2 & & y_{k-1} & y_k \end{pmatrix}.$$

Then, clearly, $\alpha = \xi_1 \circ \xi_2$, with $\xi_1 \in \mathcal{RCI}_n$, $\xi_2 \in \mathcal{RCI}_n^*$ (indeed ξ_1 is order-preserving and ξ_2 is order-reversing). Thus $\mathcal{RI}_n$ is the inverse submonoid of $\mathcal{I}_n$ generated by $\mathcal{RCI}_n$ (equivalently, $\mathcal{RCI}_n^*$), and the theorem follows by Corollary 1. $\square$

Conclusion. The authors wonder as to what are the inverse submonoids of $\mathcal{I}_n$ generated by the remaining 12 submonoids in Figure 1.

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