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Implicit-implicit and parametric-implicit surface intersections in Euclidean n-space

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Abstract. In this paper, we study the intersection problem of two surfaces in which at least one surface is given by its implicit equation in Euclidean n-space. We first obtain the curvatures of the transversal intersection curve of two implicit surfaces. Later, we study the transversal intersection of parametric-implicit surfaces in n-dimensional Euclidean space. Finally, we present examples as applications of the given methods.

1. Introduction

The analysis of surface-surface intersection problems is an important topic in CAGD. In most instances, determining the parametric equation of an intersection curve of two surfaces is difficult. Because of this, there exist some studies in literature to determine the intersection curve by using not only numerical techniques but also some differential geometric methods. The intersection problem of two surfaces has three types which are called implicit-implicit, implicit-parametric and parametric-parametric intersection depending on the equations of the intersecting surfaces. These intersection issues have been explored in 3-space using various techniques [2, 10, 13, 19, 21, 22] and have been expanded and generalized to high dimensional spaces for the intersection of $(n-1)$ hypersurfaces in n-space [1, 3–9, 14, 15, 17, 20]. Recent research [11] has also investigated the parametric-parametric surface intersection problem in n-dimensional Euclidean space. However, parametric-implicit and implicit-implicit intersection problems of two surfaces in n-space have not been studied. The linear dependence or linear independence of the normal vectors of the surfaces determines whether

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these intersection problems involve tangential intersection or transversal intersection. Due to the fact that the normal space of a surface is constructed by $(n-2)$ vectors and the vector product is described for $(n-1)$ vectors in $\mathbb{E}^n$, it is more difficult to investigate these problems in n-space than in 3-space, even for the transversal intersection.

In this paper, we consider the transversal intersection problem of two implicit surfaces as well as of a parametric surface and an implicit surface in n-dimensional Euclidean space. We obtain the curvatures and Frenet vectors of the transversal intersection curve. Finally, we give applications for implicit-implicit intersection problem in $\mathbb{E}^4$ and for implicit-parametric intersection problem in $\mathbb{E}^6$.

2. Basic concepts

Definition 1. Let $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$. The vector product of the vectors $\mathbf{\Omega}_1 = \sum\limits_{j=1}^{n} \Omega_{1j}\mathbf{e}_j$, $\mathbf{\Omega}_2 = \sum\limits_{j=1}^{n} \Omega_{2j}\mathbf{e}_j$,..., $\mathbf{\Omega}_{n-1} = \sum\limits_{j=1}^{n} \Omega_{n-1,j}\mathbf{e}_j$ is determined by (see [16])

$$\mathbf{\Omega}_1 \times \mathbf{\Omega}_2 \times \cdots \times \mathbf{\Omega}_{n-1} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & \cdots & \Omega_{2n} \\ \vdots & \vdots & & \vdots \\ \Omega_{n-1,1} & \Omega_{n-1,2} & \cdots & \Omega_{n-1,n} \end{vmatrix},$$

and its norm is given by the formula (see [16])

$$||\mathbf{\Omega}_1 \times \mathbf{\Omega}_2 \times \cdots \times \mathbf{\Omega}_{n-1}|| = ||\mathbf{\Omega}_1|| \cdot ||\mathbf{\Omega}_2|| \dots ||\mathbf{\Omega}_{n-1}|| \cdot K,$$

where

$$K = \begin{vmatrix} 1 & \Upsilon_{12} & \cdots & \Upsilon_{1,n-1} \\ \Upsilon_{21} & 1 & \cdots & \Upsilon_{2,n-1} \\ \vdots & \vdots & & \vdots \\ \Upsilon_{n-1,1} & \Upsilon_{n-1,2} & \cdots & 1 \end{vmatrix}^{1/2}$$

with $\Upsilon_{ij} = \dfrac{\langle \mathbf{\Omega}_i, \mathbf{\Omega}_j \rangle}{||\mathbf{\Omega}_i|| \cdot ||\mathbf{\Omega}_j||}$ (the generalization of the vector product to the case of an n-dimensional Euclidean space is also given by [18]).

Let us consider a regular surface $\mathcal{M}$ in n-dimensional Euclidean space and let $\{\mathbf{V}_1, \mathbf{V}_2, ..., \mathbf{V}_n\}$ denote the Frenet frame of an arc-length parametrized curve $\beta : I \subset \mathbb{R} \to \mathcal{M}$. The Frenet formulas are given by [12]

$$\begin{aligned}
\mathbf{V}_1' &= k_1 \mathbf{V}_2, \\
\mathbf{V}_i' &= -k_{i-1}\mathbf{V}_{i-1} + k_i \mathbf{V}_{i+1}, \quad 2 \leq i \leq n-1, \\
\mathbf{V}_n' &= -k_{n-1}\mathbf{V}_{n-1},
\end{aligned}$$

where $k_i, 1 \leq i \leq n-1$, indicates the ith curvature of β. The higher order derivatives of β are given by

$$\begin{aligned}
&\beta' = \mathbf{V}_1, \\
&\beta'' = k_1 \mathbf{V}_2, \\
&\beta''' = -k_1^2 \mathbf{V}_1 + k_1' \mathbf{V}_2 + k_1 k_2 \mathbf{V}_3, \\
&\beta^{(4)} = -3k_1 k_1' \mathbf{V}_1 + \left(-k_1^3 + k_1'' - k_1 k_2^2\right) \mathbf{V}_2 + \left(2k_1' k_2 + k_1 k_2'\right) \mathbf{V}_3 + k_1 k_2 k_3 \mathbf{V}_4, \\
&\vdots \\
&\beta^{(n)} = \Big\{ ... \Big\} \mathbf{V}_1 + \Big\{ ... \Big\} \mathbf{V}_2 + ... + \Big\{ ... \Big\} \mathbf{V}_{n-1} + k_1 k_2 k_3 ... k_{n-1} \mathbf{V}_n.
\end{aligned}$$

In addition to these, if the surface $\mathcal{M}$ is defined by its implicit equation

$$\mathsf{G}_i(x_1, x_2, ..., x_n) = 0, \quad 1 \leq i \leq n-2,$$

then for the curve $\beta(s) = \big(x_1(s), x_2(s), ..., x_n(s)\big)$, we may write $\mathsf{G}_i(\beta(s)) = 0$ which yields

$$\Big\langle \nabla \mathsf{G}_i, \beta' \Big\rangle = 0, \tag{1}$$

$$\Big\langle \nabla \mathsf{G}_i, \beta'' \Big\rangle = -\Big\langle \nabla \mathsf{G}_i', \beta' \Big\rangle, \tag{2}$$

$$\Big\langle \nabla \mathsf{G}_i, \beta''' \Big\rangle = -2\Big\langle \nabla \mathsf{G}_i', \beta'' \Big\rangle - \Big\langle \nabla \mathsf{G}_i'', \beta' \Big\rangle, \tag{3}$$

$$\Big\langle \nabla \mathsf{G}_i, \beta^{(4)} \Big\rangle = -3\Big\langle \nabla \mathsf{G}_i', \beta''' \Big\rangle - 3\Big\langle \nabla \mathsf{G}_i'', \beta'' \Big\rangle - \Big\langle \nabla \mathsf{G}_i''', \beta' \Big\rangle, \tag{4}$$

$$\Big\langle \nabla \mathsf{G}_i, \beta^{(5)} \Big\rangle = -4\Big\langle \nabla \mathsf{G}_i', \beta^{(4)} \Big\rangle - 6\Big\langle \nabla \mathsf{G}_i'', \beta''' \Big\rangle - 4\Big\langle \nabla \mathsf{G}_i''', \beta'' \Big\rangle - \Big\langle \nabla \mathsf{G}_i^{(4)}, \beta' \Big\rangle \tag{5}$$

or in general

$$\Big\langle \nabla \mathsf{G}_i, \beta^{(r)} \Big\rangle = -\sum_{i=1}^{r-1} \binom{r-1}{i-1} \Big\langle \nabla \mathsf{G}_i^{(r-i)}, \beta^{(i)} \Big\rangle, \quad r \geq 2. \tag{6}$$

3. Transversal intersection curve of two surfaces in $\mathbb{E}^n$

3.1. Implicit-implicit surface intersection. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be transversally intersecting regular surfaces which are given by their implicit equations

$$\mathsf{G}_i\big(x_1, x_2, ..., x_n\big) = 0, \quad \mathsf{H}_i\big(x_1, x_2, ..., x_n\big) = 0, \quad 1 \leq i \leq n-2,$$

respectively, in $\mathbb{E}^n$. Then the normal spaces of $\mathcal{M}_1$ and $\mathcal{M}_2$ are spanned by $\big\{\nabla \mathsf{G}_1, \nabla \mathsf{G}_2, ..., \nabla \mathsf{G}_{n-2}\big\}$ and $\big\{\nabla \mathsf{H}_1, \nabla \mathsf{H}_2, ..., \nabla \mathsf{H}_{n-2}\big\}$, respectively. It is assumed that these surfaces intersect throughout a regular curve $\beta(s) = \big(x_1(s), x_2(s), ..., x_n(s)\big)$ with arc-length parameter s.

Let us consider the set of vector fields

$$\mathcal{S} = \left\{\nabla\mathsf{G}_1, \nabla\mathsf{G}_2, ..., \nabla\mathsf{G}_{n-2}, \nabla\mathsf{H}_1, \nabla\mathsf{H}_2, ..., \nabla\mathsf{H}_{n-2}\right\}$$

and let $\mathcal{D}$ be the dimension of the subspace $span\{\mathcal{S}\}$ at an intersection point $\beta(s)$.

i. If $\mathcal{D} = n$, then $\beta(s)$ is an isolated point.

ii. If $\mathcal{D} = n-2$, then the surfaces intersect tangentially at the point $\beta(s)$.

iii. If $\mathcal{D} = n-1$, then the surfaces intersect transversally at the point $\beta(s)$.

Since we focus on transversal intersection in our study, we assume that $\mathcal{D} = n-1$. Let $\{\mathbf{V}_1, \mathbf{V}_2, ..., \mathbf{V}_n\}$ represent the Frenet frame of β at the intersection point $P_0 = \beta(s)$. We suppose that $\left\{\nabla\mathsf{G}_1, \nabla\mathsf{G}_2, ..., \nabla\mathsf{G}_{n-2}, \nabla\mathsf{H}_\nu\right\}$, $\nu \in \{1, 2, ..., n-2\}$, is linearly independent at P_0.

3.1.1. *Tangent vector.* Since the unit tangent vector $\mathbf{V}_1$ of the intersection curve β is perpendicular to the vectors given in $\mathcal{S}$ and the set

$$\left\{\nabla\mathsf{G}_1, \nabla\mathsf{G}_2, ..., \nabla\mathsf{G}_{n-2}, \nabla\mathsf{H}_\nu\right\}, \quad 1 \leq \nu \leq n-2,$$

is linearly independent at P_0, the unit tangent vector can be obtained by

$$\mathbf{V}_1 = \frac{\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times ... \times \nabla\mathsf{G}_{n-2} \times \nabla\mathsf{H}_\nu}{\|\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times ... \times \nabla\mathsf{G}_{n-2} \times \nabla\mathsf{H}_\nu\|}. \tag{7}$$

Since β is unit-speed, it is clear that $\mathbf{V}_1 = \beta' = \left(x_1', x_2', ..., x_n'\right)$. Thus, $x_1', x_2', ..., x_n'$ can be found by using (7).

3.1.2. *Second order derivative vector.* For the second order derivative of β,

$$\beta'' = \xi_1\nabla\mathsf{G}_1 + \xi_2\nabla\mathsf{G}_2 + ... + \xi_{n-2}\nabla\mathsf{G}_{n-2} + \xi_{n-1}\nabla\mathsf{H}_\nu \tag{8}$$

can be written. To find β'', we must compute the scalars ξ_i, $i = 1, 2,...,$ $n-1$. By taking the inner product of both sides of (8) with $\nabla\mathsf{G}_1$, $\nabla\mathsf{G}_2$,..., $\nabla\mathsf{G}_{n-2}, \nabla\mathsf{H}_\nu$, respectively, we have

$$\begin{gathered}
\langle\nabla\mathsf{G}_1, \nabla\mathsf{G}_1\rangle\xi_1 + \langle\nabla\mathsf{G}_2, \nabla\mathsf{G}_1\rangle\xi_2 + ... + \langle\nabla\mathsf{G}_{n-2}, \nabla\mathsf{G}_1\rangle\xi_{n-2} + \langle\nabla\mathsf{H}_\nu, \nabla\mathsf{G}_1\rangle\xi_{n-1} = \langle\beta'', \nabla\mathsf{G}_1\rangle \\
\langle\nabla\mathsf{G}_1, \nabla\mathsf{G}_2\rangle\xi_1 + \langle\nabla\mathsf{G}_2, \nabla\mathsf{G}_2\rangle\xi_2 + ... + \langle\nabla\mathsf{G}_{n-2}, \nabla\mathsf{G}_2\rangle\xi_{n-2} + \langle\nabla\mathsf{H}_\nu, \nabla\mathsf{G}_2\rangle\xi_{n-1} = \langle\beta'', \nabla\mathsf{G}_2\rangle \\
\vdots \\
\langle\nabla\mathsf{G}_1, \nabla\mathsf{G}_{n-2}\rangle\xi_1 + \langle\nabla\mathsf{G}_2, \nabla\mathsf{G}_{n-2}\rangle\xi_2 + ... + \langle\nabla\mathsf{H}_\nu, \nabla\mathsf{G}_{n-2}\rangle\xi_{n-1} = \langle\beta'', \nabla\mathsf{G}_{n-2}\rangle \\
\langle\nabla\mathsf{G}_1, \nabla\mathsf{H}_\nu\rangle\xi_1 + \langle\nabla\mathsf{G}_2, \nabla\mathsf{H}_\nu\rangle\xi_2 + \cdots + \langle\nabla\mathsf{G}_{n-2}, \nabla\mathsf{H}_\nu\rangle\xi_{n-2} + \langle\nabla\mathsf{H}_\nu, \nabla\mathsf{H}_\nu\rangle\xi_{n-1} = \langle\beta'', \nabla\mathsf{H}_\nu\rangle.
\end{gathered}$$

The coefficient matrix $\mathcal{R}$ of above linear equation system has nonzero determinant. Then, by applying Cramer's method, we get

$$\xi_i = \det\left(\mathcal{R}^{-1}\right)\left\{\sum_{j=1}^{n-2}\left\langle \beta'', \nabla \mathsf{G}_j\right\rangle \mathcal{R}_{ji} + \left\langle \beta'', \nabla \mathsf{H}_\nu\right\rangle \mathcal{R}_{(n-1)i}\right\}, \tag{9}$$

where $\mathcal{R}_{ji}$ indicates the (j,i) cofactor of the matrix $\mathcal{R}$, $1 \leq i \leq n-1$. Here $\left\langle \beta'', \nabla \mathsf{H}_\nu\right\rangle$ and $\left\langle \beta'', \nabla \mathsf{G}_j\right\rangle$ can be computed by using (2). Then, substitution of ξ_i into (8) gives β''. Thus, $x_1'', x_2'', ..., x_n''$ can be obtained by using β''. Besides, using the Gram–Schmidt orthogonalization method, we get

$$\mathbf{U}_2 = \beta'' - \left\langle \beta'', \mathbf{V}_1\right\rangle \mathbf{V}_1, \qquad \mathbf{V}_2 = \frac{\mathbf{U}_2}{\|\mathbf{U}_2\|}. \tag{10}$$

Hence, the first curvature of the intersection curve at the intersection point P_0 is determined by using the formula $k_1 = \langle \beta'', \mathbf{V}_2\rangle$.

3.1.3. *Third order derivative vector.* We may write

$$\begin{aligned}\beta''' &= -k_1^2\mathbf{V}_1 + k_1'\mathbf{V}_2 + k_1k_2\mathbf{V}_3\\ &= -k_1^2\mathbf{V}_1 + \rho_1\nabla\mathsf{G}_1 + \rho_2\nabla\mathsf{G}_2 + ... + \rho_{n-2}\nabla\mathsf{G}_{n-2} + \rho_{n-1}\nabla\mathsf{H}_\nu. \end{aligned}\tag{11}$$

Likewise, if we take the dot product of both sides of (11) with $\nabla\mathsf{G}_1, \nabla\mathsf{G}_2, ...,$ $\nabla\mathsf{G}_{n-2}$, $\nabla\mathsf{H}_\nu$, respectively, then we get a system of linear equations depending on ρ_i, $i = 1, 2, ..., n-1$. The solution of this system is given by

$$\rho_i = \det\left(\mathcal{R}^{-1}\right)\left\{\sum_{j=1}^{n-2}\left\langle \beta''', \nabla \mathsf{G}_j\right\rangle \mathcal{R}_{ji} + \left\langle \beta''', \nabla \mathsf{H}_\nu\right\rangle \mathcal{R}_{(n-1)i}\right\}, \tag{12}$$

$1 \leq i \leq n-1$, where $\left\langle \beta''', \nabla \mathsf{H}_\nu\right\rangle$ and $\left\langle \beta''', \nabla \mathsf{G}_j\right\rangle$ can be computed by using (3). Then, if we substitute ρ_i into (11), we find β'''. Thus, $x_1''', x_2''', ..., x_n'''$ can be obtained by using β'''. Besides, by applying the Gram–Schmidt orthogonalization method, we obtain

$$\mathbf{U}_3 = \beta''' - \left\langle \beta''', \mathbf{V}_1\right\rangle \mathbf{V}_1 - \left\langle \beta''', \mathbf{V}_2\right\rangle \mathbf{V}_2, \qquad \mathbf{V}_3 = \frac{\mathbf{U}_3}{\|\mathbf{U}_3\|}. \tag{13}$$

Also, if $k_1 \neq 0$, then the second curvature of the intersection curve at P_0 is obtained by $k_2 = \dfrac{\left\langle \beta''', \mathbf{V}_3\right\rangle}{k_1}$ and $k_1' = \left\langle \beta''', \mathbf{V}_2\right\rangle$.

3.1.4. *Higher order derivative vector.* Similarly, we may write the rth order derivative vector of the intersection curve as

$$\begin{aligned}\beta^{(r)} &= c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + ... + c_{r-1}\mathbf{V}_{r-1} + k_1k_2k_3...k_{r-1}\mathbf{V}_r\\ &= c_1\mathbf{V}_1 + \sigma_1\nabla\mathsf{G}_1 + \sigma_2\nabla\mathsf{G}_2 + ... + \sigma_{n-2}\nabla\mathsf{G}_{n-2} + \sigma_{n-1}\nabla\mathsf{H}_\nu, \end{aligned}\tag{14}$$

where

$$\sigma_i = \det\left(\mathcal{R}^{-1}\right)\left\{\sum_{j=1}^{n-2}\left\langle\beta^{(r)}, \nabla\mathsf{G}_j\right\rangle\mathcal{R}_{ji} + \left\langle\beta^{(r)}, \nabla\mathsf{H}_\nu\right\rangle\mathcal{R}_{(n-1)i}\right\}, \tag{15}$$

$1 \leq i \leq n-1$. Here $\left\langle\beta^{(r)}, \nabla\mathsf{H}_\nu\right\rangle$ and $\left\langle\beta^{(r)}, \nabla\mathsf{G}_j\right\rangle$ can be computed by using (6). While calculating the derivative vector $\beta^{(r)}$ given by (14), it is difficult to calculate the coefficient c_1 in high dimensional spaces. Therefore, this coefficient can be calculated using the MATLAB code given in [11]. Thus we get $\beta^{(r)}$ and $x_1^{(r)}, x_2^{(r)}, ..., x_n^{(r)}$. Besides, using the Gram–Schmidt orthogonalization method, we get

$$\mathbf{U}_r = \beta^{(r)} - \sum_{i=1}^{r-1}\left\langle\beta^{(r)}, \mathbf{V}_i\right\rangle\mathbf{V}_i, \qquad \mathbf{V}_r = \frac{\mathbf{U}_r}{\|\mathbf{U}_r\|}, \quad 4 \leq r \leq n-1, \tag{16}$$

and $\mathbf{V}_n = \mathbf{V}_1 \times \mathbf{V}_2 \times \cdots \times \mathbf{V}_{n-1}$. If $k_i \neq 0, i \geq 2$, is assumed, then the curvatures are derived by

$$k_{r-1} = \frac{\left\langle\beta^{(r)}, \mathbf{V}_r\right\rangle}{k_1 k_2 ... k_{r-2}}, \quad 4 \leq r \leq n. \tag{17}$$

3.2. Parametric-implicit surface intersection. Let $\mathcal{M}_1$ be a regular surface with its implicit equation $\mathsf{G}_i\left(x_1, x_2, ..., x_n\right) = 0$, $i = 1, 2, ..., n-2$, and $\mathcal{M}_2$ be a regular surface given by its parametric equation $X\left(u, v\right) = \left(g_1\left(u, v\right), g_2\left(u, v\right), ..., g_n\left(u, v\right)\right)$ in $\mathbb{E}^n$. It is obvious that the normal space of the surface $\mathcal{M}_1$ is spanned by $\left\{\nabla\mathsf{G}_1, \nabla\mathsf{G}_2, ..., \nabla\mathsf{G}_{n-2}\right\}$. On the other hand, as expressed in [11], the normal space $\left\{\mathbf{N}_1, \mathbf{N}_2, ..., \mathbf{N}_{n-2}\right\}$ of the surface $\mathcal{M}_2$ can be obtained by

$$\begin{aligned}
\mathbf{N}_1 \quad &= X_u \times X_v \times \mathbf{e}_1 \times \mathbf{e}_2 \times ... \times \mathbf{e}_{n-3}, \\
\mathbf{N}_2 \quad &= X_u \times X_v \times \mathbf{N}_1 \times \mathbf{e}_1 \times ... \times \mathbf{e}_{n-4}, \\
\mathbf{N}_3 \quad &= X_u \times X_v \times \mathbf{N}_1 \times \mathbf{N}_2 \times \mathbf{e}_1 \times ... \times \mathbf{e}_{n-5}, \\
&\vdots \\
\mathbf{N}_{n-2} \quad &= X_u \times X_v \times \mathbf{N}_1 \times \mathbf{N}_2 \times ... \times \mathbf{N}_{n-3}.
\end{aligned}$$

Let us consider the set $\left\{\nabla\mathsf{G}_1, \nabla\mathsf{G}_2, ..., \nabla\mathsf{G}_{n-2}, \mathbf{N}_1, \mathbf{N}_2, ..., \mathbf{N}_{n-2}\right\}$. Since we consider transversal intersection at an intersection point P_0, we assume that the dimension of the subspace spanned by this set is $n-1$. For this purpose, let $\left\{\nabla\mathsf{G}_1, \nabla\mathsf{G}_2, ..., \nabla\mathsf{G}_{n-2}, \mathbf{N}_\mathfrak{q}\right\}$, $\mathfrak{q} \in \{1, 2, ..., n-2\}$, be linearly independent at P_0. Then the unit tangent vector can be obtained via

$$\mathbf{V}_1 = \frac{\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times ... \times \nabla\mathsf{G}_{n-2} \times \mathbf{N}_\mathfrak{q}}{\|\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times ... \times \nabla\mathsf{G}_{n-2} \times \mathbf{N}_\mathfrak{q}\|}. \tag{18}$$

By taking the dot product of both sides of $\mathbf{V}_1 = X_u u' + X_v v'$ with X_u and X_v, respectively, yields (see [11])

$$u' = \frac{\mathcal{G}\langle \mathbf{V}_1, X_u\rangle - \mathcal{F}\langle \mathbf{V}_1, X_v\rangle}{\mathcal{E}\mathcal{G} - \mathcal{F}^2}, \quad v' = \frac{\mathcal{E}\langle \mathbf{V}_1, X_v\rangle - \mathcal{F}\langle \mathbf{V}_1, X_u\rangle}{\mathcal{E}\mathcal{G} - \mathcal{F}^2},$$

where $\mathcal{E}, \mathcal{F}, \mathcal{G}$ are the coefficients of the first fundamental form of $\mathcal{M}_2$. Also, by using (18), we get $x_1', x_2', ..., x_n'$. It is possible to write

$$\beta'' = \lambda_1 \nabla \mathsf{G}_1 + \lambda_2 \nabla \mathsf{G}_2 + ... + \lambda_{n-2} \nabla \mathsf{G}_{n-2} + \lambda_{n-1} \mathbf{N}_{\mathsf{q}} \tag{19}$$

for the second order derivative vector. Taking the dot product of both sides of (19) with $\nabla \mathsf{G}_1, \nabla \mathsf{G}_2, ..., \nabla \mathsf{G}_{n-2}, \mathbf{N}_{\mathsf{q}}$, respectively, gives us a system of linear equations whose solution is

$$\lambda_i = \frac{1}{\det \mathcal{R}} \left\{ \langle \beta'', \mathbf{N}_{\mathsf{q}} \rangle \mathcal{R}_{(n-1)i} + \sum_{j=1}^{n-2} \langle \beta'', \nabla \mathsf{G}_j \rangle \mathcal{R}_{ji} \right\}, \quad 1 \le i \le n-1, \tag{20}$$

where $\mathcal{R}_{ji}$ represents the (j, i) cofactor of the coefficient matrix $\mathcal{R}$. By using (20), we can find $x_1'', x_2'', ..., x_n''$ and u'', v''. Then we get

$$\mathbf{W}_2 = \beta'' - \langle \beta'', \mathbf{V}_1 \rangle \mathbf{V}_1, \qquad \mathbf{V}_2 = \frac{\mathbf{W}_2}{\|\mathbf{W}_2\|}, \tag{21}$$

and $k_1 = \langle \beta'', \mathbf{V}_2 \rangle$.

If we continue in the same manner, in order to get the rth order derivative of the intersection curve, we may write

$$\begin{aligned} \beta^{(r)} &= \ell_1 \mathbf{V}_1 + \ell_2 \mathbf{V}_2 + ... + \ell_{r-1} \mathbf{V}_{r-1} + k_1 k_2 k_3 ... k_{r-1} \mathbf{V}_r \\ &= \ell_1 \mathbf{V}_1 + \mu_1 \nabla \mathsf{G}_1 + \mu_2 \nabla \mathsf{G}_2 + ... + \mu_{n-2} \nabla \mathsf{G}_{n-2} + \mu_{n-1} \mathbf{N}_{\mathsf{q}}, \end{aligned} \tag{22}$$

where

$$\mu_i = \frac{1}{\det \mathcal{R}} \left\{ \langle \beta^{(r)}, \mathbf{N}_{\mathsf{q}} \rangle \mathcal{R}_{(n-1)i} + \sum_{j=1}^{n-2} \langle \beta^{(r)}, \nabla \mathsf{G}_j \rangle \mathcal{R}_{ji} \right\}, \quad 1 \le i \le n-1. \tag{23}$$

Using Gram–Schmidt orthogonalization method, we get

$$\mathbf{W}_r = \beta^{(r)} - \sum_{i=1}^{r-1} \langle \beta^{(r)}, \mathbf{V}_i \rangle \mathbf{V}_i, \qquad \mathbf{V}_r = \frac{\mathbf{W}_r}{\|\mathbf{W}_r\|}, \quad 3 \le r \le n-1. \tag{24}$$

Finally, $\mathbf{V}_n = \mathbf{V}_1 \times \mathbf{V}_2 \times \cdots \times \mathbf{V}_{n-1}$ gives the last Frenet vector $\mathbf{V}_n$. Consequenty, if $k_i \neq 0, i \ge 2$, is assumed, the curvatures of the intersection curve are derived by using

$$k_{r-1} = \frac{\langle \beta^{(r)}, \mathbf{V}_r \rangle}{k_1 k_2 ... k_{r-2}}, \quad 3 \le r \le n. \tag{25}$$

4. Illustrative examples

4.1. Example of implicit-implicit surface intersection in $\mathbb{E}^4$. Let us apply our method to the intersection of the torus surface $\mathcal{M}_1$ given by

$$\mathsf{G}_1\big(x_1, x_2, x_3, , x_4\big) = x_1^2 + x_2^2 = 2, \quad \mathsf{G}_2\big(x_1, x_2, x_3, , x_4\big) = x_3^2 + x_4^2 = 4$$

and the torus surface $\mathcal{M}_2$ given by

$$\mathsf{H}_1\big(x_1, x_2, x_3, , x_4\big) = x_2^2 + x_4^2 = 3, \quad \mathsf{H}_2\big(x_1, x_2, x_3, , x_4\big) = x_1^2 + x_3^2 = 3$$

in $\mathbb{E}^4$.

Since $\nabla\mathsf{G}_1 = \big(2x_1, 2x_2, 0, 0\big)$, $\nabla\mathsf{G}_2 = \big(0, 0, 2x_3, 2x_4\big)$, $\nabla\mathsf{H}_1 = \big(0, 2x_2, 0, 2x_4\big)$, $\nabla\mathsf{H}_2 = \big(2x_1, 0, 2x_3, 0\big)$, we have

$$\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times \nabla\mathsf{H}_1 = \big(8x_2x_3x_4, -8x_1x_3x_4, -8x_1x_2x_4, 8x_1x_2x_3\big),$$

where $\|\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times \nabla\mathsf{H}_1\| = 8\sqrt{2x_3^2x_4^2 + 4x_1^2x_2^2}$. Thus, it is clear that these surfaces intersect transversally at the points with nonzero components. Then, the tangent vector is obtained as

$$\begin{aligned}\mathbf{V}_1 &= \frac{\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times \nabla\mathsf{H}_1}{\|\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times \nabla\mathsf{H}_1\|} \\ &= \frac{1}{\sqrt{2x_3^2x_4^2 + 4x_1^2x_2^2}}\Big(x_2x_3x_4, -x_1x_3x_4, -x_1x_2x_4, x_1x_2x_3\Big),\end{aligned}$$

which yields
$x_1' = \dfrac{x_2x_3x_4}{\sqrt{2x_3^2x_4^2 + 4x_1^2x_2^2}}$, $\quad x_2' = \dfrac{-x_1x_3x_4}{\sqrt{2x_3^2x_4^2 + 4x_1^2x_2^2}}$, $\quad x_3' = \dfrac{-x_1x_2x_4}{\sqrt{2x_3^2x_4^2 + 4x_1^2x_2^2}}$, and $x_4' = \dfrac{x_1x_2x_3}{\sqrt{2x_3^2x_4^2 + 4x_1^2x_2^2}}$.

For the second order derivative vector, we may write

$$\beta'' = \xi_1\nabla\mathsf{G}_1 + \xi_2\nabla\mathsf{G}_2 + \xi_3\nabla\mathsf{H}_1.$$

If we use (9) and (2), we get

$$\begin{aligned}\xi_1 &= \frac{32}{\det\mathcal{R}(x_3^2x_4^2 + 2x_1^2x_2^2)}\big(x_3^2x_4^6 - 12x_3^2x_4^2 - 2x_1^2x_2^4x_4^2 + 6x_1^2x_2^2x_3^2\big), \\ \xi_2 &= \frac{32}{\det\mathcal{R}(x_3^2x_4^2 + 2x_1^2x_2^2)}\big(2x_1^2x_2^6 - 12x_1^2x_2^2 - x_2^2x_3^2x_4^2 + 3x_1^2x_3^2x_4^2\big), \\ \xi_3 &= \frac{128}{\det\mathcal{R}(x_3^2x_4^2 + 2x_1^2x_2^2)}\big(3x_2^2x_4^2 - 3x_1^2x_3^2\big),\end{aligned}$$

where $\det\mathcal{R} = 128(12 - 2x_2^4 - x_4^4)$. Thus, by substituting these coefficients into β'', we have

$$\beta'' = 2\Big(\xi_1x_1, (\xi_1 + \xi_3)x_2, \xi_2x_3, (\xi_2 + \xi_3)x_4\Big).$$

This gives $x_1''=2\xi_1x_1$, $\quad x_2''=2(\xi_1+\xi_3)x_2$, $\quad x_3''=2\xi_2x_3$, $\quad x_4''=2(\xi_2+\xi_3)x_4$. Then, if we use (10), we obtain

$$\mathbf{V}_2=\frac{1}{\sqrt{2\xi_1^2+4\xi_2^2+3\xi_3^2+2\xi_3(\xi_1x_2^2+\xi_2x_4^3)}}\Big(\xi_1x_1,(\xi_1+\xi_3)x_2,\xi_2x_3,(\xi_2+\xi_3)x_4\Big).$$

Thus, the first curvature of the intersection curve is calculated by

$$k_1=\langle\beta'',\mathbf{V}_2\rangle=\frac{2(\xi_1x_1)^2+2(\xi_1+\xi_3)^2x_2^2+2(\xi_2x_3)^2+2(\xi_2+\xi_3)^2x_4^2}{\sqrt{2\xi_1^2+4\xi_2^2+3\xi_3^2+4\xi_3(\xi_1x_2^2+\xi_2x_4^3)}}.$$

For the third order derivative vector, we may write

$$\beta'''=-k_1^2\mathbf{V}_1+\rho_1\nabla\mathsf{G}_1+\rho_2\nabla\mathsf{G}_2+\rho_3\nabla\mathsf{H}_1.$$

If we use (12), we find

$$\begin{aligned}
\rho_1&=\frac{192x_1x_2x_3x_4}{\det\mathcal{R}\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}}\big(12\xi_3-\xi_3x_4^4-\xi_3x_2^2x_4^2-4(\xi_2-\xi_1)x_2^2\big),\\
\rho_2&=\frac{192x_1x_2x_3x_4}{\det\mathcal{R}\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}}\big(\xi_3x_2^2x_4^2-6\xi_3+\xi_3x_2^4-2(\xi_2-\xi_1)x_4^2\big),\\
\rho_3&=\frac{384x_1x_2x_3x_4}{\det\mathcal{R}\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}}\big(-2\xi_3x_2^2+\xi_3x_4^2+4(\xi_2-\xi_1)\big).
\end{aligned}$$

Then, we get

$$\begin{aligned}
\beta'''=&2\Big(\rho_1x_1,(\rho_1+\rho_3)x_2,\rho_2x_3,(\rho_2+\rho_3)x_4\Big)\\
&-\frac{k_1^2}{\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}}\Big(x_2x_3x_4,-x_1x_3x_4,-x_1x_2x_4,x_1x_2x_3\Big).
\end{aligned}$$

This yields

$$\begin{aligned}
x_1'''&=2\rho_1x_1-\frac{k_1^2x_2x_3x_4}{\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}}, & x_2'''&=2(\rho_1+\rho_3)x_2+\frac{k_1^2x_1x_3x_4}{\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}},\\
x_3'''&=2\rho_2x_3+\frac{k_1^2x_1x_2x_4}{\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}}, & x_4'''&=2(\rho_2+\rho_3)x_4-\frac{k_1^2x_1x_2x_3}{\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}}.
\end{aligned}$$

From (13), since $\mathbf{U}_3=\beta'''-\Gamma_1\mathbf{V}_1-\Gamma_2\mathbf{V}_2$, we obtain

$$\begin{aligned}
\mathbf{U}_3=&2\Big(\rho_1x_1,(\rho_1+\rho_3)x_2,\rho_2x_3,(\rho_2+\rho_3)x_4\Big)\\
&-\frac{\Gamma_2}{\sqrt{2\xi_1^2+4\xi_2^2+3\xi_3^2+2\xi_3(\xi_1x_2^2+\xi_2x_4^3)}}\Big(\xi_1x_1,(\xi_1+\xi_3)x_2,\xi_2x_3,(\xi_2+\xi_3)x_4\Big),
\end{aligned}$$

where $\Gamma_1=-k_1^2$ and

$$\Gamma_2=\frac{4\xi_1\rho_1+8\xi_2\rho_2+6\xi_3\rho_3+2(\xi_1\rho_3+\xi_3\rho_1)x_2^2+2(\xi_2\rho_3+\xi_3\rho_2)x_4^2}{\sqrt{2\xi_1^2+4\xi_2^2+3\xi_3^2+2\xi_3(\xi_1x_2^2+\xi_2x_4^3)}}.$$

Since

$$||\mathbf{U}_3|| = \sqrt{4\left\{2\rho_1^2+4\rho_2^2+3\rho_3^2+2\rho_3(\rho_1x_2^2+\rho_2x_4^3)\right\}-(\Gamma_2)^2},$$

we get the third Frenet vector and the second curvature of the intersection curve by using

$$\begin{aligned}\mathbf{V}_3 =&\frac{1}{||\mathbf{U}_3||}\Big\{2\Big(\rho_1x_1,(\rho_1+\rho_3)x_2,\rho_2x_3,(\rho_2+\rho_3)x_4\Big)\\&-\frac{\Gamma_2}{\sqrt{2\xi_1^2+4\xi_2^2+3\xi_3^2+2\xi_3(\xi_1x_2^2+\xi_2x_4^3)}}\Big(\xi_1x_1,(\xi_1+\xi_3)x_2,\xi_2x_3,(\xi_2+\xi_3)x_4\Big)\Big\},\end{aligned}$$

and

$$k_2=\frac{\big\langle\beta''',\mathbf{V}_3\big\rangle}{k_1},$$

respectively. We also obtain $k_1'=\big\langle\beta''',\mathbf{V}_2\big\rangle=\Gamma_2$.

Similarly, the fourth order derivative can be written as

$$\beta^{(4)}=-3k_1k_1'\mathbf{V}_1+\sigma_1\nabla\mathsf{G}_1+\sigma_2\nabla\mathsf{G}_2+\sigma_3\nabla\mathsf{H}_1,$$

where

$$\begin{aligned}\sigma_1=\frac{1}{\det\mathcal{R}}\Bigg\{&(192-16x_4^2)\Bigg[16x_3x_4\left(\frac{\rho_3x_1x_2}{\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}}+\frac{k_1^2x_3x_4}{2x_3^2x_4^2+4x_1^2x_2^2}\right)\\&\qquad-24\xi_1^2x_1^2-24(\xi_1+\xi_3)^2x_2^2\Bigg]\\&-16x_2^2x_4^2\Bigg[16x_1x_2\left(\frac{\rho_3x_3x_4}{\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}}+\frac{2k_1^2x_1x_2}{2x_3^2x_4^2+4x_1^2x_2^2}\right)\\&\qquad+24\xi_2^2x_3^2+24(\xi_2+\xi_3)^2x_4^2\Bigg]\\&-64x_2^2\Bigg[8x_1x_3\left(\frac{2(\rho_1-\rho_2)x_2x_4}{\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}}+\frac{3k_1^2x_1x_3}{2x_3^2x_4^2+4x_1^2x_2^2}\right)\\&\qquad-24(\xi_1^2+\xi_3)^2x_2^2-24(\xi_2+\xi_3)^2x_4^2\Bigg]\Bigg\},\end{aligned}$$

$$\begin{aligned}\sigma_2=\frac{1}{\det\mathcal{R}}\Bigg\{&16x_2^2x_4^2\Bigg[16x_3x_4\left(\frac{\rho_3x_1x_2}{\sqrt{2x_3^2x_4^2+4x_1^2x_2^2}}+\frac{k_1^2x_3x_4}{2x_3^2x_4^2+4x_1^2x_2^2}\right)\\&\qquad-24\xi_1^2x_1^2-24(\xi_1+\xi_3)^2x_2^2\Bigg]\end{aligned}$$

$$
\begin{aligned}
&- (96 - 16x_2^4)\left[16x_1x_2\left(\frac{\rho_3x_3x_4}{\sqrt{2x_3^2x_4^2 + 4x_1^2x_2^2}} + \frac{2k_1^2x_1x_2}{2x_3^2x_4^2 + 4x_1^2x_2^2}\right)\right. \\
&\qquad \left. + 24\xi_2^2x_3^2 + 24(\xi_2 + \xi_3)^2x_4^2\right] \\
&- 32x_4^2\left[8x_1x_3\left(\frac{2(\rho_1 - \rho_2)x_2x_4}{\sqrt{2x_3^2x_4^2 + 4x_1^2x_2^2}} + \frac{3k_1^2x_1x_3}{2x_3^2x_4^2 + 4x_1^2x_2^2}\right)\right. \\
&\qquad \left.\left. - 24(\xi_1^2 + \xi_3)^2x_2^2 - 24(\xi_2 + \xi_3)^2x_4^2\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
\sigma_3 = \frac{1}{\det\mathcal{R}}\Bigg\{ &- 64x_2^2\left[16x_3x_4\left(\frac{\rho_3x_1x_2}{\sqrt{2x_3^2x_4^2 + 4x_1^2x_2^2}} + \frac{k_1^2x_3x_4}{2x_3^2x_4^2 + 4x_1^2x_2^2}\right)\right. \\
&\qquad \left. - 24\xi_1^2x_1^2 - 24(\xi_1 + \xi_3)^2x_2^2\right] \\
&+ 32x_4^2\left[16x_1x_2\left(\frac{\rho_3x_3x_4}{\sqrt{2x_3^2x_4^2 + 4x_1^2x_2^2}} + \frac{2k_1^2x_1x_2}{2x_3^2x_4^2 + 4x_1^2x_2^2}\right)\right. \\
&\qquad \left. + 24\xi_2^2x_3^2 + 24(\xi_2 + \xi_3)^2x_4^2\right] \\
&+ 128\left[8x_1x_3\left(\frac{2(\rho_1 - \rho_2)x_2x_4}{\sqrt{2x_3^2x_4^2 + 4x_1^2x_2^2}} + \frac{3k_1^2x_1x_3}{2x_3^2x_4^2 + 4x_1^2x_2^2}\right)\right. \\
&\qquad \left. - 24(\xi_1^2 + \xi_3)^2x_2^2 - 24(\xi_2 + \xi_3)^2x_4^2\right]\Bigg\}.
\end{aligned}
$$

Hence

$$\beta^{(4)} = -3k_1\Gamma_2\mathbf{V}_1 + 2\Big(\sigma_1x_1, (\sigma_1 + \sigma_3)x_2, \sigma_2x_3, (\sigma_2 + \sigma_3)x_4\Big)$$

Finally, the fourth Frenet vector $\mathbf{V}_4$ is obtained by $\mathbf{V}_4 = \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$ and the third curvature function is obtained by using $k_3 = \frac{\langle\beta^{(4)}, \mathbf{V}_4\rangle}{k_1k_2}$.

If we use the obtained formulas for the intersection point $P = \big(1, 1, \sqrt{2}, \sqrt{2}\big)$ (Figure 1), we find the Frenet vectors of the intersection curve as

$$\mathbf{V}_1 = \Big(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\Big), \quad \mathbf{V}_2 = \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{\sqrt{2}}{6}, -\frac{\sqrt{2}}{6}\right)$$

$$\mathbf{V}_3 = \left(\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{6}\right), \quad \mathbf{V}_4 = \left(-\frac{\sqrt{2}}{6}, -\frac{\sqrt{2}}{6}, \frac{2}{3}, \frac{2}{3}\right).$$

The curvatures of the intersection curve at P are

$$k_1 = \frac{1}{2}, \quad k_2 = \frac{\sqrt{2}}{3}, \quad k_3 = \frac{5}{2}.$$

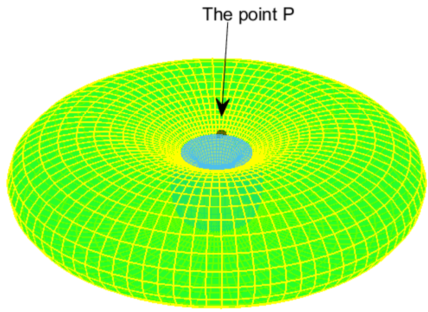

FIGURE 1. Perspective projections of the intersecting surfaces $\mathcal{M}_1$ and $\mathcal{M}_2$

4.2. Example of parametric-implicit surface intersection in $\mathbb{E}^6$. Let us consider the intersection of the surface $\mathcal{M}_1$ given by

$$\begin{aligned} \mathsf{G}_1(x_1, x_2, ..., x_6) &= x_2 - x_3 = 0 \\ \mathsf{G}_2(x_1, x_2, ..., x_6) &= x_1 - x_3^5 = 0 \\ \mathsf{G}_3(x_1, x_2, ..., x_6) &= x_5 - x_4^2 = 0 \\ \mathsf{G}_4(x_1, x_2, ..., x_6) &= x_6 - x_2^3 = 0 \end{aligned}$$

and the surface $\mathcal{M}_2$ given by $X(u, v) = (u^5, u, v, v^2, u^4, u^3)$ in $\mathbb{E}^6$.

The normal space of the surface $\mathcal{M}_1$ is spanned by

$$\begin{aligned} \{\nabla\mathsf{G}_1 &= (0, 1, -1, 0, 0, 0), & \nabla\mathsf{G}_2 &= (1, 0, -5x_3^4, 0, 0, 0), \\ \nabla\mathsf{G}_3 &= (0, 0, 0, -2x_4, 1, 0), & \nabla\mathsf{G}_4 &= (0, -3x_2^2, 0, 0, 0, 1)\}. \end{aligned}$$

The tangent vector fields of the surface $\mathcal{M}_2$ are $X_u = (5u^4, 1, 0, 0, 4u^3, 3u^2)$, $X_v = (0, 0, 1, 2v, 0, 0)$. Then, since $\{X_u, X_v, \mathbf{e}_1, \mathbf{e}_5, \mathbf{e}_6\}$ is linearly independent, we obtain the basis vectors of the normal space of $\mathcal{M}_2$ as [11]

$$
\begin{aligned}
\mathbf{N}_1 =& X_u \times X_v \times \mathbf{e}_1 \times \mathbf{e}_5 \times \mathbf{e}_6,\\
=& \big(0,0,2v,-1,0,0\big),\\
\mathbf{N}_2 =& X_u \times X_v \times \mathbf{N}_1 \times \mathbf{e}_1 \times \mathbf{e}_5,\\
=& \big(0,-16u^3v^2-4u^3,0,0,4v^2+1,0\big),\\
\mathbf{N}_3 =& X_u \times X_v \times \mathbf{N}_1 \times \mathbf{N}_2 \times \mathbf{e}_1,\\
=& (4v^2+1)\Big(0,-12u^2v^2-3u^2,0,0,-48u^5v^2-12u^5,64u^6v^2+16u^6+4v^2+1\Big),\\
\mathbf{N}_4 =& X_u \times X_v \times \mathbf{N}_1 \times \mathbf{N}_2 \times \mathbf{N}_3\\
=& (4v^2+1)^4\Big(-256u^{12}-144u^{10}-32u^6-36u^4-1,80u^{10}+5u^4,0,0,\\
& \qquad 320u^{13}+20u^7,240u^{12}+15u^6\Big).
\end{aligned}
$$

Also, we have

$$
\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times \nabla\mathsf{G}_3 \times \nabla\mathsf{G}_4 \times \mathbf{N}_1 = \big(5x_3^4,1,1,2v,4vx_4,3x_2^2\big)
$$

which means that the set $\big\{\nabla\mathsf{G}_1,\nabla\mathsf{G}_2,\nabla\mathsf{G}_3,\nabla\mathsf{G}_4,\mathbf{N}_1\big\}$ is linearly independent along the intersection curve. Thus, we compute the unit tangent vector field of the intersection curve as

$$
\mathbf{V}_1 = \frac{\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times \nabla\mathsf{G}_3 \times \nabla\mathsf{G}_4 \times \mathbf{N}_1}{\|\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times \nabla\mathsf{G}_3 \times \nabla\mathsf{G}_4 \times \mathbf{N}_1\|} = \frac{1}{\gamma}\big(5x_3^4,1,1,2v,4vx_4,3x_2^2\big),
$$

where

$$
\gamma = \|\nabla\mathsf{G}_1 \times \nabla\mathsf{G}_2 \times \nabla\mathsf{G}_3 \times \nabla\mathsf{G}_4 \times \mathbf{N}_1\| = \sqrt{2+4v^2+16v^2x_4^2+9x_2^4+25x_3^8}.
$$

So, we have $x_1' = \dfrac{5x_3^4}{\gamma}$, $x_2' = \dfrac{1}{\gamma}$, $x_3' = \dfrac{1}{\gamma}$, $x_4' = \dfrac{2v}{\gamma}$, $x_5' = \dfrac{4vx_4}{\gamma}$, $x_6' = \dfrac{3x_2^2}{\gamma}$, $u' = \dfrac{25u^4x_3^4+16u^3vx_4+9u^2x_2^2+1}{\gamma(25u^8+16u^6+9u^4+1)}$, $v' = \dfrac{1}{\gamma}$.

The second order partial derivatives of the surface $\mathcal{M}_2$ are obtained as $X_{uu} = \big(20u^3,0,0,0,12u^2,6u\big)$, $X_{uv} = \big(0,0,0,0,0,0\big)$, $X_{vv} = \big(0,0,0,2,0,0\big)$.

For the second order derivative of the intersection curve, we may write

$$
\begin{aligned}
\beta'' =& \lambda_1\nabla\mathsf{G}_1 + \lambda_2\nabla\mathsf{G}_2 + \lambda_3\nabla\mathsf{G}_3 + \lambda_4\nabla\mathsf{G}_4 + \lambda_5\mathbf{N}_1\\
=& \big(\lambda_2,\lambda_1-3x_2^2\lambda_4,2v\lambda_5-\lambda_1-5x_3^4\lambda_2,-2x_4\lambda_3-\lambda_5,\lambda_3,\lambda_4\big),
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_1 =& \frac{-2}{\gamma^4}\Big\{(9x_2^4+1)\big(10x_3^7+16v^3x_4+2v(4x_4^2+1)\big)-9x_2^3(16v^2x_4^2+4v^2+25x_3^8+1)\Big\},\\
\lambda_2 =& \frac{10x_3^3}{\gamma^4}\Big\{2\big(16v^2x_4^2+4v^2+9x_2^4+2\big)-16v^3x_3x_4-9x_2^3x_3-2vx_3(4x_4^2+1)\Big\},\\
\lambda_3 =& \frac{-4}{\gamma^4}\Big\{100vx_3^7x_4-2v^2(4v^2+9x_2^4+25x_3^8+2)+18vx_2^3x_4-x_4(9x_2^4+25x_3^8+2)\Big\},
\end{aligned}
$$

$$\lambda_4 = \frac{-6x_2}{\gamma^4}\Big\{50x_2x_3^7 + 16v^3 2x_2x_4 - 16v^2x_4^2 - 4v^2 - 25x_3^8 - 2 + 2vx_2(4x_4^2+1)\Big\},$$

$$\lambda_5 = \frac{2}{\gamma^4}\Big\{(4x_4^2+1)\big(100vx_3^7 + 18vx_2^3 - 9x_2^4 - 25x_3^8 - 2\big) - 8v^2x_4(9x_2^4 + 25x_3^8 + 2)\Big\}.$$

The curvature vector β'' yields $x_1'' = \lambda_2$, $x_2'' = \lambda_1 - 3\lambda_4x_2^2$, $x_3'' = 2\lambda_5 v - \lambda_1 - 5\lambda_2x_3^4$, $x_4'' = -2\lambda_3x_4 - \lambda_5$, $x_5'' = \lambda_3$, $x_6'' = \lambda_4$ and $u'' = \lambda_1 - 3\lambda_4x_2^2$, $v'' = 3\lambda_4x_2^2 - 2\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5 v$.

In this case, since $\langle\beta'', \mathbf{V}_1\rangle = 0$, the second Frenet vector of the intersection curve is obtained by

$$\mathbf{V}_2 = \frac{\beta''}{\|\beta''\|} = \frac{1}{\|\beta''\|}\big(\lambda_2, \lambda_1 - 3\lambda_4x_2^2, -\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5v, -2\lambda_3x_4 - \lambda_5, \lambda_3, \lambda_4\big),$$

and the first curvature is given by

$$\begin{aligned} k_1 = \|\beta''\| = &\Big\{2\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 - 3\lambda_4x_2^2(2\lambda_1 - 3\lambda_4) + 25\lambda_2^2x_4^8 \\ &+ 4\lambda_5v(\lambda_5v - \lambda_1) + 10\lambda_2x_3^4(\lambda_1 - 2\lambda_5v) + 4\lambda_3x_4(\lambda_3x_4 + \lambda_5)\Big\}^{1/2}. \end{aligned}$$

Similarly, for the third order derivative of the intersection curve, we can write

$$\begin{aligned} \beta''' &= -k_1^2\mathbf{V}_1 + c_1\nabla\mathsf{G}_1 + c_2\nabla\mathsf{G}_2 + c_3\nabla\mathsf{G}_3 + c_4\nabla\mathsf{G}_4 + c_5\mathbf{N}_1 \\ &= -\frac{k_1^2}{\gamma}\big(5x_3^4, 1, 1, 2v, 4vx_4, 3x_2^2\big) \\ &\quad + \big(c_2, c_1 - 3x_2^2c_4, 2vc_5 - c_1 - 5x_3^4c_2, -2x_4c_3 - c_5, c_3, c_4\big), \end{aligned}$$

where

$$\begin{aligned} c_1 = \frac{1}{\gamma^2}\Big\{&(9x_2^4+1)\Big\{-5x_3^4\Big(\frac{80x_3^3}{\gamma}(-\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5v) + \frac{60x_3^2}{\gamma^3}\Big) \\ &\qquad + 48v^2x_4(2\lambda_3x_4 + \lambda_5)\Big\} \\ &+ x_2^2(48v^2x_4^2 + 12v^2 + 75x_3^8 + 3)\Big(\frac{18x_2}{\gamma}(\lambda_1 - 3\lambda_4x_2^2) + \frac{6}{\gamma^3}\Big) \\ &- 12v(4x_4^2+1)(9x_2^4+1)(-2\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5v + 3\lambda_4x_2^2)\Big\}, \end{aligned}$$

$$\begin{aligned} c_2 = \frac{1}{\gamma^2}\Big\{&\big(16v^2x_4^2 + 4v^2 + 9x_2^4 + 2\big)\Big(\frac{80x_3^3}{\gamma}(-\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5v) + \frac{60x_3^2}{\gamma^3}\Big) \\ &+ 240v^2x_3^4x_4(2\lambda_3x_4 + \lambda_5) - 15x_2^2x_3^4\Big(\frac{18x_2}{\gamma}(\lambda_1 - 3\lambda_4x_2^2) + \frac{6}{\gamma^3}\Big) \\ &- 60vx_3^4(4x_4^2+1)(-2\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5v + 3\lambda_4x_2^2)\Big\}, \end{aligned}$$

$$\begin{aligned}c_3 = \frac{1}{\gamma^2}\Big\{ &- 20vx_3^4x_4\Big(\frac{80x_3^3}{\gamma}(-\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5 v) + \frac{60x_3^2}{\gamma^3}\Big)\\ &- 12vx_2^2x_4\Big(\frac{18x_2}{\gamma}(\lambda_1 - 3\lambda_4x_2^2) + \frac{6}{\gamma^3}\Big)\\ &- 12v(2\lambda_3x_4 + \lambda_5)(4v^2 + 9x_2^4 + 25x_3^8 + 2)\\ &+ 12x_4(9x_2^4 + 25x_3^2 + 2)(4x_4^2 + 1)(-2\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5v + 3\lambda_4x_2^2)\Big\},\end{aligned}$$

$$\begin{aligned}c_4 = \frac{1}{\gamma^2}\Big\{ &- 15x_2^2x_3^4\Big(\frac{80x_3^3}{\gamma}(-\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5 v) + \frac{60x_3^2}{\gamma^3}\Big)\\ &+ 144v^2x_2^2x_4(2\lambda_3x_4 + \lambda_5)\\ &+ (16v^2x_4^2 + 4v^2 + 25x_3^8 + 2)\Big(\frac{18x_2}{\gamma}(\lambda_1 - 3\lambda_4x_2^2) + \frac{6}{\gamma^3}\Big)\\ &- 36vx_2^2(4x_4^2 + 1)(-2\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5v + 3\lambda_4x_2^2)\Big\},\end{aligned}$$

$$\begin{aligned}c_5 = \frac{1}{\gamma^2}\Big\{ &10vx_3^4(4x_4^2+1)\Big(\frac{80x_3^3}{\gamma}(-\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5 v) + \frac{60x_3^2}{\gamma^3}\Big)\\ &+ 12v(2\lambda_3x_4 + \lambda_5)(18x_4x_2^4 + 50x_4x_3^8 + 4x_4)\\ &+ 6vx_2^2(4x_4^2 + 1)\Big(\frac{18x_2}{\gamma}(\lambda_1 - 3\lambda_4x_2^2) + \frac{6}{\gamma^3}\Big)\\ &- 6(4x_4^2 + 1)(9x_2^4 + 25x_3^2 + 2)(-2\lambda_1 - 5\lambda_2x_3^4 + 2\lambda_5v + 3\lambda_4x_2^2)\Big\}.\end{aligned}$$

If we use $\langle \beta''', \mathbf{V}_1\rangle = -k_1^2$ and

$$\begin{aligned}\langle \beta''', \mathbf{V}_2\rangle = &\frac{1}{\|\beta''\|}\Big\{c_2\lambda_2 + (c_1 - 3x_2^2c_4)(\lambda_1 - 3x_2^2\lambda_4)\\ &+ (2vc_5 - c_1 - 5x_3^4c_2)(2v\lambda_5 - \lambda_1 - 5x_3^4\lambda_2)\\ &+ (2x_4c_3 + c_5)(2x_4\lambda_3 + \lambda_5) + c_3\lambda_3 + c_4\lambda_4\Big\},\end{aligned}$$

we obtain the third Frenet vector $\mathbf{V}_3 = \dfrac{\mathbf{W}_3}{||\mathbf{W}_3||}$ via

$$\mathbf{W}_3 = \beta''' - \langle \beta''', \mathbf{V}_1\rangle\mathbf{V}_1 - \langle \beta''', \mathbf{V}_2\rangle\mathbf{V}_2.$$

If we continue in the same manner, by using (24) we can find the Frenet vectors $\mathbf{V}_4$ and $\mathbf{V}_5$. Finally, $\mathbf{V}_6$ can be computed by the vector product of obtained Frenet vectors. Thus, the curvatures of the intersection curve are obtained by using (25).

If we use the obtained formulas for the intersection point $P = X(0,0) = (0,0,0,0,0,0)$, we find the Frenet vectors of the intersection curve as

$$\mathbf{V}_1 = \Big(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0\Big), \quad \mathbf{V}_2 = \Big(0, 0, 0, -1, 0, 0\Big), \quad \mathbf{V}_3 = \Big(0, 0, 0, 0, 0, 1\Big),$$

$$\mathbf{V}_4 = \Big(0, 0, 0, 0, 1, 0\Big), \quad \mathbf{V}_5 = \Big(1, 0, 0, 0, 0, 0\Big), \quad \mathbf{V}_6 = \Big(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0\Big).$$

The curvatures are

$$k_1 = 1, \quad k_2 = \frac{3}{\sqrt{2}}, \quad k_3 = 2\sqrt{2}, \quad k_4 = \frac{5}{\sqrt{2}}, \quad k_5 = 0.$$

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