On the distribution of the greatest common divisor of the elements in integral part sets

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ABSTRACT. It is a classical result that the probability that two positive integers $n, m \leq x$ are relatively prime tends to $1/\zeta(2) = 6/\pi^2$ as $x \to \infty$. In this paper, the same result is still true when $n$ and $m$ are restricted to sub-sequences, i.e., Piatetski–Shapiro sequence, Beatty sequence and the floor function set.

1. Introduction and results

Let $\lfloor z \rfloor$ denote the integer part of a real number $z$. As usual, let $\mu$ be Möbius function and $\zeta$ the Riemann zeta-function.

The natural density for the set of pairs of integers which are relatively prime is a classical result in number theory. In 1849, Dirichlet [8] asserts that the proportion of coprime pairs of integers in $\{1, \ldots, n\}$,

$$\frac{1}{n^2} \# \{ (n_1, n_2) \in \{1, \ldots, n\}^2 : \gcd(n_1, n_2) = 1 \},$$

tends to $\frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 0.608$. For further details, we refer to [3, 4, 5, 7, 11]. More specially, Watson [13] proved that also for any given irrational number $\alpha$, the positive integers $n$ for which $\gcd(n, \lfloor \alpha n \rfloor) = 1$, have the natural density $\frac{6}{\pi^2}$. Later, Estermann [10] gave a different proof of a generalization of Watson’s theorem. In 1959 Erdős and Lorentz [9] established sufficient conditions for a differentiable function $f : [1, \infty) \to \mathbb{R}$ satisfying $\gcd(n, f(n)) = 1$, to have the natural density $\frac{6}{\pi^2}$. Recently, Bergelson and Richter [1] extended this problem to functions in the Hardy field $H$. They proved, under some
natural conditions on the $k$-tuple $f_1, \ldots, f_k \in H$, that the density of the set
\[\{n \in \mathbb{N} : \gcd(n, [f_1(n)], \ldots, [f_k(n)]) = 1\}\]
exists and equals $\frac{1}{\zeta(k+1)}$. Very recently, Pimbert et al. [12] studied the asymptotic formulas for the number of the integral pairs $([a^c], [b^c])$ that are coprime, $a, b \leq x$ and $1 < c < 2$. They proved that, as $x \to \infty$,
\[
\sum_{a, b \leq x, \gcd([a^c], [b^c])=1} 1 = \frac{1}{\zeta(2)} x^2 + \begin{cases} O\left(x^{(c+4)/3}\right), & \text{for } 1 < c \leq 5/4, \\
O\left(x^{c+1/2}\right), & \text{for } 5/4 \leq c < 3/2,
\end{cases}
\]
and for $k \geq 3$
\[
\sum_{a_1, \ldots, a_k \leq x, \gcd([a_1^c], [a_2^c], \ldots, [a_k^c])=1} 1 = \frac{1}{\zeta(k)} x^k + O\left(x^{k-(2-c)/3}\right).
\]
The above sums deal with the coprimality sequences $[a_1^c], [a_2^c], \ldots, [a_k^c]$ with the same parameter $c$, while those, for different parameters $c$, take the forms
\[
\sum_{a, b \leq x, \gcd([a^c_1], [b^c_2])=1} 1 = \frac{1}{\zeta(2)} x^2 + \begin{cases} O\left(x^{(c_2+4)/3}\right), & \text{for } 1 < c_1 \leq 5/4, \\
O\left(x^{1/2+(2c_1+c_2)/3}\right), & \text{for } 5/4 \leq c_1 < 3/2,
\end{cases}
\]
with $1 < c_1 < c_2 < 3/2$,
\[
\sum_{a_1, \ldots, a_k \leq x, \gcd([a_1^c_1], [a_2^c_2], \ldots, [a_k^c_k])=1} 1 = \frac{1}{\zeta(k)} x^k + O\left(x^{k-(2-c_k)/3}\right),
\]
with $k \geq 3, 1 < c_1 \leq c_2 \leq \ldots \leq c_k < 2$. Thus, it would be interesting to study the coprimality of any pairs in other sequences. Piatetksi–Shapiro sequences are defined by
\[\mathbb{N}^c = \{[n^c] \}_{n \in \mathbb{N}}, \ (c > 1, c \notin \mathbb{N}).\]
Let $\alpha > 1$ be an irrational number with bounded partial quotients and let $\beta \in [0, \alpha)$. The Beatty sequence of parameters $\alpha$ and $\beta$ is defined as
\[\{[\alpha n + \beta]\}_{n \in \mathbb{N}}.
\]
Let $S(x) := \{[\frac{z}{n}] : 1 \leq n \leq x\}$. The characteristic function of the set $S(x)$ is denoted by $\mathbb{1}_{S(x)}(n)$.

The purpose of this paper is to establish the natural density of the set
\[\{(n, m) \in \mathbb{N}^2 : n, m \leq x, \ \gcd([g_1(n)], [g_2(m)]) = 1\},
\]
where $g_i(n) = n, n^c, \alpha n + \beta, \mathbb{1}_{S(x)}(n), i = 1, 2$. We obtain the following results.
Theorem 1. Let $\alpha > 1$ be an irrational number with bounded partial quotients and let $\beta \in [0, \alpha)$ and $1 < c < 2$. As $x \to \infty$, we have

$$\sum_{\substack{a, b \leq x \\ \gcd(a, b) = 1}} 1 = \frac{1}{\zeta(2)}x^2 + O\left(x^{(c+4)/3}\right)$$

and

$$\sum_{\substack{a, b \leq x \\ \gcd(|a^c|, |ab + \beta|) = 1}} 1 = \frac{1}{\zeta(2)}x^2 + O\left(x^{(c+4)/3}\right).$$

Theorem 2. Let $\alpha > 1$ be an irrational number with bounded partial quotients and let $\beta \in [0, \alpha)$. As $x \to \infty$, we have

$$\sum_{\substack{a, b \leq x \\ \gcd(a, |\alpha b + \beta|) = 1}} 1 = \frac{1}{\zeta(2)}x^2 + O(x^{3/2}\log^{3/2+\epsilon} x)$$

and

$$\sum_{\substack{a, b \leq x \\ \gcd(|\alpha a + \beta|, |\alpha b + \beta|) = 1}} 1 = \frac{1}{\zeta(2)}x^2 + O(x^{3/2}\log^{3/2+\epsilon} x).$$

Theorem 3. Let $S(x) := \{\left\lfloor \frac{x}{n} \right\rfloor : 1 \leq n \leq x\}$. Let $\alpha > 1$ be an irrational number with bounded partial quotients and let $\beta \in [0, \alpha)$ and $1 < c < 2$. As $x \to \infty$, we have

$$\sum_{\substack{a, b \leq x \\ \gcd(a, b) = 1}} 1_{S(x)}(b) = \frac{2}{\zeta(2)}x^{3/2} + O(x^{4/3} \log x),$$

$$\sum_{\substack{a, b \leq x \\ \gcd(|a^c|, b) = 1}} 1_{S(x)}(b) = \frac{2}{\zeta(2)}x^{3/2} + \begin{cases} O\left(x^{4/3} \log x\right), & 1 < c \leq \frac{3}{2}, \\ O\left(x^{(2c+5)/6}\right), & \frac{3}{2} < c < 2, \end{cases}$$

$$\sum_{\substack{a, b \leq x \\ \gcd(|\alpha a + \beta|, b) = 1}} 1_{S(x)}(b) = \frac{2}{\zeta(2)}x^{3/2} + O(x^{4/3} \log x),$$

and

$$\sum_{\substack{a, b \leq x \\ \gcd(a, b) = 1}} 1_{S(x)}(a)1_{S(x)}(b) = \frac{4}{\zeta(2)}x + O(x^{5/6} \log x).$$
2. Lemmas

Throughout this paper, implied constants in symbols $O$ and $\ll$ may depend on the parameters $\alpha, \beta, c, \epsilon$, but are absolute otherwise. For given functions $F$ and $G$, the notations $F \ll G$ and $F = O(G)$ are both equivalent to the statement that the inequality $|F| \leq C|G|$ holds with some constant $C > 0$.

The main ingredients in the following proofs are several good estimates for the number of integers $n$ up to $x$ satisfying various floor functions $\lfloor g(n) \rfloor$ belonging to an arithmetic progression.

**Lemma 1** ([6]). For $1 < c < 2$, let $x \in \mathbb{R}$ and $a, q \in \mathbb{Z}$ be such that $0 \leq a < q \leq x^c$. Then

$$\sum_{\lfloor n^c \rfloor \equiv a \pmod{q}} 1 = \frac{x}{q} + O\left(\min\left(\frac{x^c}{q}, \frac{x^{(c+1)/3}}{q^{1/3}}\right)\right).$$

**Lemma 2** ([2]). For irrational $\alpha > 1$ with bounded partial quotients and for $\beta \in [0, \alpha)$, and for positive integers $d \geq 2, 0 \leq a < d$, we have

$$\sum_{\lfloor \alpha n + \beta \rfloor \equiv a \pmod{d}} 1 = \frac{x}{d} + O(d \log^3 x) \text{ as } x \to \infty.$$

For growing difference $d$ the result is non-trivial provided $d \ll \sqrt{x \log^{-3/2-\varepsilon} x}$, for $\varepsilon > 0$.

**Lemma 3** ([14]). Let $x$ be a positive real number, and let $q$ and $a$ be two integers such that $0 \leq a < q \leq x^{1/4} \log^{-3/2} x$. Then

$$\sum_{n \leq x \atop n \equiv a \pmod{q}} \mathbb{1}_{S(x)}(n) = \frac{2x^{1/2}}{q} + O\left(\frac{x^{1/3}}{q^{1/2} \log x}\right).$$

3. Proofs

**Proof of Theorem 1.** Let $1 < c \leq \frac{3}{2}$. In view of the well known identity

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

we have

$$\sum_{a, b \leq x \atop \gcd(a, b^c) = 1} 1 = \sum_{a, b \leq x \atop d \mid \gcd(a, b^c)} \sum_{d \leq x} \mu(d) \sum_{n \leq x} 1 \sum_{k \leq x} \mathbb{1}_{\lfloor k^c \rfloor = 0 \pmod{d}} 1.$$
By Lemma 1, we have
\[
\sum_{d \leq x} 1 = \sum_{d \leq x} \mu(d) \left( \frac{x}{d} + O(1) \right) \left( \frac{x}{d} + O\left( \frac{x^{(c+1)/3}}{d^{1/3}} \right) \right)
\]
\[
= \sum_{d \leq x^{c-1/2}} \mu(d) \left( \frac{x}{d} + O(1) \right) \left( \frac{x}{d} + O\left( \frac{x^{c}}{d^{1/3}} \right) \right)
\]
\[
+ \sum_{x^{c-1/2} < d \leq x} \mu(d) \left( \frac{x}{d} + O(1) \right) \left( \frac{x}{d} + O\left( \frac{x^{c}}{d^{1/3}} \right) \right)
\]
\[
= x^2 \sum_{d \leq x} \frac{\mu(d)}{d^2} + O\left( \sum_{d \leq x^{c-1/2}} \left( \frac{x}{d} + \frac{x^{c+1}}{d^2} + \frac{x^c}{d} \right) \right)
\]
\[
+ O\left( \sum_{x^{c-1/2} < d \leq x} \left( \frac{x}{d} + \frac{x^{c+1}}{d^2} + \frac{x^c}{d} \right) \right)
\]
\[
= x^2 \sum_{d \leq x} \frac{\mu(d)}{d^2} + O(x \log x) + O(x^{(c+4)/3})
\]
\[
+ O(x^c) + O(x^{3/2}) + O(x^{c} \log x).
\]
It is well known that \( \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} \). Thus, we have
\[
\sum_{x} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} - \sum_{d > x} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} + O\left( \frac{1}{x} \right).
\]
From (10) and (11) we conclude that
\[
\sum_{a, b \leq x \text{ gcd}(a, [b^r]) = 1} 1 = \frac{1}{\zeta(2)} x^2 + O(x^{(c+4)/3}).
\]
Let \( c > \frac{3}{2} \). In view of (9) and (11), we have
\[
\sum_{a, b \leq x \text{ gcd}(a, [b^r]) = 1} 1 = \sum_{d \leq x} \mu(d) \left( \frac{x}{d} + O(1) \right) \left( \frac{x}{d} + O\left( \frac{x^{(c+1)/3}}{d^{1/3}} \right) \right)
\]
\[
= x^2 \sum_{d \leq x} \frac{\mu(d)}{d^2} + O\left( \sum_{d \leq x} \left( \frac{x}{d} + \frac{x^{(c+4)/3}}{d^{1/3}} + \frac{x^{(c+1)/3}}{d^{1/3}} \right) \right)
\]
\[
= \frac{1}{\zeta(2)} x^2 + O(x) + O(x \log x) + O(x^{(c+4)/3}) + O(x^{(c-3)/3})
\]
\[
= \frac{1}{\zeta(2)} x^2 + O(x^{(c+4)/3}).
\]
Then, (1) follows from (12) and (13).
Now we will prove (2). Let \( \alpha > 1 \) be irrational with bounded partial quotients and let \( \beta \in [0, \alpha) \). Similarly to proving (1), we write
\[
1 = \sum_{\substack{a, b \leq x \\ \gcd([a^e], [ab + \beta]) = 1}} \mu(d) = \sum_{d \leq \alpha x} \mu(d) \sum_{a \leq x} \sum_{b \leq x \atop d \mid [ab + \beta]} 1.
\]
In view of Lemma 1 and 2, we have
\[
\sum_{\substack{a, b \leq x \\ \gcd([a^e], [ab + \beta]) = 1}} 1 = \sum_{d \leq x^{1/2} \log^{-3/2 - \varepsilon} x} \mu(d) \left( \frac{x}{d} + O\left( \min\left( \frac{x^e}{d}, \frac{x^{(e+1)/3}}{d^{1/3}} \right) \right) \right) \times \\
\times \left( \frac{x}{d} + O\left( d \log^3 x \right) \right) + \sum_{x^{1/2} \log^{-3/2 - \varepsilon} x < d \leq \alpha x} \mu(d) \sum_{a \leq x \atop d \mid [ab + \beta]} \sum_{b \leq x \atop b \equiv \beta (\mod d)} 1
\]
\[
= \sum_{d \leq x^{1/2} \log^{-3/2 - \varepsilon} x} \mu(d) \left( \frac{x}{d} + O\left( \frac{x^{(e+1)/3}}{d^{1/3}} \right) \right) \times \\
\times \left( \frac{x}{d} + O\left( d \log^3 x \right) \right) + O\left( x^{2} \sum_{x^{1/2} \log^{-3/2 - \varepsilon} x < d \leq \alpha x} \frac{1}{d^2} \right)
\]
\[
= x^{2} \sum_{d \leq x^{1/2} \log^{-3/2 - \varepsilon} x} \frac{\mu(d)}{d^2} + O\left( x^{(e+4)/3} \right).
\]
Then, (2) follows from (11) and (14). \( \square \)

**Proof of Theorem 2.** Let \( \alpha > 1 \) be irrational with bounded partial quotients and let \( \beta \in [0, \alpha) \). We write
\[
1 = \sum_{\substack{a, b \leq x \\ \gcd([a^e], [ab + \beta]) = 1}} \mu(d) = \sum_{d \leq \alpha x} \mu(d) \sum_{a \leq x \atop a \equiv 0 (\mod d)} \sum_{b \leq x \atop b \equiv \beta (\mod d)} 1.
\]
In view of Lemma 2, we have
\[
\sum_{\substack{a, b \leq x \\ \gcd([a^e], [ab + \beta]) = 1}} 1 = \sum_{d \leq x^{1/2} \log^{-3/2 - \varepsilon} x} \mu(d) \left( \frac{x}{d} + O(1) \right) \left( \frac{x}{d} + O\left( d \log^3 x \right) \right)
\]
\[
+ \sum_{x^{1/2} \log^{-3/2 - \varepsilon} x < d \leq x} \mu(d) \left( \frac{x}{d} + O(1) \right) O\left( \frac{x}{d} \right)
\]
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\[ = x^2 \sum_{d \leq x^{1/2} \log^{-3/2-\varepsilon} x} \frac{\mu(d)}{d^2} \]

\[ + O \left( \sum_{d \leq x^{1/2} \log^{-3/2-\varepsilon} x} \left( x \log^3 x + \frac{x}{d} + d \log^3 x \right) \right) \]

\[ + O \left( \sum_{x^{1/2} \log^{-3/2-\varepsilon} x < d \leq x} \left( \frac{x^2}{d^2} + \frac{x}{d} \right) \right) \]

\[ = x^2 \sum_{d \leq x^{1/2} \log^{-3/2-\varepsilon} x} \frac{\mu(d)}{d^2} + O(x^{3/2} \log^{3/2+\varepsilon} x). \quad (15) \]

Due to (15) and (11) we have

\[ \sum_{a,b \leq x, \gcd(a, \lfloor \alpha b + \beta \rfloor) = 1} 1 = \frac{1}{\zeta(2)} x^2 + O(x^{3/2} \log^{3/2+\varepsilon} x). \]

The proof of (3) follows. Next we will prove (4). We have

\[ \sum_{a,b \leq x, \gcd(\lfloor \alpha a + \beta \rfloor, \lfloor \alpha b + \beta \rfloor) = 1} 1 = \sum_{d \leq x} \mu(d) \sum_{\substack{a \leq x \lfloor \alpha a + \beta \rfloor \equiv 0 \pmod{d} \lfloor \alpha b + \beta \rfloor \equiv 0 \pmod{d}}} 1. \]

In view of Lemma 2, we have

\[ \sum_{a,b \leq x, \gcd(\lfloor \alpha a + \beta \rfloor, \lfloor \alpha b + \beta \rfloor) = 1} 1 = \sum_{d \leq x^{1/2} \log^{-3/2-\varepsilon} x} \mu(d) \left( \frac{x}{d} + O \left( d \log^3 x \right) \right)^2 \]

\[ + O \left( x^2 \sum_{x^{1/2} \log^{-3/2-\varepsilon} x < d \leq x} \frac{1}{d^2} \right) \]

\[ = x^2 \sum_{d \leq x^{1/2} \log^{-3/2-\varepsilon} x} \frac{\mu(d)}{d^2} \]

\[ + O \left( \sum_{x^{1/2} \log^{-3/2-\varepsilon} x \leq d \leq x} \left( x \log^3 x + d^2 \log^6 x \right) \right) \]

\[ + O(x^{3/2} \log^{3/2+\varepsilon} x) \]

\[ = x^2 \sum_{d \leq x^{1/2} \log^{-3/2-\varepsilon} x} \frac{\mu(d)}{d^2} + O(x^{3/2} \log^{3/2+\varepsilon} x). \quad (16) \]
Because of (16) and (11), we have
\[
\sum_{\gcd(\lfloor \alpha a + \beta \rfloor, \lfloor \alpha b + \beta \rfloor) = 1} 1 = \frac{1}{\zeta(2)} x^2 + O\left(x^{3/2} \log^{3/2 + \epsilon} x \right).
\]
The proof of (4) follows.

\textbf{Proof of Theorem 3.} Similarly to proving (1), we write
\[
\sum_{\gcd(a, b) = 1} \mathbb{1}_{S(x)}(b) = \sum_{d \mid a, b \leq x} \mathbb{1}_{S(x)}(b) \sum_{d \mid \gcd(a, b)} \mu(d)
\]
\[
= \sum_{d \leq x} \mu(d) \sum_{\substack{a \leq x \mod d \atop a \equiv 0 \mod d}} \mathbb{1}_{S(x)}(b)
\]
By Lemma 3, we have
\[
\sum_{\gcd(a, b) = 1} \mathbb{1}_{S(x)}(b) = \sum_{d \leq x^{1/4} \log^{-3/4} x} \mu(d) \left(\frac{x}{d} + O(1)\right) \left(\frac{2x^{1/2}}{d} + O\left(\frac{x^{1/3} \log x}{d^{1/3}}\right)\right)
\]
\[
+ \sum_{x^{1/4} \log^{-3/4} x < d \leq x} \mu(d) \left(\frac{x}{d} + O(1)\right) O\left(\frac{x^{1/2}}{d}\right)
\]
\[
= 2x^{3/2} \sum_{d \leq x^{1/4} \log^{-3/4} x} \mu(d) \frac{d^2}{d^2}
\]
\[
+ O\left(\sum_{d \leq x^{1/4} \log^{-3/4} x} \left(\frac{x^{4/3} \log x}{d^{4/3}} + \frac{x^{1/2}}{d} + \frac{x^{1/3} \log x}{d^{1/3}}\right)\right)
\]
\[
+ O\left(\sum_{x^{1/4} \log^{-3/4} x < d \leq x} \left(\frac{x^{3/2}}{d^2} + \frac{x^{1/2}}{d}\right)\right)
\]
\[
= x^{3/2} \sum_{d \leq x^{1/2} \log^{-3/2 - \epsilon} x} \frac{\mu(d)}{d^2} + O(x^{4/3} \log x).
\]
(17)
The proof of (5) follows from (11) and (17).

Now we prove (6). We write
\[
\sum_{\gcd(\lfloor \alpha^x \rfloor, b) = 1} \mathbb{1}_{S(x)}(b) = \sum_{d \mid \lfloor \alpha^x \rfloor} \mathbb{1}_{S(x)}(b) \sum_{d \mid \lfloor \alpha^x \rfloor} \mu(d)
\]
\[
= \sum_{d \leq x} \mu(d) \sum_{\substack{b \leq x \mod d \atop b \equiv 0 \mod d}} \mathbb{1}_{S(x)}(b).
\]
In view of Lemma 1 and 3, we have
\[
\sum_{a, b \leq x, \gcd([a^\alpha], b) = 1} 1_S(x)(b) = \sum_{d \leq x^{1/4} \log^{-3/2} x} \mu(d) \left( \frac{x}{d} + O\left( \min\left( \frac{x^c}{d}, \frac{x^{(c+1)/3}}{d^{1/3}} \right) \right) \right) \times
\left( \frac{2x^{1/2}}{d} + O\left( \frac{x^{1/3} \log x}{d^{1/3}} \right) \right)
\]
\[
+ \sum_{x^{1/4} \log^{-3/2} x < d \leq x} \mu(d) \sum_{a \leq x \mid [a^\alpha] \equiv 0 \pmod{d}} 1 \sum_{b \equiv 0 \pmod{d}} 1_S(x)(b)
\]
\[
= \sum_{d \leq x^{1/4} \log^{-3/2} x} \mu(d) \left( \frac{x}{d} + O\left( \frac{x^{(c+1)/3}}{d^{1/3}} \right) \right) \times
\left( \frac{2x^{1/2}}{d} + O\left( \frac{x^{1/3} \log x}{d^{1/3}} \right) \right) + O\left( x^{3/2} \sum_{x^{1/4} \log^{-3/2} x < d \leq x} \frac{1}{d^2} \right)
\]
\[
= 2x^{3/2} \sum_{d \leq x^{1/4} \log^{-3/2} x} \frac{\mu(d)}{d^2} + O\left( \sum_{d \leq x^{1/4} \log^{-3/2} x} \left( \frac{x^{4/3} \log x}{d^{1/3}} + \frac{x^{(2c+5)/6}}{d^{1/3}} + \frac{x^{(c+2)/3} \log x}{d^{2/3}} \right) \right)
\]
\[
+ O\left( x^{5/4} \log^{3/2} x \right)
\]
\[
= 2x^{3/2} \sum_{d \leq x^{1/4} \log^{-3/2} x} \frac{\mu(d)}{d^2} + O(x^{4/3} \log x) + O(x^{(2c+5)/6}). \quad (18)
\]

Now (6) follows from (11) and (18).

Next we prove (7). We write
\[
\sum_{a, b \leq x, \gcd([a^\alpha + b^\beta], b) = 1} 1_S(x)(b) = \sum_{a, b \leq x} \sum_{d \mid [a^\alpha + b^\beta] \pmod{d}} \mu(d)
\]
\[
= \sum_{d \leq x} \mu(d) \sum_{a \leq x \mid [a^\alpha + b^\beta] \equiv 0 \pmod{d}} 1 \sum_{b \equiv 0 \pmod{d}} 1_S(x)(b).
\]

In view of Lemma 3 and 2, we have
\[
\sum_{a, b \leq x, \gcd([a^\alpha + b^\beta], b) = 1} 1_S(x)(b) = \sum_{d \leq x^{1/4} \log^{-3/2} x} \mu(d) \left( \frac{x}{d} + O\left( d \log^3 x \right) \right) \times
\left( \frac{2x^{1/2}}{d} + O\left( \frac{x^{1/3} \log x}{d^{1/3}} \right) \right)
\]
\begin{align*}
&+ \sum_{x^{1/4} \log^{-3/2} x < d \leq x} \mu(d) \sum_{\substack{a,b \leq x \mod d \equiv \alpha \mod d \equiv 0 \mod d}} 1 \sum_{b \leq x} \mathbb{1}_{S(x)}(b) \\
&= \sum_{d \leq x^{1/4} \log^{-3/2} x} \mu(d) \left( \frac{x}{d} + O( \log^3 x ) \right) \\
&\quad \times \left( \frac{2x^{1/2}}{d} + O \left( \frac{x^{1/3} \log x}{d^{1/3}} \right) \right) \\
&\quad + O \left( x^{3/2} \sum_{x^{1/4} \log^{-3/2} x < d \leq x} \frac{1}{d^2} \right) \\
&= 2x^{3/2} \sum_{d \leq x^{1/4} \log^{-3/2} x} \frac{\mu(d)}{d^2} \\
&\quad + O \left( x^{3/2} \sum_{d \leq x^{1/4} \log^{-3/2} x} \left( \frac{x^{4/3} \log x}{d^{1/3}} + x^{1/2} \log x + x^{1/3} d^{2/3} \log^4 x \right) \right) \\
&\quad + O \left( x^{3/2} \log x \right) \\
&= 2x^{3/2} \sum_{d \leq x^{1/4} \log^{-3/2} x} \frac{\mu(d)}{d^2} + O(x^{4/3} \log x). \quad (19)
\end{align*}

Now (7) follows from (11) and (19).

Lastly, we prove (8). We write
\begin{align*}
\sum_{a,b \leq x \gcd(a,b)=1} \mathbb{1}_{S(x)}(a) \mathbb{1}_{S(x)}(b) &= \sum_{a,b \leq x} \mathbb{1}_{S(x)}(a) \mathbb{1}_{S(x)}(b) \sum_{d | a} \mu(d) \\
&= \sum_{d \leq x} \mu(d) \sum_{a \leq x \gcd(a,d)=1} \mathbb{1}_{S(x)}(a) \sum_{b \leq x \gcd(b,d)=1} \mathbb{1}_{S(x)}(b).
\end{align*}

In view of Lemma 3, we have
\begin{align*}
\sum_{a,b \leq x \gcd(a,b)=1} \mathbb{1}_{S(x)}(a) \mathbb{1}_{S(x)}(b) &= \sum_{d \leq x^{1/4} \log^{-3/2} x} \mu(d) \left( \frac{2x^{1/2}}{d} + O \left( \frac{x^{1/3}}{d^{1/3}} \log x \right) \right)^2 \\
&\quad + \sum_{x^{1/4} \log^{-3/2} x < d \leq x} \mu(d) \sum_{a \leq x \gcd(a,d)=1} \mathbb{1}_{S(x)}(a) \sum_{b \leq x \gcd(b,d)=1} \mathbb{1}_{S(x)}(b) \\
&= 4x \sum_{d \leq x^{1/4} \log^{-3/2} x} \frac{\mu(d)}{d^2} \\
&\quad + O \left( \sum_{d \leq x^{1/4} \log^{-3/2} x} \left( \frac{x^{5/6} \log x}{d^{1/3}} + \frac{x^{2/3} \log^2 x}{d^{2/3}} \right) \right).
\end{align*}
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\[ + O \left( x \sum_{x^{1/4} \log^{-3/2} x < d \leq x} \left( \frac{1}{d^2} \right) \right) \]

\[ = 4x \sum_{d \leq x^{1/2} \log^{-3/2} x} \frac{\mu(d)}{d^2} + O(x^{5/6} \log x). \tag{20} \]

The proof of (8) follows from (20) and (11). \qed

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