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On the distribution of the greatest common divisor of the elements in integral part sets

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Abstract. It is a classical result that the probability that two positive integers $n, m \leq x$ are relatively prime tends to $1/\zeta(2) = 6/\pi^2$ as $x \to \infty$. In this paper, the same result is still true when n and m are restricted to sub-sequences, i.e, Piatetski–Shapiro sequence, Beatty sequence and the floor function set.

1. Introduction and results

Let $\lfloor z \rfloor$ denote the integer part of a real number z. As usual, let μ be Möbius function and ζ the Riemann zeta-function.

The natural density for the set of pairs of integers which are relatively prime is a classical result in number theory. In 1849, Dirichlet [8] asserts that the proportion of coprime pairs of integers in $\{1, ..., n\}$,

$$\frac{1}{n^2}\#\{(n_1, n_2) \in \{1, ..., n\}^2 \ : \ \gcd(n_1, n_2) = 1\},$$

tends to $\frac{1}{\zeta(2)} = \frac{6}{\pi^2} \sim 0.608$. For further details, we refer to [3, 4, 5, 7, 11]. More specially, Watson [13] proved that also for any given irrational number α, the positive integers n for which $\gcd(n, \lfloor \alpha n \rfloor) = 1$, have the natural density $\frac{6}{\pi^2}$. Later, Estermann [10] gave a different proof of a generalization of Watson's theorem. In 1959 Erdős and Lorentz [9] established sufficient conditions for a differentiable function $f : [1, \infty) \to \mathbb{R}$ satisfying $\gcd(n, f(n)) = 1$, to have the natural density $\frac{6}{\pi^2}$. Recently, Bergelson and Richter [1] extended this problem to functions in the Hardy field H. They proved, under some

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natural conditions on the k-tuple $f_1, ..., f_k \in H$, that the density of the set

$$\{n \in \mathbb{N} : \gcd(n, \lfloor f_1(n) \rfloor, ..., \lfloor f_k(n) \rfloor) = 1\}$$

exists and equals $\frac{1}{\zeta(k+1)}$. Very recently, Pimsert et al. [12] studied the asymptotic formulas for the number of the integral pairs $(\lfloor a^c \rfloor, \lfloor b^c \rfloor)$ that are coprime, $a, b \le x$ and $1 < c < 2$. They proved that, as $x \to \infty$,

$$\sum_{\substack{a,b \le x \\ \gcd(\lfloor a^c \rfloor, \lfloor b^c \rfloor)=1}} 1 = \frac{1}{\zeta(2)} x^2 + \begin{cases} O\Big(x^{(c+4)/3}\Big), & \text{for } 1 < c \le 5/4, \\ O\Big(x^{c+1/2}\Big), & \text{for } 5/4 \le c < 3/2, \end{cases}$$

and for $k \ge 3$

$$\sum_{\substack{a_1,...,a_k \le x \\ \gcd(\lfloor a_1^c \rfloor, \lfloor a_2^c \rfloor, ..., \lfloor a_k^c \rfloor)=1}} 1 = \frac{1}{\zeta(k)} x^k + O\Big(x^{k-(2-c)/3}\Big).$$

The above sums deal with the coprimality sequences $\lfloor a_1^c \rfloor, \lfloor a_2^c \rfloor, ..., \lfloor a_k^c \rfloor$ with the same parameter c, while those, for different parameters c, take the forms

$$\sum_{\substack{a,b \le x \\ \gcd(\lfloor a^{c_1} \rfloor, \lfloor b^{c_2} \rfloor)=1}} 1 = \frac{1}{\zeta(2)} x^2 + \begin{cases} O\Big(x^{(c_2+4)/3}\Big), & \text{for } 1 < c_1 \le 5/4, \\ O\Big(x^{1/2+(2c_1+c_2)/3}\Big), & \text{for } 5/4 \le c_1 < 3/2, \end{cases}$$

with $1 < c_1 < c_2 < 3/2$,

$$\sum_{\substack{a_1,...,a_k \le x \\ \gcd(\lfloor a_1^{c_1} \rfloor, \lfloor a_2^{c_2} \rfloor, ..., \lfloor a_k^{c_k} \rfloor)=1}} 1 = \frac{1}{\zeta(k)} x^k + O\Big(x^{k-(2-c_k)/3}\Big),$$

with $k \ge 3$, $1 < c_1 \le c_2 \le ... \le c_k < 2$. Thus, it would be interesting to study the coprimality of any pairs in other sequences. Piatetski–Shapiro sequences are defined by

$$\mathbb{N}^c = \{\lfloor n^c \rfloor\}_{n \in \mathbb{N}}, \ (c > 1, c \notin \mathbb{N}).$$

Let $\alpha > 1$ be an irrational number with bounded partial quotients and let $\beta \in [0, \alpha)$. The Beatty sequence of parameters α and β is defined as

$$\{\lfloor \alpha n + \beta \rfloor\}_{n \in \mathbb{N}}.$$

Let $S(x) := \{\lfloor \frac{x}{n} \rfloor : 1 \le n \le x\}$. The characteristic function of the set $S(x)$ is denoted by $\mathbb{1}_{S(x)}(n)$.

The purpose of this paper is to establish the natural density of the set

$$\{(n, m) \in \mathbb{N}^2 : n, m \le x, \ \gcd(\lfloor g_1(n) \rfloor, \lfloor g_2(m) \rfloor) = 1\},$$

where $g_i(n) = n, n^c, \alpha n + \beta, \mathbb{1}_{S(x)}(n), \ i = 1, 2$. We obtain the following results.

Theorem 1. *Let $\alpha > 1$ be an irrational number with bounded partial quotients and let $\beta \in [0, \alpha)$ and $1 < c < 2$. As $x \to \infty$, we have*

$$\sum_{\substack{a,b \le x \\ \gcd(a, \lfloor b^c \rfloor)=1}} 1 = \frac{1}{\zeta(2)} x^2 + O\Big(x^{(c+4)/3}\Big) \tag{1}$$

and

$$\sum_{\substack{a,b \le x \\ \gcd(\lfloor a^c \rfloor, \lfloor \alpha b + \beta \rfloor)=1}} 1 = \frac{1}{\zeta(2)} x^2 + O\Big(x^{(c+4)/3}\Big). \tag{2}$$

Theorem 2. *Let $\alpha > 1$ be an irrational number with bounded partial quotients and let $\beta \in [0, \alpha)$. As $x \to \infty$, we have*

$$\sum_{\substack{a,b \le x \\ \gcd(a, \lfloor \alpha b + \beta \rfloor)=1}} 1 = \frac{1}{\zeta(2)} x^2 + O(x^{3/2} \log^{3/2+\epsilon} x) \tag{3}$$

and

$$\sum_{\substack{a,b \le x \\ \gcd(\lfloor \alpha a + \beta \rfloor, \lfloor \alpha b + \beta \rfloor)=1}} 1 = \frac{1}{\zeta(2)} x^2 + O(x^{3/2} \log^{3/2+\epsilon} x). \tag{4}$$

Theorem 3. *Let $S(x) := \{\lfloor \frac{x}{n} \rfloor : 1 \le n \le x\}$. Let $\alpha > 1$ be an irrational number with bounded partial quotients and let $\beta \in [0, \alpha)$ and $1 < c < 2$. As $x \to \infty$, we have*

$$\sum_{\substack{a,b \le x \\ \gcd(a,b)=1}} \mathbb{1}_{S(x)}(b) = \frac{2}{\zeta(2)} x^{3/2} + O(x^{4/3} \log x), \tag{5}$$

$$\sum_{\substack{a,b \le x \\ \gcd(\lfloor a^c \rfloor, b)=1}} \mathbb{1}_{S(x)}(b) = \frac{2}{\zeta(2)} x^{3/2} + \begin{cases} O\Big(x^{4/3} \log x\Big), & 1 < c \le \frac{3}{2}, \\ O\Big(x^{(2c+5)/6}\Big), & \frac{3}{2} < c < 2, \end{cases} \tag{6}$$

$$\sum_{\substack{a,b \le x \\ \gcd(\lfloor \alpha a + \beta \rfloor, b)=1}} \mathbb{1}_{S(x)}(b) = \frac{2}{\zeta(2)} x^{3/2} + O(x^{4/3} \log x), \tag{7}$$

and

$$\sum_{\substack{a,b \le x \\ \gcd(a,b)=1}} \mathbb{1}_{S(x)}(a) \mathbb{1}_{S(x)}(b) = \frac{4}{\zeta(2)} x + O(x^{5/6} \log x). \tag{8}$$

2. Lemmas

Throughout this paper, implied constants in symbols O and $\ll$ may depend on the parameters $\alpha, \beta, c, \epsilon$, but are absolute otherwise. For given functions F and G, the notations $F \ll G$ and $F = O(G)$ are both equivalent to the statement that the inequality $|F| \leq C|G|$ holds with some constant $C > 0$.

The main ingredients in the following proofs are several good estimates for the number of integers n up to x satisfying various floor functions $\lfloor g(n) \rfloor$ belonging to an arithmetic progression.

Lemma 1 ([6]). *For $1 < c < 2$, let $x \in \mathbb{R}$ and $a, q \in \mathbb{Z}$ be such that $0 \leq a < q \leq x^c$. Then*

$$\sum_{\substack{n \leq x \\ \lfloor n^c \rfloor \equiv a (\bmod q)}} 1 = \frac{x}{q} + O\Big(\min \Big(\frac{x^c}{q}, \frac{x^{(c+1)/3}}{q^{1/3}} \Big) \Big).$$

Lemma 2 ([2]). *For irrational $\alpha > 1$ with bounded partial quotients and for $\beta \in [0, \alpha)$, and for positive integers $d \geq 2, 0 \leq a < d$, we have*

$$\sum_{\substack{n \leq x \\ \lfloor \alpha n + \beta \rfloor \equiv a (\bmod d)}} 1 = \frac{x}{d} + O(d \log^3 x) \quad \text{as } x \to \infty.$$

For growing difference d the result is non-trivial provided $d \ll \sqrt{x} \log^{-3/2-\varepsilon} x$, for $\varepsilon > 0$.

Lemma 3 ([14]). *Let x be a positive real number, and let q and a be two integers such that $0 \leq a < q \leq x^{1/4} \log^{-3/2} x$. Then*

$$\sum_{\substack{n \leq x \\ n \equiv a \ (\bmod q)}} \mathbb{1}_{S(x)}(n) = \frac{2x^{1/2}}{q} + O\Big(\frac{x^{1/3}}{q^{1/3}} \log x \Big).$$

3. Proofs

Proof of Theorem 1. Let $1 < c \leq \frac{3}{2}$. In view of the well known identity

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

we have

$$\sum_{\substack{a,b \leq x \\ \gcd(a, \lfloor b^c \rfloor) = 1}} 1 = \sum_{a,b \leq x} \sum_{\substack{d|a \\ d|\lfloor b^c \rfloor}} \mu(d) = \sum_{d \leq x} \mu(d) \sum_{\substack{a \leq x \\ a \equiv 0 (\bmod d)}} 1 \sum_{\substack{b \leq x \\ \lfloor b^c \rfloor \equiv 0 (\bmod d)}} 1.$$

By Lemma 1, we have

$$\sum_{\substack{a,b\le x\\ \gcd(a,\lfloor b^c\rfloor)=1}} 1 = \sum_{d\le x}\mu(d)\Big(\frac{x}{d}+O(1)\Big)\Big(\frac{x}{d}+O\Big(\min\Big(\frac{x^c}{d},\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)\Big) \tag{9}$$

$$= \sum_{d\le x^{c-1/2}}\mu(d)\Big(\frac{x}{d}+O(1)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)$$

$$+ \sum_{x^{c-1/2}<d\le x}\mu(d)\Big(\frac{x}{d}+O(1)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^c}{d}\Big)\Big)$$

$$= x^2\sum_{d\le x}\frac{\mu(d)}{d^2}+O\Big(\sum_{d\le x^{c-1/2}}\Big(\frac{x}{d}+\frac{x^{(c+4)/3}}{d^{4/3}}+\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)$$

$$+O\Big(\sum_{x^{c-1/2}<d\le x}\Big(\frac{x}{d}+\frac{x^{c+1}}{d^2}+\frac{x^c}{d}\Big)\Big)$$

$$= x^2\sum_{d\le x}\frac{\mu(d)}{d^2}+O(x\log x)+O(x^{(c+4)/3})$$

$$+O(x^c)+O(x^{3/2})+O(x^c\log x). \tag{10}$$

It is well known that $\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}=\frac{1}{\zeta(2)}$. Thus, we have

$$\sum_{d\le x}\frac{\mu(d)}{d^2}=\frac{1}{\zeta(2)}-\sum_{d>x}\frac{\mu(d)}{d^2}=\frac{1}{\zeta(2)}+O\Big(\frac{1}{x}\Big). \tag{11}$$

From (10) and (11) we conclude that

$$\sum_{\substack{a,b\le x\\ \gcd(a,\lfloor b^c\rfloor)=1}} 1 = \frac{1}{\zeta(2)}x^2+O(x^{(c+4)/3}). \tag{12}$$

Let $c>\frac{3}{2}$. In view of (9) and (11), we have

$$\sum_{\substack{a,b\le x\\ \gcd(a,\lfloor b^c\rfloor)=1}} 1 = \sum_{d\le x}\mu(d)\Big(\frac{x}{d}+O(1)\Big)\Big(\frac{x}{d}+O\Big(\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)$$

$$= x^2\sum_{d\le x}\frac{\mu(d)}{d^2}+O\Big(\sum_{d\le x}\Big(\frac{x}{d}+\frac{x^{(c+4)/3}}{d^{4/3}}+\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)$$

$$= \frac{1}{\zeta(2)}x^2+O(x)+O(x\log x)+O(x^{(c+4)/3})+O(x^{(c+3)/3})$$

$$= \frac{1}{\zeta(2)}x^2+O(x^{(c+4)/3}). \tag{13}$$

Then, (1) follows from (12) and (13).

Now we will prove (2). Let $\alpha > 1$ be irrational with bounded partial quotients and let $\beta \in [0, \alpha)$. Similarly to proving (1), we write

$$\begin{aligned}\sum_{\substack{a,b\le x\\ \gcd(\lfloor a^c\rfloor, \lfloor \alpha b+\beta\rfloor)=1}} 1 &= \sum_{a,b\le x}\sum_{\substack{d\mid\lfloor a^c\rfloor\\ d\mid\lfloor \alpha b+\beta\rfloor}} \mu(d)\\ &= \sum_{d\le \alpha x}\mu(d)\sum_{\substack{a\le x\\ \lfloor a^c\rfloor\equiv 0(\bmod d)}} 1 \sum_{\substack{b\le x\\ \lfloor \alpha b+\beta\rfloor\equiv 0(\bmod d)}} 1.\end{aligned}$$

In view of Lemma 1 and 2, we have

$$\begin{aligned}\sum_{\substack{a,b\le x\\ \gcd(\lfloor a^c\rfloor, \lfloor \alpha b+\beta\rfloor)=1}} 1 &= \sum_{d\le x^{1/2}\log^{-3/2-\varepsilon} x}\mu(d)\Big(\frac{x}{d}+O\Big(\min\Big(\frac{x^c}{d},\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)\Big)\times\\ &\quad\times\Big(\frac{x}{d}+O\Big(d\log^3 x\Big)\Big)\\ &\quad+\sum_{x^{1/2}\log^{-3/2-\varepsilon} x<d\le\alpha x}\mu(d)\sum_{\substack{a\le x\\ \lfloor a^c\rfloor\equiv 0(\bmod d)}} 1 \sum_{\substack{b\le x\\ \lfloor \alpha b+\beta\rfloor\equiv 0(\bmod d)}} 1\\ &= \sum_{d\le x^{1/2}\log^{-3/2-\varepsilon} x}\mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)\times\\ &\quad\times\Big(\frac{x}{d}+O\Big(d\log^3 x\Big)\Big)+O\Big(x^2\sum_{x^{1/2}\log^{-3/2-\varepsilon} x<d\le\alpha x}\frac{1}{d^2}\Big)\\ &= x^2\sum_{d\le x^{1/2}\log^{-3/2-\varepsilon} x}\frac{\mu(d)}{d^2}+O(x^{(c+4)/3}). \qquad (14)\end{aligned}$$

Then, (2) follows from (11) and (14). $\square$

Proof of Theorem 2. Let $\alpha > 1$ be irrational with bounded partial quotients and let $\beta \in [0, \alpha)$. We write

$$\begin{aligned}\sum_{\substack{a,b\le x\\ \gcd(a, \lfloor \alpha b+\beta\rfloor)=1}} 1 &= \sum_{a,b\le x}\sum_{\substack{d\mid a\\ d\mid\lfloor \alpha b+\beta\rfloor}} \mu(d)\\ &= \sum_{d\le x}\mu(d)\sum_{\substack{a\le x\\ a\equiv 0(\bmod d)}} 1 \sum_{\substack{b\le x\\ \lfloor \alpha b+\beta\rfloor\equiv 0(\bmod d)}} 1.\end{aligned}$$

In view of Lemma 2, we have

$$\begin{aligned}\sum_{\substack{a,b\le x\\ \gcd(a, \lfloor \alpha b+\beta\rfloor)=1}} 1 &= \sum_{d\le x^{1/2}\log^{-3/2-\varepsilon} x}\mu(d)\Big(\frac{x}{d}+O(1)\Big)\Big(\frac{x}{d}+O\Big(d\log^3 x\Big)\Big)\\ &\quad+\sum_{x^{1/2}\log^{-3/2-\varepsilon} x<d\le x}\mu(d)\Big(\frac{x}{d}+O(1)\Big)O\Big(\frac{x}{d}\Big)\end{aligned}$$

$$\begin{aligned}
&= x^2 \sum_{d \le x^{1/2} \log^{-3/2-\varepsilon} x} \frac{\mu(d)}{d^2} \\
&\quad + O\Big(\sum_{d \le x^{1/2} \log^{-3/2-\varepsilon} x} \Big(x \log^3 x + \frac{x}{d} + d \log^3 x \Big)\Big) \\
&\quad + O\Big(\sum_{x^{1/2} \log^{-3/2-\varepsilon} x < d \le x} \Big(\frac{x^2}{d^2} + \frac{x}{d} \Big)\Big) \\
&= x^2 \sum_{d \le x^{1/2} \log^{-3/2-\varepsilon} x} \frac{\mu(d)}{d^2} + O(x^{3/2} \log^{3/2+\epsilon} x). \qquad (15)
\end{aligned}$$

Due to (15) and (11) we have

$$\sum_{\substack{a,b \le x \\ \gcd(a, \lfloor \alpha b + \beta \rfloor) = 1}} 1 = \frac{1}{\zeta(2)} x^2 + O(x^{3/2} \log^{3/2+\epsilon} x).$$

The proof of (3) follows.

Next we will prove (4). We have

$$\begin{aligned}
\sum_{\substack{a,b \le x \\ \gcd(\lfloor \alpha a + \beta \rfloor, \lfloor \alpha b + \beta \rfloor) = 1}} 1 &= \sum_{a,b \le x} \sum_{\substack{d \mid \lfloor \alpha a + \beta \rfloor \\ d \mid \lfloor \alpha b + \beta \rfloor}} \mu(d) \\
&= \sum_{d \le \alpha x} \mu(d) \sum_{\substack{a \le x \\ \lfloor \alpha a + \beta \rfloor \equiv 0 (\bmod d)}} 1 \sum_{\substack{b \le x \\ \lfloor \alpha b + \beta \rfloor \equiv 0 (\bmod d)}} 1.
\end{aligned}$$

In view of Lemma 2, we have

$$\begin{aligned}
\sum_{\substack{a,b \le x \\ \gcd(\lfloor \alpha a + \beta \rfloor, \lfloor \alpha b + \beta \rfloor) = 1}} 1 &= \sum_{d \le x^{1/2} \log^{-3/2-\varepsilon} x} \mu(d) \Big(\frac{x}{d} + O\Big(d \log^3 x \Big)\Big)^2 \\
&\quad + O\Big(x^2 \sum_{x^{1/2} \log^{-3/2-\varepsilon} x < d \le \alpha x} \frac{1}{d^2} \Big) \\
&= x^2 \sum_{d \le x^{1/2} \log^{-3/2-\varepsilon} x} \frac{\mu(d)}{d^2} \\
&\quad + O\Big(\sum_{d \le x^{1/2} \log^{-3/2-\varepsilon} x} \Big(x \log^3 x + d^2 \log^6 x \Big)\Big) \\
&\quad + O(x^{3/2} \log^{3/2+\epsilon} x) \\
&= x^2 \sum_{d \le x^{1/2} \log^{-3/2-\varepsilon} x} \frac{\mu(d)}{d^2} + O(x^{3/2} \log^{3/2+\epsilon} x). \qquad (16)
\end{aligned}$$

Because of (16) and (11), we have

$$\sum_{\substack{a,b\le x\\ \gcd(\lfloor \alpha a+\beta\rfloor,\lfloor \alpha b+\beta\rfloor)=1}} 1=\frac{1}{\zeta(2)}x^2+O(x^{3/2}\log^{3/2+\epsilon}x).$$

The proof of (4) follows.

□

Proof of Theorem 3. Similarly to proving (1), we write

$$\begin{aligned}\sum_{\substack{a,b\le x\\ \gcd(a,b)=1}}\mathbb{1}_{S(x)}(b)&=\sum_{a,b\le x}\mathbb{1}_{S(x)}(b)\sum_{\substack{d|a\\ d|b}}\mu(d)\\ &=\sum_{d\le x}\mu(d)\sum_{\substack{a\le x\\ a\equiv 0(\bmod d)}}1\sum_{\substack{b\le x\\ b\equiv 0(\bmod d)}}\mathbb{1}_{S(x)}(b).\end{aligned}$$

By Lemma 3, we have

$$\begin{aligned}\sum_{\substack{a,b\le x\\ \gcd(a,b)=1}}\mathbb{1}_{S(x)}(b)=&\sum_{d\le x^{1/4}\log^{-3/2}x}\mu(d)\Big(\frac{x}{d}+O(1)\Big)\Big(\frac{2x^{1/2}}{d}+O\Big(\frac{x^{1/3}\log x}{d^{1/3}})\Big)\Big)\\ &+\sum_{x^{1/4}\log^{-3/2}x<d\le x}\mu(d)\Big(\frac{x}{d}+O(1)\Big)O\Big(\frac{x^{1/2}}{d}\Big)\\ =&\,2x^{3/2}\sum_{d\le x^{1/4}\log^{-3/4}x}\frac{\mu(d)}{d^2}\\ &+O\Big(\sum_{d\le x^{1/4}\log^{-3/4}x}\Big(\frac{x^{4/3}\log x}{d^{4/3}}+\frac{x^{1/2}}{d}+\frac{x^{1/3}\log x}{d^{1/3}}\Big)\Big)\\ &+O\Big(\sum_{x^{1/4}\log^{-3/4}x<d\le x}\Big(\frac{x^{3/2}}{d^2}+\frac{x^{1/2}}{d}\Big)\Big)\\ =&\,x^{3/2}\sum_{d\le x^{1/2}\log^{-3/2-\varepsilon}x}\frac{\mu(d)}{d^2}+O(x^{4/3}\log x).\end{aligned}\tag{17}$$

The proof of (5) follows from (11) and (17).

Now we prove (6). We write

$$\begin{aligned}\sum_{\substack{a,b\le x\\ \gcd(\lfloor a^c\rfloor,b)=1}}\mathbb{1}_{S(x)}(b)&=\sum_{a,b\le x}\mathbb{1}_{S(x)}(b)\sum_{\substack{d|\lfloor a^c\rfloor\\ d|b}}\mu(d)\\ &=\sum_{d\le x}\mu(d)\sum_{\substack{a\le x\\ \lfloor a^c\rfloor\equiv 0(\bmod d)}}1\sum_{\substack{b\le x\\ b\equiv 0(\bmod d)}}\mathbb{1}_{S(x)}(b).\end{aligned}$$

In view of Lemma 1 and 3, we have

$$\begin{aligned}
\sum_{\substack{a,b\le x\\ \gcd(\lfloor a^c\rfloor,b)=1}} \mathbb{1}_{S(x)}(b) &= \sum_{d\le x^{1/4}\log^{-3/2}x} \mu(d)\Big(\frac{x}{d}+O\Big(\min\Big(\frac{x^c}{d},\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)\Big)\times\\
&\quad\times\Big(\frac{2x^{1/2}}{d}+O\Big(\frac{x^{1/3}\log x}{d^{1/3}})\Big)\Big)\\
&\quad+\sum_{x^{1/4}\log^{-3/2}x<d\le x}\mu(d)\sum_{\substack{a\le x\\ \lfloor a^c\rfloor\equiv 0(\bmod d)}}1\sum_{\substack{b\le x\\ b\equiv 0(\bmod d)}}\mathbb{1}_{S(x)}(b)\\
&= \sum_{d\le x^{1/4}\log^{-3/2}x}\mu(d)\Big(\frac{x}{d}+O\Big(\frac{x^{(c+1)/3}}{d^{1/3}}\Big)\Big)\times\\
&\quad\times\Big(\frac{2x^{1/2}}{d}+O\Big(\frac{x^{1/3}\log x}{d^{1/3}}\Big)\Big)+O\Big(x^{3/2}\sum_{x^{1/4}\log^{-3/2}x<d\le x}\frac{1}{d^2}\Big)\\
&= 2x^{3/2}\sum_{d\le x^{1/4}\log^{-3/2}x}\frac{\mu(d)}{d^2}\\
&\quad+O\Big(\sum_{d\le x^{1/4}\log^{-3/2}x}\Big(\frac{x^{4/3}\log x}{d^{4/3}}+\frac{x^{(2c+5)/6}}{d^{4/3}}+\frac{x^{(c+2)/3}\log x}{d^{2/3}}\Big)\Big)\\
&\quad+O\Big(x^{5/4}\log^{3/2}x\Big)\\
&= 2x^{3/2}\sum_{d\le x^{1/4}\log^{-3/2}x}\frac{\mu(d)}{d^2}+O(x^{4/3}\log x)+O(x^{(2c+5)/6}). \qquad (18)
\end{aligned}$$

Now (6) follows from (11) and (18).

Next we prove (7). We write

$$\begin{aligned}
\sum_{\substack{a,b\le x\\ \gcd(\lfloor \alpha a+\beta\rfloor,b)=1}} \mathbb{1}_{S(x)}(b) &= \sum_{a,b\le x}\mathbb{1}_{S(x)}(b)\sum_{\substack{d\mid\lfloor\alpha a+\beta\rfloor\\ d\mid b}}\mu(d)\\
&= \sum_{d\le x}\mu(d)\sum_{\substack{a\le x\\ \lfloor\alpha a+\beta\rfloor\equiv 0(\bmod d)}}1\sum_{\substack{b\le x\\ b\equiv 0(\bmod d)}}\mathbb{1}_{S(x)}(b).
\end{aligned}$$

In view of Lemma 3 and 2, we have

$$\begin{aligned}
\sum_{\substack{a,b\le x\\ \gcd(\lfloor \alpha a+\beta\rfloor,b)=1}} \mathbb{1}_{S(x)}(b) &= \sum_{d\le x^{1/4}\log^{-3/2}x}\mu(d)\Big(\frac{x}{d}+O\Big(d\log^3 x\Big)\Big)\times\\
&\quad\times\Big(\frac{2x^{1/2}}{d}+O\Big(\frac{x^{1/3}\log x}{d^{1/3}}\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{x^{1/4}\log^{-3/2} x<d\le x} \mu(d) \sum_{\substack{a\le x\\ \lfloor \alpha a+\beta\rfloor\equiv 0(\bmod d)}} 1 \sum_{\substack{b\le x\\ b\equiv 0(\bmod d)}} \mathbb{1}_{S(x)}(b)\\
&= \sum_{d\le x^{1/4}\log^{-3/2} x} \mu(d)\Big(\frac{x}{d}+O\Big(d\log^3 x\Big)\Big)\times\\
&\times\Big(\frac{2x^{1/2}}{d}+O\Big(\frac{x^{1/3}\log x}{d^{1/3}})\Big)\Big)\\
&+O\Big(x^{3/2}\sum_{x^{1/4}\log^{-3/2} x<d\le x}\frac{1}{d^2}\Big)\\
&=2x^{3/2}\sum_{d\le x^{1/4}\log^{-3/2} x}\frac{\mu(d)}{d^2}\\
&+O\Big(\sum_{d\le x^{1/4}\log^{-3/2} x}\Big(\frac{x^{4/3}\log x}{d^{4/3}}+x^{1/2}\log^3 x+x^{1/3}d^{2/3}\log^4 x\Big)\Big)\\
&+O\Big(x^{5/4}\log^{3/2} x\Big)\\
&=2x^{3/2}\sum_{d\le x^{1/4}\log^{-3/2} x}\frac{\mu(d)}{d^2}+O(x^{4/3}\log x). \qquad (19)
\end{aligned}
$$

Now (7) follows from (11) and (19).

Lastly, we prove (8). We write

$$
\begin{aligned}
\sum_{\substack{a,b\le x\\ \gcd(a,b)=1}} \mathbb{1}_{S(x)}(a)\mathbb{1}_{S(x)}(b) &= \sum_{a,b\le x}\mathbb{1}_{S(x)}(a)\mathbb{1}_{S(x)}(b)\sum_{\substack{d|a\\ d|b}}\mu(d)\\
&=\sum_{d\le x}\mu(d)\sum_{\substack{a\le x\\ a\equiv 0(\bmod d)}}\mathbb{1}_{S(x)}(a)\sum_{\substack{b\le x\\ b\equiv 0(\bmod d)}}\mathbb{1}_{S(x)}(b).
\end{aligned}
$$

In view of Lemma 3, we have

$$
\begin{aligned}
\sum_{\substack{a,b\le x\\ \gcd(a,b)=1}} \mathbb{1}_{S(x)}(a)\mathbb{1}_{S(x)}(b) &= \sum_{d\le x^{1/4}\log^{-3/2} x}\mu(d)\Big(\frac{2x^{1/2}}{d}+O\Big(\frac{x^{1/3}}{d^{1/3}}\log x\Big)\Big)^2\\
&+\sum_{x^{1/4}\log^{-3/2} x<d\le x}\mu(d)\sum_{\substack{a\le x\\ a\equiv 0(\bmod d)}}\mathbb{1}_{S(x)}(a)\sum_{\substack{b\le x\\ b\equiv 0(\bmod d)}}\mathbb{1}_{S(x)}(b)\\
&=4x\sum_{d\le x^{1/4}\log^{-3/2} x}\frac{\mu(d)}{d^2}\\
&+O\Big(\sum_{d\le x^{1/4}\log^{-3/2} x}\Big(\frac{x^{5/6}\log x}{d^{4/3}}+\frac{x^{2/3}\log^2 x}{d^{2/3}}\Big)\Big)
\end{aligned}
$$

$$+O\Big(x \sum_{x^{1/4}\log^{-3/2} x<d\le x} \Big(\frac{1}{d^2}\Big)\Big)$$
$$= 4x \sum_{d\le x^{1/2}\log^{-3/2} x} \frac{\mu(d)}{d^2} + O(x^{5/6}\log x). \qquad (20)$$

The proof of (8) follows from (20) and (11).

□

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