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# Two types of the second Hankel determinant for the class $\mathcal{U}$ and the general class $\mathcal{S}$

Milutin Obradović and Nikola Tuneski

Abstract. In this paper we determine the upper bounds of the Hankel determinants of special type $H_2(3)(f)$ and $H_2(4)(f)$ for the general class of univalent functions and for the class $\mathcal{U}$.

## 1. Introduction and preliminaries

Let the class $\mathcal{A}$ consist of functions which are analytic in the unit disk $\mathbb{D} := \{|z| < 1\}$ and which are normalized such that

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots , \tag{1}$$

i.e., $f(0) = 0 = f'(0) - 1$; and let $\mathcal{S}$ be the class of functions from $\mathcal{A}$ that are univalent in $\mathbb{D}$.

In his paper [7] Zaprawa considered the following Hankel determinant of the second order

$$H_2(n)(f) = \begin{vmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{vmatrix} = a_na_{n+2} - a_{n+1}^2,$$

defined for the coefficients of the function given by (1) for the case when $n = 3$. The author studied the upper bound of $|H_2(3)(f)| = |a_3a_5 - a_4^2|$ in the cases when $f$ from $\mathcal{A}$ is starlike ($\operatorname{Re}[zf'(z)/f(z)] > 0$, $z \in \mathcal{U}$), convex ($\operatorname{Re}[1 + zf''(z)/f'(z)] > 0$, $z \in \mathcal{U}$), and with bounded turning ($\operatorname{Re} f'(z) > 0$, $z \in \mathcal{U}$). These types of functions were studied separately, under the condition that the functions are missing their second coefficient, i.e., $a_2 = 0$. For the general class $\mathcal{S}$, he proved that $|H_2(3)(f)| > 1$. In [6] the authors gave sharp bounds of the modulus of the second Hankel determinant of type $H_2(2)$ of inverse coefficients for various classes of univalent functions.

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Another interesting subclass of $\mathcal{S}$ that has attracted significant interest in the past two decades is

$$\mathcal{U}=\left\{f\in\mathcal{A}:\left|\left[\frac{z}{f(z)}\right]^2 f'(z)-1\right|<1,\, z\in\mathbb{D}\right\}.$$

More details can be found in [3] and Chapter 12 from [5].

The objective of this paper is to find upper bounds (preferably sharp) of the modulus of the Hankel determinants $H_2(3)(f)=a_3a_5-a_4^2$ and $H_2(4)(f)=a_4a_6-a_5^2$ for the class $\mathcal{U}$, as well as for the general class $\mathcal{S}$.

## 2. Class $\mathcal{U}$

For the functions $f$ from the class $\mathcal{U}$ in [4], as a part of the proof of Theorem 1, it was proven that there exists a function $\omega_1$, such that

$$\frac{z}{f(z)}=1-a_2z-z\omega_1(z), \tag{2}$$

where $|\omega_1(z)|\le|z|<1$ and $|\omega_1'(z)|\le 1$ for all $z\in\mathbb{D}$, and additionally, for $\omega_1(z)=c_1z+c_2z^2+\cdots$,

$$|c_1|\le 1,\quad |c_2|\le\frac{1}{2}(1-|c_1|^2)\quad\text{and}\quad |c_3|\le\frac{1}{3}\left[1-|c_1|^2-\frac{4|c_2|^2}{1+|c_1|}\right]. \tag{3}$$

In a similar way, since $|\omega_1'(z)|\le 1$, one can verify that

$$|c_4|\le\frac{1}{4}(1-|c_1|^2-4|c_2|^2).$$

Further, from (2), we have

$$z=f(z)\left[1-\left(a_2z+c_1z^2+c_2z^3+\cdots\right)\right]$$

and, after equating the coefficients,

$$\begin{aligned} a_3&=c_1+a_2^2,\\ a_4&=c_2+2a_2c_1+a_2^3,\\ a_5&=c_3+2a_2c_2+c_1^2+3a_2^2c_1+a_2^4,\\ a_6&=c_4+2a_2c_3+2c_1c_2+3a_2^2c_2+3a_2c_1^2+4a_2^3c_1+a_2^5.\end{aligned} \tag{4}$$

Now we can prove the estimates for the class $\mathcal{U}$.

**Theorem 1.** *Let $f\in\mathcal{U}$. Then*

(a) $|H_2(3)(f)|\le 1$ *if* $a_2=0$, *and the result is sharp due to the function* $f(z)=\frac{z}{1-z^2}=z+z^3+z^5+\cdots$.
(b) $|H_2(3)(f)|\le 1.4846575\ldots$ *for every* $f\in\mathcal{U}$.

*Proof.* Using (4), after some calculations we obtain

$$H_2(3)(f) = a_3a_5 - a_4^2 = (c_1 + a_2^2)c_3 - 2a_2c_1c_2 + c_1^3 - c_2^2,$$

and from here

$$|H_2(3)(f)| \le |c_1 + a_2^2||c_3| + 2|a_2||c_1||c_2| + |c_1|^3 + |c_2|^2. \tag{5}$$

(a) If $a_2 = 0$, from (5) we obtain

$$|H_2(3)(f)| \le |c_1||c_3| + |c_1|^3 + |c_2|^2,$$

and using (3),

$$\begin{aligned} |H_2(3)(f)| &\le |c_1| \cdot \frac{1}{3} \cdot \left[1 - |c_1|^2 - \frac{4|c_2|^2}{1+|c_1|}\right] + |c_1|^3 + |c_2|^2 \\ &= \frac{1}{3}\left(|c_1| - |c_1|^3\right) + \frac{3-|c_1|}{3(1+|c_1|)}|c_2|^2 + |c_1|^3 \\ &\le \frac{1}{3}|c_1| + \frac{2}{3}|c_1|^3 + \frac{3-|c_1|}{3(1+|c_1|)}\frac{1}{4}(1-|c_1|^2)^2 \\ &= \frac{1}{12}\left(3 - 2|c_1|^2 + 12|c_1|^3 - |c_1|^4\right) \equiv h_1(|c_1|), \end{aligned}$$

where $h_1(t) = \frac{1}{12}\left(3 - 2t^2 + 12t^3 - t^4\right)$ and $t = |c_1| \le 1$ (see (3)). Now, $h_1'(t) = -\frac{1}{3}c(1 - 9c + c^2)$ vanishes in only one point on the interval $(0, 1)$ and that is a minimum of $h_1$ on the interval since $h_1(t) < 0$ for small enough positive numbers (let us say, for $t = 0.1$). Therefore

$$\max\{h_1(t) : t \in [0, 1]\} = \max\{h_1(0), h_1(1)\} = h_1(1) = 1,$$

i.e. $|H_2(3)(f)| \le 1$. The sharpness of the estimate follows from the function $f(z) = \frac{z}{1-z^2}$ with $a_2 = a_4 = 0$ and $a_3 = a_5 = 1$.

(b) Since $\mathcal{U} \subset \mathcal{S}$, we have $|a_2| \le 2$ and $|a_3| = |c_1 + a_2^2| \le 3$ From (5) we have

$$|H_2(3)(f)| \le 3|c_3| + 4|c_1||c_2| + |c_1|^3 + |c_2|^2 \equiv \varphi_1(|c_1|, |c_2|, |c_3|),$$

where $\varphi_1(x, y, z) = 3z + 4xy + x^3 + y^2$ with (due to (3))

$$0 \le x \le 1, \quad 0 \le y \le \frac{1}{2}(1 - x^2), \quad 0 \le z \le \frac{1}{3}\left(1 - x^2 - \frac{4y^2}{1+x}\right).$$

It is evident that

$$\begin{aligned}
\varphi_1(x,y,z) &\le 3\cdot\frac{1}{3}\left(1-x^2-\frac{4y^2}{1+x}\right)+4xy+x^3+y^2\\
&=1-x^2+\left(4-\frac{4}{1+x}\right)y^2-3y^2+4xy+x^3\\
&\le 1-x^2+\frac{4x}{1+x}\cdot\frac{1}{4}\left(1-x^2\right)^2-3y^2+4xy+x^3\\
&=1+x-2x^2+x^4+4xy-3y^2\equiv\psi(x,y).
\end{aligned}$$

It remains to find the maximal value of the function $\psi$ on the domain $\Omega_1 = \left\{(x,y) : 0\le x\le 1,\, 0\le y\le \frac{1}{2}(1-x^2)\right\}$. Since $\psi'_y(x,y)=4x-6y$ vanishes for $x=\frac{3}{2}y$, and $\psi'_x(3y/2,y)=1-2y+\frac{27}{2}y^3$ vanishes only for $y=-0.535\ldots$ we realize that $\psi$ attains its maximal value on the boundary of $\Omega_1$. Finally, when $x=0$ or $x=1$, the maximum is 1, while for $y=0$, the maximum is $1.1295\ldots$ for $x=0.26959\ldots$, and for $y=\frac{1}{2}(1-x^2)$, the maximum is $1.4846575\ldots$ for $x=0.6618\ldots$. This completes the proof. $\square$

**Theorem 2.** *Let $f\in\mathcal{U}$ and $a_2=0$. Then $|H_2(4)(f)|\le 1$ and the estimate is sharp due to the function $f(z)=\frac{z}{1-z^2}=z+z^3+z^5+z^7+\cdots$.*

*Proof.* If $f\in\mathcal{U}$ and $a_2=0$, then from (4) we obtain

$$a_4=c_2,\quad a_5=c_3+c_1^2,\quad c_6=c_4+2c_1c_2.$$

Further,

$$H_2(4)(f)=a_4a_6-a_5^2=c_2c_4+2c_1c_2^2-c_3^2-2c_1^2c_3+c_1^4$$

and, using (3), we have

$$\begin{aligned}
&|H_2(4)(f)|\\
&\le |c_2||c_4|+2|c_1||c_2|^2+|c_3|^2+2|c_1|^2|c_3|+|c_1|^4\\
&\le \frac{1}{2}(1-|c_1|^2)\cdot\frac{1}{4}(1-|c_1|^2-4|c_2|^2)+2|c_1||c_2|^2+\frac{1}{9}\left(1-|c_1|^2-\frac{4|c_2|^2}{1+|c_1|}\right)^2\\
&\quad+2|c_1|^2\cdot\frac{1}{3}\left(1-|c_1|^2-\frac{4|c_2|^2}{1+|c_1|}\right)+|c_1|^4\\
&=A|c_2|^4+B|c_2|^2+C\equiv h_2(|c_2|),
\end{aligned}$$

where $h_2(t) = At^4 + Bt^2 + C$,

$$A = \frac{16}{9(1+|c_1|)^2},$$
$$B = 2|c_1| - \frac{1}{2}(1-|c_1|^2) - \frac{8}{9}(1-|c_1|) - \frac{8}{3}\frac{|c_1|^2}{1+|c_1|},$$
$$C = \frac{17}{72}(1-|c_1|^2)^2 + \frac{2}{3}|c_1|^2(1-|c_1|^2) + |c_1|^4,$$

with $A > 0$, $0 \le |c_2| \le \frac{1}{2}(1-|c_1|^2)$ and $|c_1| \le 1$. Therefore, $h_2$ attains its maximal value on the boundary, i.e.

$$\max h_2(|c_2|) = \max\left\{h_2(0), h_2\left(\frac{1}{2}(1-|c_1|^2)\right)\right\}.$$

We note that $h_2(0) = C \equiv g_1(|c_1|)$, where $g_1(t) = \frac{1}{72}(41t^4 + 14t^2 + 17)$, has a maximal value 1 when $0 \le t = |c_1| \le 1$, attained for $t = 1$.

Further, let $g_2(|c_1|) \equiv h_2\left(\frac{1}{2}(1-|c_1|^2)\right)$, where

$$g_2(t) = \frac{1}{72}(17t^6 - 12t^5 + 38t^4 - 24t^3 + 17t^2 + 36t),$$

$0 \le t = |c_1| \le 1$. In order to complete the proof of the theorem it is enough to show that this function is increasing on the interval $[0,1]$, which will lead to the conclusion that $h_2\left(\frac{1}{2}(1-|c_1|^2)\right) = g_2(|c_1|) \le g_2(1) = 1$.

Indeed, $g_2'''(t) = \frac{1}{72}\left(1020t^2 - 288t + 228\right) > 0$ for all $t \in [0,1]$, meaning that $g_2''(t) = \frac{1}{72}\left(340t^3 - 144t^2 + 228t - 48\right)$ is increasing on the same interval. Since $g_2''(0) < 0$ and $g_2''(1) > 0$, there is only one real solution of $g_2''(t) = 0$ on $[0,1]$, i.e., only one local extreme (minimum) on $[0,1]$ for $t_* = 0.22554\ldots$ with value $g_2'(t_*) = 39.028\ldots > 0$. Thus, $g_2'(t) > 0$ for all $t \in [0,1]$. □

Theorem 1(a) and Theorem 2 are the motivation for the following conjecture for the functions from $\mathcal{U}$ with missing second coefficient.

*Conjecture* 1. Let $f \in \mathcal{U}$ and $a_2 = 0$. Then $|H_2(n)(f)| = |a_n a_{n+2} - a_{n+1}^2| \le 1$ for any integer $n \ge 3$. The estimate is a sharp due to the function $f(z) = \frac{z}{1-z^2} = \sum_{n=1}^{\infty} z^{2n-1}$.

## 3. General class $\mathcal{S}$

For obtaining the estimates of the modulus of $H_2(3)(f)$ for the general class $\mathcal{S}$ we will use a method based on the Grunsky coefficients based on the results and notations given in the book by Lebedev ([2]) as follows.

Let $f \in \mathcal{S}$ and let

$$\log \frac{f(t)-f(z)}{t-z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,$$

where $\omega_{p,q}$ are the Grunsky's coefficients with property $\omega_{p,q} = \omega_{q,p}$. For those coefficients the next Grunsky's inequality ([1, 2]) holds:

$$\sum_{q=1}^{\infty} q \left| \sum_{p=1}^{\infty} \omega_{p,q} x_p \right|^2 \le \sum_{p=1}^{\infty} \frac{|x_p|^2}{p}, \tag{6}$$

where $x_p$ are arbitrary complex numbers such that the last series converges.

Further, it is well-known that if the function $f$ given by (1) belongs to $\mathcal{S}$, then also

$$\tilde{f}_2(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \cdots \tag{7}$$

belongs to the class $\mathcal{S}$. Then, for the function $\tilde{f}_2$ we have the appropriate Grunsky's coefficients of the form $\omega_{2p-1,2q-1}^{(2)}$ and the inequality (6) has the form:

$$\sum_{q=1}^{\infty} (2q-1) \left| \sum_{p=1}^{\infty} \omega_{2p-1,2q-1} x_{2p-1} \right|^2 \le \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1}. \tag{8}$$

Here and further in the paper we omit the upper index (2) in $\omega_{2p-1,2q-1}^{(2)}$ if compared with Lebedev's notation.

If in the inequality (8) we put $x_1 = 1$ and $x_{2p-1} = 0$ for $p = 2, 3, \ldots$, then we obtain

$$|\omega_{11}|^2 + 3|\omega_{13}|^2 + 5|\omega_{15}|^2 + 7|\omega_{17}|^2 \le 1. \tag{9}$$

As it has been shown in [2, p. 57], if $f$ is given by (1), then the coefficients $a_2$, $a_3$, $a_4$ and $a_5$ are expressed by the Grunsky's coefficients $\omega_{2p-1,2q-1}$ of the function $\tilde{f}_2$ given by (7) in the following way:

$$\begin{aligned}
a_2 &= 2\omega_{11}, \\
a_3 &= 2\omega_{13} + 3\omega_{11}^2, \\
a_4 &= 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3, \\
a_5 &= 2\omega_{35} + 8\omega_{11}\omega_{33} + 5\omega_{13}^2 + 18\omega_{11}^2\omega_{13} + \frac{7}{3}\omega_{11}^4, \\
0 &= 3\omega_{15} - 3\omega_{11}\omega_{13} + \omega_{11}^3 - 3\omega_{33}, \\
0 &= \omega_{17} - \omega_{35} - \omega_{11}\omega_{33} - \omega_{13}^2 + \frac{1}{3}\omega_{11}^4.
\end{aligned} \tag{10}$$

We note that in the cited book of Lebedev there exists a typing mistake for the coefficient $a_5$. Namely, instead of the term $5\omega_{13}^2$, there is $5\omega_{15}^2$.

**Theorem 3.** *Let $f \in \mathcal{S}$ be given by* (1). *Then*
(a) $|H_2(3)(f)| \le 2.02757\dots$ *if* $a_2 = 0$;
(b) $|H_2(3)(f)| \le 4.8986977\dots$ *for every* $f \in \mathcal{S}$.

*Proof.* From the fifth relation of (10) we have

$$\omega_{33} = \omega_{15} - \omega_{11}\omega_{13} + \frac{1}{3}\omega_{11}^3.$$

This, together with the sixth relation from (10) yields

$$\omega_{35} = \omega_{17} - \omega_{11}\omega_{15} + \omega_{11}^2\omega_{13} - \omega_{13}^2.$$

By applying the two expressions from above in the relations for $a_4$ and $a_5$ from (10), we obtain

$$\begin{aligned} a_4 &= 2\omega_{15} + 6\omega_{11}\omega_{13} + 4\omega_{11}^3, \\ a_5 &= 2\omega_{17} + 6\omega_{11}\omega_{15} + 12\omega_{11}^2\omega_{13} + 3\omega_{13}^2 + 5\omega_{11}^4. \end{aligned}$$

Finally, these two relations, together with the relation for $a_3$ from (10) give

$$\begin{aligned} H_2(3)(f) &= a_3a_5 - a_4^2 \\ &= 2(2\omega_{13} + 3\omega_{11}^2)\omega_{17} - 12\omega_{11}\omega_{13}\omega_{15} - 3\omega_{11}^2\omega_{13}^2 + 6\omega_{13}^3 \qquad (11) \\ &\quad - 2\omega_{11}^4\omega_{13} + 2\omega_{11}^3\omega_{15} - \omega_{11}^6 - 4\omega_{15}^2. \end{aligned}$$

(a) If $a_2 = 2\omega_{11} = 0$, then $\omega_{11} = 0$, and we conclude that

$$H_2(3)(f) = 4\omega_{13}\omega_{17} + 6\omega_{13}^3 - 4\omega_{15}^2,$$

with the following constraints on $\omega_{13}$, $\omega_{15}$ and $\omega_{17}$ obtained from (9):

$$|\omega_{13}| \le \frac{1}{\sqrt{3}}, \quad |\omega_{15}| \le \frac{1}{\sqrt{5}}\sqrt{1 - 3|\omega_{13}|^2}$$

and

$$|\omega_{17}| \le \frac{1}{\sqrt{7}}\sqrt{1 - 3|\omega_{13}|^2 - 5|\omega_{15}|^2}.$$

So,

$$\begin{aligned} |H_2(3)(f)| &= 4|\omega_{13}||\omega_{17}| + 6|\omega_{13}|^3 + 4|\omega_{15}|^2 \\ &\le \frac{4}{\sqrt{7}}|\omega_{13}|\sqrt{1 - 3|\omega_{13}|^2 - 5|\omega_{15}|^2} + 6|\omega_{13}|^3 + 4|\omega_{15}|^2 \\ &= \psi_1(|\omega_{13}|, |\omega_{15}|), \end{aligned}$$

where $\psi_1(y,z) = \frac{4}{\sqrt{7}}y\sqrt{1-3y^2-5z^2} + 6y^3 + 4z^2$ with $0 \le y = |\omega_{13}| \le \frac{1}{\sqrt{3}}$, $0 \le z = |\omega_{15}| \le \frac{1}{\sqrt{5}}\sqrt{1-3y^2}$. It remains to find an upper bound of the function $\psi_1(y,z)$ on its domain

$$\Omega = \left\{ (y,z) : 0 \le y \le \frac{1}{\sqrt{3}},\ 0 \le z \le \frac{1}{\sqrt{5}}\sqrt{1-3y^2} \right\}.$$

Not being able to do better and leaving the sharp bound as an open problem, we continue with what is easy to get:

$$\begin{aligned}\psi_1(y,z) &\le \frac{4}{\sqrt{7}}y + 6y^3 + \frac{4}{5}(1-3y^2) = \frac{4}{5} + \frac{4}{\sqrt{7}}y - \frac{12}{5}y^2 + 6y^3\\ &\le \frac{4}{\sqrt{21}} + \frac{2}{\sqrt{3}} = 2.02757\dots,\end{aligned}$$

obtained for $y = \frac{1}{\sqrt{3}}$.

(b) In the general case, if $a_2 \neq 0$, since $|a_2| \le 2$ and $|c_1 + a_2^2| = |a_3| \le 3$, from (11) we get

$$\begin{aligned}|H_2(3)(f)| &= 6|\omega_{17}| + 12|\omega_{11}||\omega_{13}||\omega_{15}| + 3|\omega_{11}|^2|\omega_{13}|^2\\ &\quad + 6|\omega_{13}|^3 + 2|\omega_{11}|^4|\omega_{13}| + 2|\omega_{11}|^3|\omega_{15}|\\ &\quad + |\omega_{11}|^6 + 4|\omega_{15}|^2\\ &\le 6\cdot\frac{1}{\sqrt{7}}\sqrt{1-|\omega_{11}|^2-3|\omega_{13}|^2-5|\omega_{15}|^2} + 12|\omega_{11}||\omega_{13}||\omega_{15}|\\ &\quad + 3|\omega_{11}|^2|\omega_{13}|^2 + 6|\omega_{13}|^3 + 2|\omega_{11}|^4|\omega_{13}|\\ &\quad + 2|\omega_{11}|^3|\omega_{15}| + |\omega_{11}|^6 + 4|\omega_{15}|^2\\ &= \psi_2(|\omega_{11}|,|\omega_{13}|,\omega_{15}|),\end{aligned}$$

where

$$\begin{aligned}\psi_2(x,y,z) = \frac{6}{\sqrt{7}}\sqrt{1-x^2-3y^2-5z^2} + 12xyz + 3x^2y^2\\ + 6y^3 + 2x^4y + 2x^3z + x^6 + 4z^2\end{aligned}$$

with $0 \le x = |\omega_{11}| \le 1$, $0 \le y = |\omega_{13}| \le \frac{1}{\sqrt{3}}\sqrt{1-x^2}$, $0 \le z = |\omega_{15}| \le \frac{1}{\sqrt{5}}\sqrt{1-x^2-3y^2}$. Similarly as in the part (a), finding an upper bound of the function $\psi_2(x,y,z)$ on its domain

$$\left\{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le \frac{1}{\sqrt{3}}\sqrt{1-x^2},\right.$$
$$\left. 0 \le z \le \frac{1}{\sqrt{5}}\sqrt{1-x^2-3y^2}\right\},$$

is still an open problem, even though analysis suggests that it is 1. Easy way around, leading to a non-sharp upper bound is:

$$\begin{aligned}\psi_2(x,y,z) \le \frac{6}{\sqrt{7}}\sqrt{1-x^2} + 12xyz + 3x^2y^2\\ + 6y^3 + 2x^4y + 2x^3z + x^6 + 4z^2,\end{aligned}$$

which after applying $y \le \frac{1}{\sqrt{3}}\sqrt{1-x^2}$ and $z \le \frac{1}{\sqrt{5}}\sqrt{1-x^2}$ leads to

$$\begin{aligned}\psi_2(x,y,z) \le & \frac{6}{\sqrt{7}}\sqrt{1-x^2} + \frac{12}{\sqrt{15}}x(1-x^2) + x^2(1-x^2) \\ & + \frac{6}{3\sqrt{3}}(1-x^2)\sqrt{1-x^2} + \frac{2}{\sqrt{3}}x^4\sqrt{1-x^2} \\ & + \frac{2}{\sqrt{5}}x^3\sqrt{1-x^2} + x^6 + \frac{4}{5}(1-x^2) \equiv h_*(x).\end{aligned}$$

Our numerical computations show that this function has the maximal value $4.8986977\ldots$ obtained for $x = 0.3945667\ldots$. □

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Department of Mathematics, Faculty of Civil Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11000, Belgrade, Serbia.

*E-mail address*: `obrad@grf.bg.ac.rs`

Department of Mathematics and Informatics, Faculty of Mechanical Engineering, Ss. Cyril and Methodius University in Skopje, Karpoš II b.b., 1000 Skopje, Republic of North Macedonia.

*E-mail address*: `nikola.tuneski@mf.ukim.edu.mk`