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(ψ, ϕ)-Wardowski contraction for three maps in G_b-metric spaces

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Abstract. Introducing $(\psi, \phi) - G_b$-Wardowski contraction for three maps, a common fixed point result is obtained for complete G_b-metric spaces. An application related to discontinuous activation function in a neural network is also established.

1. Introduction and preliminaries

One of the interesting problems of fixed point theory is the Rhoades' problem on discontinuity at a fixed point. Rhoades [8] mentioned the question "whether there exists a contractive condition that is strong enough to generate a fixed point but that does not force the map to be continuous at the fixed point?" After the first solution given by Pant [6], several solutions of this open problem have been presented via different approaches.

Here we solve this problem for a G_b-metric spaces.

Definition 1 ([1])**.** Let X be a nonempty set, $s \geq 1$ and $G_b : X \times X \times X \to \mathbb{R}^+$ a function satisfying the following properties:

$(GB1)$ $G_b(x, y, z) = 0$, if $x = y = z$,
$(GB2)$ $G_b(x, x, y) > 0$, for all $x, y \in X$ with $x \neq y$,
$(GB3)$ $G_b(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,
$(GB4)$ $G_b(x, y, z) = G_b(p\{x, y, z\})$, where p is a permutation of x, y, z,
$(GB5)$ $G_b(x, y, z) \leq s[G_b(x, a, a) + G_b(a, y, z)]$, for all $x, y, z, a \in X$.

Then G_b is called a generalized b-metric on X and the pair (X, G_b) is called a G_b-metric space.

Note that, for $s = 1$, a G_b-metric space reduces to a G-metric space.

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Example 1. Let $X = \mathbb{R}$. Define a mapping $G : X^3 \to \mathbb{R}^+$ by

$$G(x, y, z) = max\{|x - y|^2, |y - z|^2, |z - x|^2\}.$$

Then (X, G) is a G_b-metric space, but not a G-metric space.

Definition 2 ([1])**.** A G_b-metric space (X, G_b) is said to be symmetric if $G_b(x, y, y) = G_b(y, x, x)$, for all $x, y \in X$.

Definition 3 ([1])**.** For a sequence $\{x_n\}$ and a point x in (X, G_b), we say that:

(1) $\{x_n\}$ G_b-converges to x, if $\lim_{n,m\to\infty} G_b(x_n, x_m, x) = 0$, that is, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G_b(x_n, x_m, x) < \varepsilon$, for all $n, m \geq n_0$;
(2) $\{x_n\}$ is G_b-Cauchy if $\lim_{n,m,k\to\infty} G_b(x_n, x_m, x_k) = 0$, that is, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G_b(x_n, x_m, x_k) < \varepsilon$, for all $n, m, k \geq n_0$;
(3) (X, G_b) is complete if every G_b-Cauchy sequence in X is G_b-convergent in X.

Proposition 1 ([1])**.** *For a sequence $\{x_n\}$ and a point x in (X, G_b), the following are equivalent:*

(a) $\{x_n\}$ *G_b-converges to x,*
(b) $\lim_{n\to\infty} G_b(x_n, x_n, x) = 0$,
(c) $\lim_{n\to\infty} G_b(x_n, x, x) = 0$.

Proposition 2 ([1])**.** *For a sequence $\{x_n\}$ and a point x in (X, G_b), $\{x_n\}$ is G_b-Cauchy if and only if* $\lim_{n,m\to\infty} G_b(x_n, x_m, x_m) = 0$.

Definition 4. Let (X, G) and (X, G') be two G_b-metric spaces. Then a function $f : X \to X'$ is G_b-continuous at a point $x \in X$ if and only if $\{f(x_n)\} \to f(x)$, whenever $\{x_n\} \to x$.

Proposition 3 ([1])**.** *Let (X, G_b) be a G_b-metric space. Then, for each $x, y, z, a \in X$:*

(1) $G_b(x, y, z) = 0 \implies x = y = z$,
(2) $G_b(x, y, z) \leq s[G_b(x, x, y) + G_b(x, x, z)]$,
(3) $G_b(x, y, y) \leq 2sG_b(y, x, x)$,
(4) $G_b(x, y, z) \leq s[G_b(x, a, z) + G_b(a, y, z)]$.

In 1997, Matkowski [3] introduced the concept of comparison functions. A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a comparison function if it satisfies the following:

(a) ψ is monotone increasing,
(b) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, where ψ^n is n^{th} iterate of ψ.

The collection of all comparison functions is denoted by F_{com}. Notice that, if ψ is a comparison function, then $\psi(t) < t$ for each $t > 0$.

In the sequel, Φ denotes the collection of non-decreasing, continuous functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \phi(t_n) = 0$ if and only if $\lim_{n\to\infty} t_n = 0$.

In 2012, Wardowski [9] introduced the F-contraction and proved fixed point results for such mappings. Later, Liu et al. [2] introduced the (ψ,ϕ)-type contraction for metric spaces as follows.

Definition 5. Let T be a self-map defined on the metric space (X,d). Then T is said to be a (ψ,ϕ)-type contraction, if there exists $\phi \in \Phi$ and $\psi \in F_{com}$, such that

$$d(Tx,Ty) > 0 \implies \phi(d(Tx,Ty)) \leq \psi(\phi(M(x,y))), \forall\ x,y \in X,$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}d(x,Ty), d(y,Tx)\}.$$

The objective of this article is to find a contractive condition which does not force the mapping to be continuous at their common fixed points. For this, we first introduce generalized $(\psi,\phi) - G_b$-Wardowski contraction for three maps and establish a common fixed point theorem in the setting of complete G_b-metric spaces.

2. Main result

Definition 6. Let f be a self-map defined on the G_b-metric space (X,G). Suppose that there exist $\phi \in \Phi$ and $\psi \in F_{com}$, such that

$$G(fx,fy,fz) > 0 \implies \phi(2s^4 G(fx,fy,fz)) \leq \psi(\phi(M_1(x,y,z))),$$

for all $x,y,z \in X$, where

$$M_1(x,y,z) = \max\Big\{G(x,y,z), G(x,fx,fy), G(y,fy,fz), G(z,fz,fx), \frac{1}{4s}[G(fx,y,z) + G(x,fy,z) + G(x,y,fz)]\Big\}.$$

Then f is said to be a $(\psi,\phi) - G_b$-Wardowski contraction.

Definition 7. Let f,g,h be self-maps defined on the G_b-metric space (X,G). Suppose that there exist $\phi \in \Phi$ and $\psi \in F_{com}$, such that

$$G(fx,gy,hz) > 0 \implies \phi(2s^4 G(fx,gy,hz)) \leq \psi(\phi(M_2(x,y,z))), \quad (1)$$

for all $x,y,z \in X$, where

$$M_2(x,y,z) = \max\Big\{G(x,y,z), G(x,fx,gy), G(y,gy,hz), G(z,hz,fx),$$

$$\frac{1}{4s}[G(fx,y,z)+G(x,gy,z)+G(x,y,hz)]\Big\}.$$

Then we say that (f,g,h) is a generalized $(\psi,\phi)-G_b$-Wardowski contraction.

Now, we establish a common fixed point theorem for three maps related to a generalized $(\psi,\phi)-G_b$-Wardowski contraction.

Theorem 1. *Let $f,g,h : X \to X$ be a generalized $(\psi,\phi)-G_b$-Wardowski contraction in a complete G_b-metric space. Then f,g,h have a unique common fixed point, say u, and $f^n x \to u$, $g^n x \to u$ and $h^n x \to u$, for each $x \in X$. Further, at least one of f,g and h is not continuous at u if and only if*

$$\lim_{x\to u} M_2(x,u,u) \neq 0 \text{ or } \lim_{y\to u} M_2(u,y,u) \neq 0 \text{ or } \lim_{z\to u} M_2(u,u,z) \neq 0.$$

Proof. For any initial point $x_0 \in X$, we can construct a sequence $\{x_n\}$ by setting

$$x_{3n+1} = fx_{3n},\ x_{3n+2} = gx_{3n+1},\ x_{3n+3} = hx_{3n+2},\ n \geq 0.$$

Suppose that $x_n = x_{n+1}$, for some $n \in \mathbb{N}$.
If $x_{3n} = x_{3n+1}$, then x_{3n} is a fixed point of f.
If $x_{3n+1} = x_{3n+2}$, then x_{3n+1} is a fixed point of g.
If $x_{3n+2} = x_{3n+3}$, then x_{3n+2} is a fixed point of h.
Thus, at least, one of the mappings f,g or h has a fixed point.
We assume that $x_n \neq x_{n+1}$, for all n. Let $d_n = G(x_n,x_{n+1},x_{n+2}) > 0$, for all n.
Hence

$$G(fx_{3n},gx_{3n+1},hx_{3n+2}) = G(x_{3n+1},x_{3n+2},x_{3n+3}) = d_{3n+1} > 0$$

implies that

$$\begin{aligned}\phi(2s^4 d_{3n+1}) &= \phi(2s^4 G(x_{3n+1},x_{3n+2},x_{3n+3})) \\ &\leq \psi(\phi(M_2(x_{3n},x_{3n+1},x_{3n+2}))), \end{aligned} \tag{2}$$

where

$$\begin{aligned}&M_2(x_{3n},x_{3n+1},x_{3n+2}) \\ &= \max\Big\{G(x_{3n},x_{3n+1},x_{3n+2}), G(x_{3n},fx_{3n},gx_{3n+1}), G(x_{3n+1},gx_{3n+1},hx_{3n+2}), \\ &\qquad G(x_{3n+2},hx_{3n+2},fx_{3n}), \frac{1}{4s}[G(fx_{3n},x_{3n+1},x_{3n+2}) \\ &\qquad + G(x_{3n},gx_{3n+1},x_{3n+2}) + G(x_{3n},x_{3n+1},hx_{3n+2})]\Big\} \\ &= \max\Big\{G(x_{3n},x_{3n+1},x_{3n+2}), G(x_{3n+1},x_{3n+2},x_{3n+3}),\end{aligned}$$

$$\frac{1}{4s}[G(x_{3n+1}, x_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+2}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, x_{3n+3})]\Big\}.$$

We have

$$G(x_{3n+1}, x_{3n+1}, x_{3n+2}) \leq G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = d_{3n+1},$$

$$G(x_{3n}, x_{3n+2}, x_{3n+2}) \leq G(x_{3n}, x_{3n+1}, x_{3n+2}) = d_{3n},$$

$$\begin{aligned} G(x_{3n}, x_{3n+1}, x_{3n+3}) &\leq s[G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3})] \\ &= s[d_{3n} + d_{3n+1}]. \end{aligned}$$

Hence

$$\begin{aligned} M_2 &= max\{d_{3n}, d_{3n+1}, \frac{s+1}{4s}(d_{3n} + d_{3n+1})\} \\ &= \max\{d_{3n}, d_{3n+1}\}. \end{aligned}$$

If $M_2 = d_{3n+1}$, then, from (2), we have

$$\phi(2s^4 d_{3n+1}) \leq \psi(\phi(d_{3n+1})) < \phi(d_{3n+1}),$$

which is not possible. Hence $M_2 = d_{3n}$.

Using (2), we obtain

$$\phi(2s^4 d_{3n+1}) \leq \psi(\phi(d_{3n})) < \phi(d_{3n}),\ \ for\ all\ n \in \mathbb{N}. \tag{3}$$

Also, we have

$$\begin{aligned} \phi(2s^4 d_{3n+2}) &= \phi(2s^4 G(x_{3n+2}, x_{3n+3}, x_{3n+4})) \\ &= \phi(G(gx_{3n+1}, hx_{3n+2}, fx_{3n+3})) \\ &\leq \psi(\phi(M_2(x_{3n+3}, x_{3n+1}, x_{3n+2}))), \end{aligned} \tag{4}$$

where

$$\begin{aligned} &M_2(x_{3n+3}, x_{3n+1}, x_{3n+2}) \\ &= \max\Big\{G(x_{3n+3}, x_{3n+1}, x_{3n+2}), G(x_{3n+3}, fx_{3n+3}, gx_{3n+1}), \\ &\qquad G(x_{3n+1}, gx_{3n+1}, hx_{3n+2}), G(x_{3n+2}, hx_{3n+2}, fx_{3n+3}), \\ &\qquad \frac{1}{4s}[G(fx_{3n+3}, x_{3n+1}, x_{3n+2}) + G(x_{3n+3}, gx_{3n+1}, x_{3n+2}) \\ &\qquad + G(x_{3n+3}, x_{3n+1}, hx_{3n+2})]\Big\} \\ &= \max\Big\{G(x_{3n+3}, x_{3n+1}, x_{3n+2}), G(x_{3n+3}, x_{3n+4}, x_{3n+2}), \end{aligned}$$

$$\frac{1}{4s}[G(x_{3n+4},x_{3n+1},x_{3n+2})+G(x_{3n+3},x_{3n+2},x_{3n+2})$$
$$+G(x_{3n+3},x_{3n+1},x_{3n+3})]\Big\}$$
$$\leq \max\Big\{G(x_{3n+1},x_{3n+2},x_{3n+3}),G(x_{3n+2},x_{3n+3},x_{3n+4}),$$
$$\frac{s+1}{4s}[G(x_{3n+1},x_{3n+2},x_{3n+3})+G(x_{3n+2},x_{3n+3},x_{3n+4})]\Big\}$$
$$=\max\{d_{3n+1},d_{3n+2}\}.$$

If $M_2 = d_{3n+2}$, then from (4) we get

$$\phi(2s^4 d_{3n+2}) \leq \psi(\phi(d_{3n+2})) < \phi(d_{3n+2}),$$

which is not possible. Hence $M_2 = d_{3n+1}$.

Using (4), we have

$$\phi(2s^4 d_{3n+2}) \leq \psi(\phi(d_{3n+1})) < \phi(d_{3n+1}). \tag{5}$$

Similarly, we can obtain

$$\phi(2s^4 d_{3n+3}) \leq \psi(\phi(d_{3n+2})) < \phi(d_{3n+2}). \tag{6}$$

From (3),(5) and (6), we have

$$\phi(d_{n+1}) \leq \phi(2s^4 d_{n+1}) \leq \psi(\phi(d_n)) \leq \psi^2(\phi(d_{n-1})) \leq ... \leq \psi^n(\phi(d_1)).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} \psi^n(\phi(d_1)) = 0$.
Thus $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = 0$.
Since $x_n \neq x_{n+1}$ for every n, so by property $(GB3)$, we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}).$$

Hence

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

Since $G(x_n, x_n, x_{n+1}) \leq sG(x_n, x_{n+1}, x_{n+1})$, for all $n \geq 0$,

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0.$$

Now, we prove that $\{x_n\}$ is a G_b-Cauchy sequence in X. It is sufficient to show that $\{x_{3n}\}$ is G_b-Cauchy in X. On contrary, assume that $\{x_{3n}\}$ is not a G_b-Cauchy sequence. There exists $\varepsilon > 0$ for which we can find subsequences $\{x_{3m_k}\}$ and $\{x_{3n_k}\}$ of $\{x_{3n}\}$ such that m_k is the smallest index for which $3m_k > 3n_k > k$ and

$$G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) < \varepsilon \leq G(x_{3n_k}, x_{3m_k}, x_{3m_k}).$$

Since

$$\varepsilon \leq G(x_{3n_k}, x_{3m_k}, x_{3m_k})$$

$$\leq s[G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+1}) + G(x_{3n_k+1}, x_{3m_k}, x_{3m_k})]$$
$$\leq s[G(x_{3n_k}, x_{3n_k+1}, x_{3n_k+1}) + G(x_{3n_k+1}, x_{3m_k}, x_{3m_k-1})],$$

taking upper limit as $k \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} G(x_{3n_k+1}, x_{3m_k}, x_{3m_k-1}), \tag{7}$$

which implies that $G(x_{3n_k+1}, x_{3m_k}, x_{3m_k-1}) > 0$, for all $k \in \mathbb{N}$.
Hence, from (1), we have

$$\begin{aligned}\phi(2s^4G(x_{3n_k+1}, x_{3m_k-1}, x_{3m_k})) &= \phi(2s^4G(fx_{3n_k}, gx_{3m_k-2}, hx_{3m_k-1})) \\ &\leq \psi(\phi(M_2(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}))),\end{aligned} \tag{8}$$

where

$$\begin{aligned}&M_2(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) \\ &= \max\Big\{G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}), G(x_{3n_k}, fx_{3_k}, gx_{3m_k-2}), \\ &\quad G(x_{3m_k-2}, gx_{3m_k-2}, hx_{3m_k-1}), G(x_{3m_k-1}, hx_{3m_k-1}, fx_{3n_k}), \\ &\quad \frac{1}{4s}[G(fx_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) + G(x_{3n_k}, gx_{3m_k-2}, x_{3m_k-1}) \\ &\quad + G(x_{3n_k}, x_{3m_k-2}, hx_{3m_k-1})]\Big\} \\ &= \max\Big\{G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}), G(x_{3n_k}, x_{3n_k+1}, x_{3m_k-1}), \\ &\quad G(x_{3m_k-2}, x_{3m_k-1}, x_{3m_k}), G(x_{3m_k-1}, x_{3m_k}, x_{3n_k+1}), \\ &\quad \frac{1}{4s}[G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1}) + G(x_{3n_k}, x_{3m_k-1}, x_{3m_k-1}) \\ &\quad + G(x_{3n_k}, x_{3m_k-2}, x_{3m_k})]\Big\}.\end{aligned}$$

Since

$$\begin{aligned}G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) \leq &s[G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) \\ &+ G(x_{3m_k-3}, x_{3m_k-2}, x_{3m_k-1})],\end{aligned}$$

taking upper limit as $k \to \infty$, we get

$$\limsup_{k\to\infty} G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) \leq s\varepsilon. \tag{9}$$

Also,

$$\begin{aligned}&G(x_{3n_k}, x_{3n_k+1}, x_{3m_k-1}) \\ &\leq s[G(x_{3m_k-1}, x_{3m_k-3}, x_{3m_k-3}) + G(x_{3m_k-3}, x_{3n_k}, x_{3n_k+1})] \\ &\leq sG(x_{3m_k-1}, x_{3m_k-3}, x_{3m_k-3}) + s^2G(x_{3m_k-3}, x_{3n_k}, x_{3n_k})\end{aligned}$$

$$+ s^2G(x_{3n_k}, x_{3n_k}, x_{3n_k+1})$$
$$\leq sG(x_{3m_k-1}, x_{3m_k-3}, x_{3m_k-3}) + 2s^3G(x_{3m_k-3}, x_{3m_k-3}, x_{3n_k})$$
$$+ s^2G(x_{3n_k}, x_{3n_k}, x_{3n_k+1}).$$

Taking upper limit as $k \to \infty$, we get

$$\limsup_{k\to\infty} G(x_{3n_k}, x_{3m_k+1}, x_{3m_k-1}) \leq 2s^3\varepsilon. \tag{10}$$

Again,

$$G(x_{3m_k-1}, x_{3m_k}, x_{3n_k+1})$$
$$\leq sG(x_{3n_k+1}, x_{3n_k}, x_{3n_k}) + s^2G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3})$$
$$+ s^2G(x_{3m_k-3}, x_{3m_k}, x_{3m_k-1}).$$

Hence

$$\limsup_{k\to\infty} G(x_{3m_k-1}, x_{3m_k}, x_{3n_k+1}) \leq s^2\varepsilon. \tag{11}$$

Also,

$$G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1})$$
$$\leq s^2G(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + s^2G(x_{3n_k}, x_{3n_k}, x_{3n_k+1})$$
$$+ sG(x_{3m_k-3}, x_{3m_k-2}, x_{3m_k-1})$$

implies

$$\limsup_{k\to\infty} G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1}) \leq s^2\varepsilon. \tag{12}$$

Also,

$$G(x_{3n_k+1}, x_{3m_k-1}, x_{3m_k-1}) \leq G(x_{3n_k+1}, x_{3m_k-2}, x_{3m_k-1})$$

implies

$$\limsup_{k\to\infty} G(x_{3n_k+1}, x_{3m_k-1}, x_{3m_k-1}) \leq s^2\varepsilon. \tag{13}$$

Again,

$$G(x_{3n_k}, x_{3m_k-1}, x_{3m_k-1}) \leq G(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1})$$

implies

$$\limsup_{k\to\infty} G(x_{3n_k}, x_{3m_k-1}, x_{3m_k-1}) \leq s\varepsilon. \tag{14}$$

Also,

$$G(x_{3n_k}, x_{3m_k-2}, x_{3m_k})$$
$$\leq sG(x_{3n_k}, x_{3m_k-3}, x_{3m_k-3}) + sG(x_{3m_k-3}, x_{3m_k-2}, x_{3m_k}).$$

Hence

$$\limsup_{k\to\infty} G(x_{3n_k}, x_{3m_k-2}, x_{3m_k}) \leq s\varepsilon. \tag{15}$$

Using (9)-(15), we get

$$\begin{aligned}\limsup_{k\to\infty} M_2(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1}) &\le \max\{s\varepsilon, 2s^3\varepsilon, s^2\varepsilon, \frac{1}{4s}(2s^2\varepsilon + s\varepsilon)\}\\ &= 2s^3\varepsilon.\end{aligned}$$

Now, using (7) and (8), we get

$$\begin{aligned}\phi(2s^4\frac{\varepsilon}{s}) &\le \phi(2s^4 \limsup_{k\to\infty} G(x_{3n_k+1}, x_{3m_k-1}, x_{3m_k}))\\ &= \phi(2s^4 \limsup_{k\to\infty} G(fx_{3n_k}, gx_{3m_k-2}, hx_{3m_k-1}))\\ &\le \psi(\phi(\limsup_{k\to\infty} M_2(x_{3n_k}, x_{3m_k-2}, x_{3m_k-1})))\\ &\le \psi(\phi(2s^3\varepsilon))\\ &< \phi(2s^3\varepsilon),\end{aligned}$$

which is a contradiction. Hence, $\{x_{3n}\}$ is Cauchy in X and so $\{x_n\}$ is Cauchy in X. Since X is a complete metric space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Therefore

$$\begin{aligned}&\lim_{n\to\infty} x_{3n+1} = \lim_{n\to\infty} fx_{3n} = \lim_{n\to\infty} x_{3n+2}\\ &= \lim_{n\to\infty} gx_{3n+1} = \lim_{n\to\infty} x_{3n+3} = \lim_{n\to\infty} hx_{3n+2} = u.\end{aligned}$$

We will prove that $u = hu$. We have,

$$G(fx_{3n}, gx_{3n+1}, hx_{3n+2}) \le s[G(fx_{3n}, gx_{3n+1}, hu) + G(hu, hu, hx_{3n+2})].$$

Suppose $G(fx_{3n}, gx_{3n+1}, hu) = 0$ and $G(hu, hu, hx_{3n+2}) = 0$, for some $n \in \mathbb{N}$, then $G(fx_{3n}, gx_{3n+1}, hx_{3n+2}) = 0$, a contradiction to our assumption. Therefore, we take $G(fx_{3n}, gx_{3n+1}, hu) > 0$, for all n.
From (1) we get

$$\phi(2s^4 G(fx_{3n}, gx_{3n+1}, hu)) \le \psi(\phi(M_2(x_{3n}, x_{3n+1}, u))), \tag{16}$$

where

$$\begin{aligned}&M_2(x_{3n}, x_{3n+1}, u)\\ &= \max\{G(x_{3n}, x_{3n+1}, u), G(x_{3n}, fx_{3n}, gx_{3n+1}),\\ &\qquad G(x_{3n+1}, gx_{3n+1}, hu), G(u, hu, fx_{3n}),\\ &\qquad \frac{1}{4s}[G(fx_{3n}, x_{3n+1}, u) + G(x_{3n}, gx_{3n+1}, u) + G(x_{3n}, x_{3n+1}, hu)]\}.\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty} M_2(x_{3n}, x_{3n+1}, u) &= \max\{G(u, u, u), G(u, u, hu), \frac{1}{4s}G(u, u, hu)\}\\ &= G(u, u, hu).\end{aligned}$$

Taking limit as $n \to \infty$ in (16), we get

$$\phi(2s^4 G(u,u,hu)) \leq \psi(\phi(G(u,u,hu))) < \phi(G(u,u,hu)),$$

which implies that

$$2s^4 G(u,u,hu) \leq G(u,u,hu),$$

a contradiction. Hence, $u = hu$, that is u is a fixed point of h.

Similarly, we can prove that u is a fixed point of both f and g. Therefore u is a common fixed point of f, g and h.

To prove that u is the unique common fixed point of f, g and h, let v be another common fixed point of f, g and h. Then $fu = gu = hu = u$ and $fv = gv = hv = v$. We have $G(u,u,v) = G(fu,gu,hv) > 0$ and $G(u,v,v) = G(fu,gv,hv) > 0$. From (1), we have

$$\phi(2s^4 G(fu,gu,hv)) \leq \psi(\phi(M_2(u,u,v))) < \phi(M_2(u,u,v)), \tag{17}$$

where

$$M_2(u,u,v) = \max\{G(u,u,v), G(u,v,v)\}.$$

If $M_2(u,u,v) = G(u,v,v)$, then from (17) we get

$$\phi(2s^4 G(u,u,v)) < \phi(G(u,v,v)),$$

which implies

$$2s^4 G(u,u,v) < G(u,v,v) \leq 2sG(u,u,v),$$

a contradiction.

Similarly, if $M_2(u,u,v) = G(u,u,v)$, then from (17) we get

$$\phi(2s^4 G(u,u,v)) < \phi(G(u,u,v)),$$

which implies

$$2s^4 G(u,u,v) < G(u,u,v),$$

a contradiction. Hence f, g and h have a unique common fixed point in X.

Further, we prove that at least one of f, g and h is not continuous at u if and only if

$$\lim_{x \to u} M_2(x,u,u) \neq 0 \text{ or } \lim_{y \to u} M_2(u,y,u) \neq 0 \text{ or } \lim_{z \to u} M_2(u,u,z) \neq 0.$$

Equivalently, we prove that f, g and h are continuous at u if and only if

$$\lim_{x \to u} M_2(x,u,u) = 0 \text{ and } \lim_{y \to u} M_2(u,y,u) = 0 \text{ and } \lim_{z \to u} M_2(u,u,z) = 0.$$

We suppose that

$$\lim_{x \to u} M_2(x,u,u) = 0 \text{ and } \lim_{y \to u} M_2(u,y,u) = 0 \text{ and } \lim_{z \to u} M_2(u,u,z) = 0.$$

Now

$$\lim_{x_n \to u} M_2(x_n,u,u)$$

$$= \lim_{x_n \to u} \max\Big\{ G(x_n, u, u), G(x_n, fx_n, gu), G(u, gu, hu), G(u, hu, fx_n),$$

$$\frac{1}{4s}[G(fx_n, u, u) + G(x_n, gu, u) + G(x_n, u, hu)] \Big\} = 0.$$

Thus $\lim_{x_n \to u} G(x_n, fx_n, u) = 0$. This implies that $fx_n \to u = fu$, that is, f is continuous at u. Similarly we can prove that g and h are continuous at u.

On the other hand, if f, g and h are continuous at their common fixed point u, that is $\lim_{x_n \to u} fx_n = fu$, $\lim_{x_n \to u} gx_n = gu$ and $\lim_{x_n \to u} hx_n = hu$. Then

$$\lim_{x_n \to u} M_2(x_n, u, u)$$
$$= \lim_{x_n \to u} \max\Big\{ G(x_n, u, u), G(x_n, fx_n, gu), G(u, gu, hu), G(u, hu, fx_n),$$
$$\frac{1}{4s}[G(fx_n, u, u) + G(x_n, gu, u) + G(x_n, u, hu)] \Big\} = 0,$$

$$\lim_{x_n \to u} M_2(u, x_n, u)$$
$$= \lim_{x_n \to u} \max\Big\{ G(u, x_n, u), G(u, fu, gx_n), G(x_n, gx_n, hu), G(u, hu, fu),$$
$$\frac{1}{4s}[G(fu, x_n, u) + G(u, gx_n, u) + G(u, x_n, hu)] \Big\} = 0$$

and

$$\lim_{x_n \to u} M_2(u, u, x_n)$$
$$= \lim_{x_n \to u} \max\Big\{ G(u, u, x_n), G(u, fu, gu), G(u, gu, hx_n), G(x_n, hx_n, fu),$$
$$\frac{1}{4s}[G(fu, u, x_n) + G(u, gu, x_n) + G(u, u, hx_n)] \Big\} = 0.$$

□

The subsequent example affirms the result obtained by us.

Example 2. Let $X = [0, \infty)$ and define $G : X^3 \to [0, \infty)$ by

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{otherwise.} \end{cases}$$

Then (X, G) is a complete G_b-metric space with $s = 1$.

We define $f, g, h : X \to X$ by

$$fx = \begin{cases} \frac{x}{16}, & x \in [0, 1], \\ 0, & x \in (1, \infty), \end{cases} \quad gx = \begin{cases} \frac{x}{12}, & x \in [0, 1], \\ 0, & x \in (1, \infty), \end{cases} \quad hx = \begin{cases} \frac{x}{10}, & x \in [0, 1], \\ 0, & x \in (1, \infty). \end{cases}$$

Also, take $\phi(t) = t$ and $\psi(t) = \frac{t}{2}$. Then f, g, h satisfy all the conditions of Theorem 1 and $x = 0$ is the only common fixed point of f, g and h.

Corollary 1. *Let $f : X \to X$ be a $(\psi, \phi) - G_b$-Wardowski contraction in a complete G_b-metric space. Then f has a unique fixed point, say u, and $f^n x \to u$, for each $x \in X$. Further, f is discontinuous at u if and only if*

$$\lim_{x \to u} M_1(x, u, u) \neq 0.$$

Proof. By taking $f = g = h$ in Theorem 1, we get the result. □

Corollary 2. *Let (X, G_b) be a complete G_b-metric space and let $f : X \to X$ satisfy*

$$G(fx, fy, fz) > 0 \implies \phi(2s^4 G(fx, fy, fz)) \leq \psi(\phi(G(x, y, z))),$$

for all $x, y, z \in X$, where $\phi \in \Phi$ and $\Psi \in F_{com}$. Then f has a unique fixed point, say u, and $f^n x \to u$, for each $x \in X$. Further, f is discontinuous at u if and only if

$$\lim_{x \to u} G(x, u, u) \neq 0.$$

Proof. Taking $M_1(x, y, z) = G(x, y, z)$, the conclusion follows from Corollary 1. □

The following result is for Wardowski type contractions in G_b-metric spaces.

Corollary 3. *Let (X, G_b) be a complete G_b-metric space and let $f : X \to X$ satisfy*

$$G(fx, fy, fz) > 0 \implies \tau + F(2s^4 G(fx, fy, fz)) \leq F(G(x, y, z)),$$

for all $x, y, z \in X$. Then f has a unique fixed point, say u, and $f^n x \to u$, for each $x \in X$. Further, f is discontinuous at u if and only if

$$\lim_{x \to u} G(x, u, u) \neq 0.$$

Proof. In Corollary 1, we take $M_1(x, y, z) = G(x, y, z)$ and $\psi(t) = e^{-\tau} t$, where $\tau > 0$ and $\phi(t) = e^{F(t)}$, where F is an F-contraction, then we get the result. □

3. Application

In fixed point theorems, contractive mappings that admit discontinuity at the fixed point have found applications in neural networks with discontinuous activation functions (e.g. Özgür and Tas [5] and Rashid et al. [7]). Here we give an application of our result by considering discontinuous activation functions occurring in the neural networks. Nie and Zheng [4] generalized the class of discontinuous activation functions as follows:

$$f_i(x) = \begin{cases} u_i, & -\infty < x < p_i, \\ l_{i,1}x + c_{i,1}, & p_i \le x \le r_i, \\ l_{i,2}x + c_{i,2}, & r_i < x \le q_i, \\ v_i, & q_i < x < +\infty, \end{cases}$$

where $p_i, r_i, q_i, u_i, v_i, l_{i,1}, l_{i,2}, c_{i,1}, c_{i,2}$ are constants with

$$\begin{aligned} & -\infty < p_i < r_i < q_i < +\infty, \\ & l_{i,1} > 0,\ l_{i,2} < 0, \\ & u_i = l_{i,1}p_i + c_{i,1} = l_{i,2}q_i + c_{i,2}, \\ & l_{i,1}r_i + c_{i,1} = l_{i,2}r_i + c_{i,2}, \\ & v_i > f_i(r_i),\ i = 1, 2, ..., n. \end{aligned}$$

The function f_i is continuous at every real number except the value $x = q_i$.

Here we consider the discontinuous activation functions f, g and h:

$$f(x) = \begin{cases} 4, & -\infty < x < -2, \\ x + 6, & -2 \le x \le 1, \\ -x + 8, & 1 < x \le 4, \\ 8, & 4 < x < +\infty, \end{cases}$$

where

$$\begin{aligned} & u_i = 4,\ v_i = 3,\ p_i = -2,\ r_i = 1,\ q_i = 4, \\ & l_{i,1} = 1,\ c_{i,1} = 6,\ l_{i,2} = -1,\ c_{i,2} = 8, \end{aligned}$$

$$g(x) = \begin{cases} -3, & -\infty < x < -2, \\ 2x + 1, & -2 \le x \le -\frac{1}{2}, \\ -2x - 1, & -\frac{1}{2} < x \le 1, \\ 4, & 1 < x < +\infty, \end{cases}$$

where

$$\begin{aligned} & u_i = -3,\ v_i = 4,\ p_i = -2,\ r_i = -\frac{1}{2},\ q_i = 1, \\ & l_{i,1} = 2,\ c_{i,1} = 1,\ l_{i,2} = -2,\ c_{i,2} = -1 \end{aligned}$$

and

$$h(x) = \begin{cases} -2, & -\infty < x < -4, \\ 2x + 6, & -4 \le x \le -3, \\ -2x - 6, & -3 < x \le -2, \\ 4, & -2 < x < +\infty, \end{cases}$$

where

$$\begin{aligned} & u_i = -2,\ v_i = 4,\ p_i = -4,\ r_i = -3,\ q_i = -2, \\ & l_{i,1} = 2,\ c_{i,1} = 6,\ l_{i,2} = -2,\ c_{i,2} = -6. \end{aligned}$$

The function g has four fixed points, $u_1 = -3, u_2 = -1, u_3 = \frac{-1}{3}$ and $u_4 = 4$, and the functions f and h have only one fixed point at $x = 4$. So $x = 4$ is the common fixed point of f, g and h. Since

$$\lim_{x \to 4} M_2(x, 4, 4) \neq 0,$$

f is discontinuous at $x = 4$.

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