ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 27, Number 1, June 2023
Available online at https://ojs.utlib.ee/index.php/ACUTM

Some Galois connections arising from Morita contexts of semigroups

Alvin Lepik

Abstract. We show that a unitary surjective Morita context connecting two semigroups yields Galois connections between certain lattices of compatible relations whenever either semigroup has common weak local units. In the event both semigroups have common weak local units, we obtain mutually inverse lattice isomorphisms that preserve reflexivity, symmetricity and transitivity between the lattices of compatible relations on the semigroups.

1. Introduction

Two rings are said to be Morita equivalent if the categories of modules over those rings are equivalent. A similar concept of Morita equivalence of monoids has been investigated by Knauer [6], regarding two monoids to be Morita equivalent if the categories of acts over those monoids are equivalent. A useful tool in the study of Morita theory of rings is a Morita context, which is covered by Jacobson [4]. Morita contexts are later adapted to the semigroup case by Talwar [13, 14]. Relations between the lattices of submodules that arise from a Morita context connecting two rings has been studied by Kashu [5] and his results were adapted to the semigroup case by Paseka [7]. The main results in both of these papers are obtained for nondegenerate Morita contexts imposing no restrictions on the rings or semigroups.

In this note, we make use of some of the mappings considered by Paseka [7]. Instead of requiring nondegeneracy of the Morita context, we work with unitary surjective Morita contexts connecting two semigroups. To compensate, we assume the presence of common weak local units in the semigroups.

Received February 7, 2023.
2020 *Mathematics Subject Classification.* 20M30.
Key words and phrases. Galois connection, Morita equivalence, Morita context, semigroup.
https://doi.org/10.12697/ACUTM.2023.27.07

Given a unitary surjective Morita context connecting two semigroups, our aim is to collect information on certain sublattices of relations on the semigroups and the biacts present in the Morita context.

We obtain Galois connections whenever either semigroup satisfies the property of having one-sided common weak local units. When both semigroups have common weak local units, we obtain pairs of mutually inverse lattice isomorphisms that preserve weak congruences, compatible preorders, tolerances and congruences between the sublattices of compatible relations on the semigroups. This generalises a result on Morita invariants of Laan and Márki [9] regarding isomorphism of congruence lattices of strongly Morita equivalent semigroups with common joint weak local units.

Let S be a semigroup and ${}_SA$ be a left S-act. The act ${}_SA$ is called **unitary** if $SA = A$, that is, for every element $a \in A$ there exist $a' \in A$ and $s \in S$ such that $sa' = a$. Dually, a right act A_S is unitary if $AS = A$. By an S-**compatible relation** we mean a binary relation ρ on A such that $sa_1 \,\rho\, sa_2$ whenever $a_1 \,\rho\, a_2$ and $s \in S$. We denote the lattice of S-compatible relations on A by $\mathsf{Comp}({}_SA)$ and dually, $\mathsf{Comp}(A_S)$ for a right act A_S. Lattices of compatible relations of algebras have been considered by Chajda, Šešelja and Tepavčević [2]. In general, unions of compatible relations need not be compatible, but since acts over semigroups are algebras with only unary operations it is readily verified that both $\mathsf{Comp}({}_SA)$ and $\mathsf{Comp}(A_S)$ are sublattices of the lattice $(\mathcal{P}(A \times A), \subseteq)$. Additionally, we consider **tolerances**, which are reflexive, symmetric and compatible relations, **compatible preorders**, i.e, reflexive, transitive and compatible relations and **weak congruences**, i.e, relations that are symmetric, transitive and compatible.

An (S,T)-biact ${}_SA_T$ is called **unitary** if $SA = A = AT$ and we have $\mathsf{Comp}({}_SA_T) = \mathsf{Comp}({}_SA) \cap \mathsf{Comp}(A_T)$. In case of an (S,S)-biact ${}_SA_S$, we call the elements of $\mathsf{Comp}({}_SA)$ **left** S-**compatible** and, dually, the elements of $\mathsf{Comp}(A_S)$ **right** S-**compatible**.

Let X and Y be posets and $f \in Y^X$ and $g \in X^Y$ a pair of mappings. It is said that the pair (f,g) forms a **monotone Galois connection** if

$$(\forall x \in X)(\forall y \in Y)(f(x) \leqslant y \Leftrightarrow x \leqslant g(y)).$$

In this case, f is called the **left adjoint** of g and one writes $f \dashv g$. It holds that $f \dashv g$ if and only if $fg \leqslant \mathtt{id}_Y$ and $\mathtt{id}_X \leqslant gf$ with respect to the pointwise order of mappings. It is clear that if $f \dashv g \dashv f$, then f and g are mutually inverse order isomorphisms. A primer on Galois connections can be found in [3] among others.

2. Results

Our main tool is that of a Morita context, which was adapted to the semigroup case by Talwar [13]. Recall that the **tensor product** of a right act A_S and a left act ${}_SB$, where S is a semigroup, is the quotient set $A \otimes_S B :=$

$(A \times B)/\vartheta$, where ϑ is the smallest equivalence relation on $A \times B$ containing the set $\{((as, b), (a, sb)) \mid a \in A, b \in B, s \in S\}$. The ϑ-class of a pair (a, b) is denoted by $a \otimes b$, so $A \otimes_S B = \{a \otimes b \mid a \in A, b \in B\}$.

Definition 1. A **Morita context** connecting semigroups S and T is a six-tuple $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$, where ${}_SP_T$ is an (S, T)-biact, ${}_TQ_S$ is a (T, S)-biact and

$$\theta : {}_S(P \otimes_T Q)_S \longrightarrow {}_SS_S \quad \text{and} \quad \phi : {}_T(Q \otimes_S P)_T \longrightarrow {}_TT_T$$

are biact morphisms satisfying the identities

$$\theta(p \otimes q)p' = p\phi(q \otimes p') \quad \text{and} \quad q'\theta(p \otimes q) = \phi(q' \otimes p)q.$$

A Morita context is called

i) **unitary** if the biacts are unitary;
ii) **surjective** if the biact morphisms are surjective.

A semigroup S is called **factorisable** if $S^2 = S$. If two semigroups are connected by a unitary surjective Morita context, then they are necessarily factorisable [8]. Two factorisable semigroups are **Morita equivalent** if and only if they are connected by a unitary surjective Morita context [11].

Let $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ be a unitary surjective Morita context connecting semigroups S and T. We consider the pairs of mappings (see p. 2251 in [7] and Theorem 2.5 in [12])

$$\mathcal{P}(P \times P) \underset{\beta_P}{\overset{\alpha_P}{\rightleftarrows}} \mathcal{P}(T \times T) \quad \text{and} \quad \mathcal{P}(Q \times Q) \underset{\beta_Q}{\overset{\alpha_Q}{\rightleftarrows}} \mathcal{P}(T \times T)$$

that are defined by

$$\begin{aligned}
\alpha_P(\sigma) &:= \{(t, t') \mid (\forall p \in P)(pt \,\sigma\, pt')\},\\
\beta_P(\psi) &:= \{(p, p') \mid (\forall q \in Q)(\phi(q \otimes p) \,\psi\, \phi(q \otimes p'))\},\\
\alpha_Q(\tau) &:= \{(t, t') \mid (\forall q \in Q)(tq \,\tau\, t'q)\},\\
\beta_Q(\psi) &:= \{(q, q') \mid (\forall p \in P)(\phi(q \otimes p) \,\psi\, \phi(q' \otimes p))\}.
\end{aligned}$$

It is clear that the maps α_P, β_P, α_Q and β_Q are order preserving. These maps also have the following properties.

Proposition 1. *Let $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ be a unitary surjective Morita context connecting semigroups S and T. Then all of the maps $\alpha_P, \beta_P, \alpha_Q$ and β_Q preserve reflexivity, symmetricity and transitivity. Additionally, the following assertions hold:*

(1) $\alpha_P(\mathcal{P}(P \times P)) \subseteq \mathsf{Comp}({}_TT)$ *and* $\alpha_P(\mathsf{Comp}(P_T)) \subseteq \mathsf{Comp}(T_T)$;
(2) $\beta_P(\mathcal{P}(T \times T)) \subseteq \mathsf{Comp}({}_SP)$ *and* $\beta_P(\mathsf{Comp}(T_T)) \subseteq \mathsf{Comp}(P_T)$;
(3) $\alpha_Q(\mathcal{P}(Q \times Q)) \subseteq \mathsf{Comp}(T_T)$ *and* $\alpha_Q(\mathsf{Comp}({}_TQ)) \subseteq \mathsf{Comp}({}_TT)$;

(4) $\beta_Q(\mathcal{P}(T \times T)) \subseteq \mathsf{Comp}(Q_S)$ *and* $\beta_Q(\mathsf{Comp}({}_TT)) \subseteq \mathsf{Comp}({}_TQ)$.

Proof. Preservation of symmetricity is clear in all cases. We verify preservation of reflexivity and transitivity for β_P. Preservation properties for the other maps are verified analogously. Let $\psi \subseteq T \times T$ be reflexive and take $p \in P$. Then $\phi(q{\otimes}p)\,\psi\,\phi(q{\otimes}p)$ for all $q \in Q$ due to reflexivity of ψ. Therefore, $(p,p) \in \beta_P(\psi)$. Now let ψ be transitive and take $(p_1,p_2),(p_2,p_3) \in \beta_P(\psi)$. It follows that

$$\phi(q \otimes p_1)\,\psi\,\phi(q \otimes p_2)\,\psi\,\phi(q \otimes p_3),$$

whence $\phi(q \otimes p_1)\,\psi\,\phi(q \otimes p_3)$ for all $q \in Q$ due to transitivity of ψ. Thus, $(p_1,p_3) \in \beta_P(\psi)$. For item (1) let $\sigma \subseteq P \times P$. Take $(t_1,t_2) \in \alpha_P(\sigma)$ and $t \in T$. Note that $pt_1\,\sigma\,pt_2$ must hold for all $p \in P$. Firstly, we check $\alpha_P(\sigma)$ is left T-compatible. Let $p \in P$. Then by the above we have $p(tt_1) = (pt)t_1\,\sigma\,(pt)t_2 = p(tt_2)$. Thus, $(tt_1,tt_2) \in \alpha_P(\sigma)$.

Secondly, assume σ is T-compatible. From T-compatibility we conclude $(pt_1)t\,\sigma\,(pt_2)t$ for all $p \in P$. Thus, $(t_1t,t_2t) \in \alpha_P(\sigma)$.

For item (2) let $\psi \subseteq T \times T$. Take $(p_1,p_2) \in \beta_P(\psi)$ and $q \in Q$. Firstly, we check $\beta_P(\psi)$ is S-compatible. Take $s \in S$, then by definition of $\beta_P(\psi)$ we have

$$\phi(q \otimes sp_1) = \phi(qs \otimes p_1)\,\psi\,\phi(qs \otimes p_2) = \phi(q \otimes sp_2),$$

which implies that $(sp_1,sp_2) \in \beta_P(\psi)$. Secondly, assume ψ is right T-compatible. Then

$$\phi(q \otimes p_1t) = \phi(q \otimes p_1)t\,\psi\,\phi(q \otimes p_2)t = \phi(q \otimes p_2t)$$

for every $t \in T$, which implies that $(p_1t,p_2t) \in \beta_P(\psi)$. Items (3) and (4) are proved similarly. □

A semigroup S is said to have **common weak left local units** if, for any two elements $s_1,s_2 \in S$, there exists $u \in S$ such that $us_1 = s_1$ and $us_2 = s_2$. Dually, one defines **common weak right local units**. A semigroup S is said to have **common weak local units** if S has both common weak left local units and common weak right local units [10].

Example 1. Every monoid is a semigroup with common weak local units. Every lattice has common weak local units with respect to both joins and meets.

The following result can be checked easily.

Lemma 1. *Let ${}_SA$ be a unitary S-act, where S is a semigroup with common weak left local units. Then, for every $a_1,a_2 \in A$, there exists $u \in S$ such that $ua_1 = a_1$ and $ua_2 = a_2$.*

Proposition 2. *Let $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ be a unitary surjective Morita context connecting semigroups S and T. The following assertions hold for the pair of maps*

$$\alpha_P : \mathsf{Comp}({}_SP) \longrightarrow \mathsf{Comp}({}_TT) \quad \textit{and} \quad \beta_P : \mathsf{Comp}({}_TT) \longrightarrow \mathsf{Comp}({}_SP).$$

1. *If T has common weak left local units, then $\alpha_P \dashv \beta_P$ and α_P is surjective.*
2. *If S has common weak left local units, then $\beta_P \dashv \alpha_P$ and β_P is surjective.*

Proof. We restrict the mapping α_P from Proposition 1 from $\mathcal{P}(P \times P)$ to its subset $\mathsf{Comp}({}_SP)$, but we still denote the restriction by α_P. We use a similar convention for β_P. The maps α_P and β_P are well defined by Proposition 1. Let $\sigma \in \mathsf{Comp}({}_SP)$ and $\psi \in \mathsf{Comp}({}_TT)$.
1. Assume $\alpha_P(\sigma) \subseteq \psi$ and let $(p_1, p_2) \in \sigma$. To show $(p_1, p_2) \in \beta_P(\psi)$, we must show that $\phi(q \otimes p_1)\, \psi\, \phi(q \otimes p_2)$ for every $q \in Q$. Let $q \in Q$ and take $p \in P$. Note that

$$p\phi(q \otimes p_1) = \theta(p \otimes q)p_1 \quad \text{and} \quad p\phi(q \otimes p_2) = \theta(p \otimes q)p_2.$$

Since $\theta(p \otimes q) \in S$ and σ is S-compatible, we have

$$p\phi(q \otimes p_1)\, \sigma\, p\phi(q \otimes p_2).$$

This implies that $(\phi(q \otimes p_1), \phi(q \otimes p_2)) \in \alpha_P(\sigma) \subseteq \psi$, as required.

Conversely, assume $\sigma \subseteq \beta_P(\psi)$ and let $(t_1, t_2) \in \alpha_P(\sigma)$. Then $pt_1\, \sigma\, pt_2$ for every $p \in P$. By assumption, we have $(pt_1, pt_2) \in \beta_P(\psi)$, so

$$(\forall p \in P)(\forall q \in Q)(\phi(q \otimes pt_1)\, \psi\, \phi(q \otimes pt_2)).$$

Equivalently, $tt_1\, \psi\, tt_2$ for every $t \in T$ due to surjectivity of ϕ. In particular, we have $t_1 = vt_1\, \psi\, vt_2 = t_2$ for some $v \in T$ due to the presence of common weak local units in T. Hence, $\alpha_P(\sigma) \subseteq \psi$, as required.

We check surjectivity of α_P. Let $\psi \in \mathsf{Comp}({}_TT)$. The containment $\alpha_P(\beta_P(\psi)) \subseteq \psi$ holds due to $\alpha_P \dashv \beta_P$. Conversely, let $t_1\, \psi\, t_2$. Left T-compatibility implies that

$$\phi(q \otimes pt_1) = \phi(q \otimes p)t_1\, \psi\, \phi(q \otimes p)t_2 = \phi(q \otimes pt_2)$$

for every $p \in P$ and $q \in Q$. Equivalently, $(pt_1, pt_2) \in \beta_P(\psi)$ for every $p \in P$. So we have $\psi \subseteq \alpha_P(\beta_P(\psi))$. Therefore $\alpha_P(\beta_P(\psi)) = \psi$ and α_P is surjective.
2. Assume $\psi \subseteq \alpha_P(\sigma)$ and let $(p_1, p_2) \in \beta_P(\psi)$. It follows that

$$\begin{aligned}
&(\forall q \in Q)(\phi(q \otimes p_1)\, \psi\, \phi(q \otimes p_2)) \\
\Rightarrow &(\forall q \in Q)(\phi(q \otimes p_1)\, \alpha_P(\sigma)\, \phi(q \otimes p_2)) \\
\Leftrightarrow &(\forall q \in Q)(\forall p \in P)(p\phi(q \otimes p_1)\, \sigma\, p\phi(q \otimes p_2)).
\end{aligned}$$

By Lemma 1, take $u \in S$ such that $up_1 = p_1$ and $up_2 = p_2$. We have $u = \theta(p \otimes q)$ for some $p \in P$ and $q \in Q$ due to surjectivity of θ. It follows that

$$p_1 = up_1 = \theta(p \otimes q)p_1 = p\phi(q \otimes p_1)\,\sigma\, p\phi(q \otimes p_2) = p_2.$$

So $(p_1, p_2) \in \sigma$ and $\beta_P(\psi) \subseteq \sigma$.

Conversely, assume $\beta_P(\psi) \subseteq \sigma$ and let $(t_1, t_2) \in \psi$. To show $(t_1, t_2) \in \alpha_P(\sigma)$, we must show that $pt_1\,\sigma\, pt_2$ for every $p \in P$. Let $p \in P$. Since ψ is left T-compatible, we have

$$(\forall q \in Q)(\phi(q \otimes pt_1)\,\psi\,\phi(q \otimes pt_2)).$$

This implies that $(pt_1, pt_2) \in \beta_P(\psi) \subseteq \sigma$, as required.

We check surjectivity of β_P. Let $\sigma \in \mathsf{Comp}({}_SP)$ and $p_1\,\sigma\, p_2$. Take $p \in P$ and $q \in Q$. By S-compatibility we have $\theta(p\otimes q)p_1\,\sigma\,\theta(p\otimes q)p_2$, which implies that

$$p\phi(q \otimes p_1)\,\sigma\, p\phi(q \otimes p_2).$$

It follows that $(\phi(q\otimes p_1), \phi(q\otimes p_2)) \in \alpha_P(\sigma)$ so $(p_1, p_2) \in \beta_P(\alpha_P(\sigma))$. Thus, $\sigma \subseteq \beta_P(\alpha_P(\sigma))$. The containment $\beta_P(\alpha_P(\sigma)) \subseteq \sigma$ follows from $\beta_P \dashv \alpha_P$. □

Remark 1. It is well known that if $f \dashv g$, then f is surjective if and only if g is injective.

Similarly to the proof of Proposition 2, one can prove the following.

Proposition 3. *Let $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ be a unitary surjective Morita context connecting semigroups S and T. The following assertions hold for the pair of maps*

$$\alpha_Q : \mathsf{Comp}(Q_S) \longrightarrow \mathsf{Comp}(T_T) \quad \textit{and} \quad \beta_Q : \mathsf{Comp}(T_T) \longrightarrow \mathsf{Comp}(Q_S).$$

1. *If T has common weak right local units, then $\alpha_Q \dashv \beta_Q$ and α_Q is surjective.*
2. *If S has common weak right local units, then $\beta_Q \dashv \alpha_Q$ and β_Q is surjective.*

The assertions made for one-sided acts also hold in case of biacts.

Corollary 1. *Let $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ be a unitary surjective Morita context connecting semigroups S and T. The following assertions hold for the pairs of maps*

$$\alpha_P : \mathsf{Comp}({}_SP_T) \longrightarrow \mathsf{Comp}({}_TT_T) \quad \textit{and} \quad \beta_P : \mathsf{Comp}({}_TT_T) \longrightarrow \mathsf{Comp}({}_SP_T),$$

$$\alpha_Q : \mathsf{Comp}({}_TQ_S) \longrightarrow \mathsf{Comp}({}_TT_T) \quad \textit{and} \quad \beta_Q : \mathsf{Comp}({}_TT_T) \longrightarrow \mathsf{Comp}({}_TQ_S).$$

1. *If T has common weak left local units, then $\alpha_P \dashv \beta_P$ and α_P is surjective.*
2. *If S has common weak left local units, then $\beta_P \dashv \alpha_P$ and β_P is surjective.*

3. *If T has common weak right local units, then $\alpha_Q \dashv \beta_Q$ and α_Q is surjective.*
4. *If S has common weak right local units, then $\beta_Q \dashv \alpha_Q$ and β_Q is surjective.*

Proof. The maps are well defined by Proposition 1. The adjunctions pertaining to α_P and β_P follow from Proposition 2. The claims for α_Q and β_Q are proved analogously. $\square$

Results obtained with the Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ can be reproduced dually using the Morita context $(T, S, {}_TQ_S, {}_SP_T, \phi, \theta)$, for which we have the dual mappings

$$\mathcal{P}(Q \times Q) \underset{\overline{\beta}_Q}{\overset{\overline{\alpha}_Q}{\rightleftarrows}} \mathcal{P}(S \times S) \quad \text{and} \quad \mathcal{P}(P \times P) \underset{\overline{\beta}_P}{\overset{\overline{\alpha}_P}{\rightleftarrows}} \mathcal{P}(S \times S)$$

that are defined by

$$\begin{aligned}
\overline{\alpha}_Q(\tau) &:= \left\{(s, s') \mid (\forall q \in Q)(qs \, \tau \, qs')\right\}, \\
\overline{\beta}_Q(\rho) &:= \left\{(q, q') \mid (\forall p \in P)(\theta(p \otimes q) \, \rho \, \theta(p \otimes q'))\right\}, \\
\overline{\alpha}_P(\sigma) &:= \left\{(s, s') \mid (\forall p \in P)(sp \, \sigma \, s'p)\right\}, \\
\overline{\beta}_P(\rho) &:= \left\{(p, p') \mid (\forall q \in Q)(\theta(p \otimes q) \, \rho \, \theta(p' \otimes q))\right\}.
\end{aligned}$$

Theorem 1. *Let semigroups S and T with common weak local units be connected by a unitary surjective Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then we have the following pairs*

$$\mathsf{Comp}({}_SP) \underset{\beta_P}{\overset{\alpha_P}{\rightleftarrows}} \mathsf{Comp}({}_TT) \qquad \mathsf{Comp}(P_T) \underset{\overline{\beta}_P}{\overset{\overline{\alpha}_P}{\rightleftarrows}} \mathsf{Comp}(S_S)$$

$$\mathsf{Comp}({}_TQ) \underset{\overline{\beta}_Q}{\overset{\overline{\alpha}_Q}{\rightleftarrows}} \mathsf{Comp}({}_SS) \qquad \mathsf{Comp}(Q_S) \underset{\beta_Q}{\overset{\alpha_Q}{\rightleftarrows}} \mathsf{Comp}(T_T)$$

of mutually inverse lattice isomorphisms that preserve weak congruences, compatible preorders, tolerances and congruences.

Proof. We have pairs of mutually inverse order isomorphisms by Proposition 2 and its dual. Preservation of weak congruences, compatible preorders, tolerances and congruences follows by Proposition 1. It is well known that an order isomorphism between lattices is a lattice isomorphism. $\square$

Theorem 3 in [9] yields that the lattices of ideals of strongly Morita equivalent semigroups S and T are isomorphic whenever S and T have weak local units. In other words, the sublattices of Rees congruences on S and T are isomorphic. In view of Corollary 1 and its dual, we may conclude the following.

Theorem 2. *Let semigroups S and T with common weak local units be connected by a unitary surjective Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$. Then we have the following pairs*

$$\mathsf{Comp}({}_SP_T) \underset{\beta_P}{\overset{\alpha_P}{\rightleftarrows}} \mathsf{Comp}({}_TT_T) \underset{\alpha_Q}{\overset{\beta_Q}{\rightleftarrows}} \mathsf{Comp}({}_TQ_S) \underset{\overline{\beta}_Q}{\overset{\overline{\alpha}_Q}{\rightleftarrows}} \mathsf{Comp}({}_SS_S)$$

of mutually inverse lattice isomorphisms that preserve weak congruences, compatible preorders, tolerances and congruences.

A relation $\psi \subseteq T \times T$ is a congruence of the biact ${}_TT_T$ if and only if it is a congruence of the semigroup T. That is, $\mathsf{Con}({}_TT_T) = \mathsf{Con}(T)$. A similar assertion holds for compatible preorders. Hence we obtain the following result.

Corollary 2. *If S and T are strongly Morita equivalent semigroups with common weak local units, then the lattices of weak congruences, compatible preorders, tolerances and congruences of the biacts ${}_SS_S$ and ${}_TT_T$ are isomorphic. Explicitly, if $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ is a unitary surjective Morita context, then the mutually inverse isomorphisms are the restrictions of the mappings*

$$\Theta : \mathsf{Comp}({}_SS_S) \longrightarrow \mathsf{Comp}({}_TT_T) \quad \text{and} \quad \Phi : \mathsf{Comp}({}_TT_T) \longrightarrow \mathsf{Comp}({}_SS_S)$$

defined by

$$\Theta(\rho) := \left\{(t, t') \mid (\forall q \in Q)(\forall p \in P)(\theta(pt \otimes q)\, \rho\, \theta(pt' \otimes q))\right\},$$
$$\Phi(\psi) := \left\{(s, s') \mid (\forall q \in Q)(\forall p \in P)(\phi(q \otimes sp)\, \psi\, \phi(q \otimes s'p))\right\}.$$

In particular, the lattices of compatible preorders and congruences of the semigroups S and T are isomorphic.

These mappings are also similar to the mappings Θ and Φ of Theorem 2 in [15].

Remark 2. Let S be a semigroup. Since tolerances are reflexive, it follows that every tolerance of the semigroup S is also a tolerance of ${}_SS_S$. Also, due to transitivity, every weak congruence of ${}_SS_S$ is a weak congruence of the semigroup S.

Remark 3. The above corollary generalises Theorem 6 in [9]. It turns out that the mappings in said paper coincide with the ones given in Corollary 2. Laan and Márki [9] consider the transitive closure of the relation

$$\Pi_\rho := \left\{(\phi(q \otimes sp), \phi(q \otimes s'p)) \mid s\, \rho\, s', p \in P, q \in Q\right\},$$

where $\rho \in \mathsf{Con}(S)$. It holds that

$$\Pi_\rho = \Theta(\rho) = \left\{(t,t') \mid (\forall q \in Q)(\forall p \in P)(\theta(pt \otimes q)\,\rho\,\theta(pt' \otimes q))\right\}.$$

Proof. We show $\Theta(\rho) \subseteq \Pi_\rho$. Let $(t,t') \in \Theta(\rho)$. Since ϕ is surjective, there exist $p_t, p_{t'} \in P$ and $q_t, q_{t'} \in Q$ such that $t = \phi(q_t \otimes p_t)$ and $t' = \phi(q_{t'} \otimes p_{t'})$. Lemma 2.3 and its dual imply that there exist $u = \phi(q_u \otimes p_u) \in T$ and $v = \phi(q_v \otimes p_v)$ such that $uq_t = q_t$, $uq_{t'} = q_{t'}$, $p_t v = p_t$ ja $p_{t'} v = p_{t'}$. Denote

$$s := \theta(p_u t \otimes q_v)\ \rho\ \theta(p_u t' \otimes q_v) =: s'.$$

It follows that

$$\begin{aligned}
t = \phi(q_t \otimes p_t) &= \phi(uq_t \otimes p_t v)\\
&= \phi(\phi(q_u \otimes p_u)q_t \otimes p_t\phi(q_v \otimes p_v))\\
&= \phi(q_u \otimes p_u)\phi(q_t \otimes p_t)\phi(q_v \otimes p_v)\\
&= \phi(q_u \otimes p_u t\phi(q_v \otimes p_v))\\
&= \phi(q_u \otimes \theta(p_u t \otimes q_v)p_v)\\
&= \phi(q_u \otimes sp_v)
\end{aligned}$$

and similarly $t' = \phi(q_u \otimes s'p_v)$. Thus, we have $(t,t') \in \Pi_\rho$.

Conversely, let $\phi(q_0 \otimes sp_0)\,\Pi_\rho\,\phi(q_0 \otimes s'p_0)$, where $s\,\rho\,s'$. Take $q \in Q$ and $p \in P$. It follows that

$$\begin{aligned}
\theta(p\phi(q_0 \otimes sp_0) \otimes q) &= \theta(\theta(p \otimes q_0)sp_0 \otimes q)\\
&= \theta(p \otimes q_0)\,s\,\theta(p_0 \otimes q)\\
&\ \rho\,\theta(p \otimes q_0)\,s'\,\theta(p_0 \otimes q)\\
&= \theta(p\phi(q_0 \otimes s'p_0) \otimes q).
\end{aligned}$$

Thus, $\Pi_\rho \subseteq \Theta(\rho)$. □

3. Conclusion

Morita invariants are properties which are shared by all semigroups in the same equivalence class. Corollary 2 implies that all properties defined in terms of the tolerance lattices of semigroups with common weak local units, regarded as biacts, are Morita invariant. One such property is **tolerance triviality** [1], i.e, every tolerance of the biact ${}_SS_S$ is a congruence.

Recall that a left (right) S-act ${}_SA$ (A_S) is called **faithful** if for any $s, s' \in S$ it holds that $s = s'$ whenever $sa = s'a$ ($as = as'$) for all $a \in A$.

A Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ connecting semigroups S and T is called **nondegenerate** (see [7]) if ${}_SP, P_T, {}_TQ$ and Q_S are faithful and the following conditions are satisfied for any $p_1, p_2 \in P$ and $q_1, q_2 \in Q$:

(1) $p_1 = p_2$ whenever $\phi(q \otimes p_1) = \phi(q \otimes p_2)$ for all $q \in Q$;
(2) $p_1 = p_2$ whenever $\theta(p_1 \otimes q) = \theta(p_2 \otimes q)$ for all $q \in Q$;
(3) $q_1 = q_2$ whenever $\phi(q_1 \otimes p) = \phi(q_2 \otimes p)$ for all $p \in P$;

(4) $q_1 = q_2$ whenever $\theta(p \otimes q_1) = \theta(p \otimes q_2)$ for all $p \in P$.

The results of Paseka [7] are obtained for nondegenerate Morita contexts. It would be interesting to know more about such Morita contexts and where they appear. Presently, we do not possess an abundant supply of them.

Problem 1. What are examples of nondegenerate Morita contexts, where at least one of the semigroups is not singleton? When are two semigroups connected by a nondegenerate Morita context?

Acknowledgements

I would like to thank the anonymous referee(s) for careful reading and useful suggestions given. This work was partially supported by the Estonian Research Council grant PRG1204.

References

[1] I. Chajda, *Algebraic theory of tolerance relations*, Monograph, Univerzita Palackého Olomouc, Olomouc, 1991.
[2] I. Chajda, B. Šešelja, and A. Tepavčević, *Lattices of compatible relations satisfying a set of formulas*, Algebra Univers. **40** (1998), 51–58.
[3] M. Erné, *Adjunctions and Galois connections: origins, history and development.* In: K. Denecke, M. Erné, and S. L. Wismath (eds.), Galois Connections and Applications, Kluwer Academic Publishers, Dordrecht–Boston–London, 2004, pp. 1–138.
[4] N. Jacobson, *Basic Algebra II*, W. H. Freeman and Co, New York, 1989.
[5] A. I. Kashu, *The composition of dualities in a nondegenerate Morita context*, J. Pure Appl. Algebra **133** (1998), 143–149.
[6] U. Knauer, *Projectivity of acts and Morita equivalence of monoids*, Semigroup Forum **3** (1971), 359–370.
[7] J. Paseka, *Morita contexts and their lattices of relations*, Internat. J. Theoret. Phys. **44** (2005), 2249–2258.
[8] V. Laan and L. Márki, *Strong Morita equivalence of semigroups with local units*, J. Pure Appl. Algebra **215** (2011), 2538–2546.
[9] V. Laan and L. Márki, *Morita invariants for semigroups with local units*, Monatsh. Math. **166** (2012), 441–451.
[10] V. Laan, L. Márki, and Ü. Reimaa, *Morita equivalence of semigroups revisited: firm semigroups*, J. Algebra **505** (2018), 247–270.
[11] V. Laan and Ü. Reimaa, *Morita equivalence of factorizable semigroups*, Internat. J. Algebra Comput. **29** (2019), 723–741.
[12] S. K. Sardar and S. Gupta, *A note on Morita invariants of semigroups*, Semigroup Forum **92** (2016), 71–76.
[13] S. Talwar, *Morita equivalence for semigroups*, J. Aust. Math. Soc. A **59** (1995), 81–111.
[14] S. Talwar, *Strong Morita equivalence and a generalisation of the Rees theorem*, Internat. J. Algebra **181** (1996), 371–394.
[15] L. Tooming, *Morita contexts, ideals, and congruences for semirings with local units*, Proc. Est. Acad. Sci. **67** (2018), 252–259.

Institute of Mathematics and Statistics, University of Tartu, Narva rd 18, Tartu, Estonia
E-mail address: alvin.lepik@ut.ee