On the identities of Ramanujan – a $q$-series approach

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Dedicated to my parents Professor H. M. Srivastava and Pushpa Srivastava

ABSTRACT. The idea of this paper is to prove two general theorems using $q$-series and deduce some partial theta identities and false theta identities. Some of the identities of Ramanujan come as special cases.

1. Introduction

Andrews and Warnaar in [1] used Bailey transform to prove some identities found on page 13 of Ramanujan’s Lost Notebook [2] and some more identities. On an empirical exploration of these identities, Andrews and Warnaar using MACSYMA got the identity

$$\sum_{n=0}^{\infty} \frac{(-zq; q^2)_n(-z^{-1}q^2; q^2)_n}{(-q; q)_{2n+1}} q^n = \sum_{n=0}^{\infty} \frac{1 - z^{2n+1}}{1 - z} z^{-n} q^{n(n+1)} = \sum_{n=0}^{\infty} \frac{1 + z + z^2 + \ldots + z^{2n}}{z^n} q^{n(n+1)}.$$

They proved this theorem using Bailey transform. They put the query: “Where does it fit in the classical theory of $q$-series”? This paper is an answer to this query. It is to be seen that Theorems 1 and 7 of [1] rely on Symmetric Bilateral Bailey Transform [1, Lemma 2.1, p. 175]

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$
and my (3.1) of [4] can also be deduced from the above transform by a suitable choice of \( \alpha_n, \beta_n \) and \( \gamma_n \). This implies that (3.1) of [4] is in fact a precursor of some of the results of [1].

The aim in writing this paper is to give a simple proof of this theorem using \( q \)-series, and deduce some partial theta identities, false theta identities and identities of Ramanujan.

2. The \( q \)-notations and Jacobi theta functions

Throughout the paper we shall be employing the standard \( q \)-hypergeometric notation. For \( |q| < 1 \),

\[
(a; q^n)_n = \prod_{r=1}^{n} (1 - aq^{r-1}), \quad n \geq 1,
\]

\[
(a; q^0) = 1,
\]

\[
(a; q^\infty)_n = \prod_{r=0}^{\infty} (1 - aq^{kr}).
\]

The partial products for the four classical Jacobi’s theta functions are

\[
\theta_{1;N}(z, q) = 2q^{1/4} \sin z \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{m=1}^{N} (1 - 2q^{2m} \cos 2z + q^{4m}),
\]

\[
\theta_{2;N}(z, q) = 2q^{1/4} \cos z \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{m=1}^{N} (1 + 2q^{2m} \cos 2z + q^{4m}),
\]

\[
\theta_{3;N}(z, q) = \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{m=1}^{N} (1 + 2q^{2m-1} \cos 2z + q^{4m-2}),
\]

\[
\theta_{4;N}(z, q) = \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{m=1}^{N} (1 - 2q^{2m-1} \cos 2z + q^{4m-2}).
\]

We suppose throughout that \( q = \exp(2\pi i \tau) \), \( \text{Im}(\tau) > 0 \). Rogers–Fine identity [3, p. 334] is

\[
\sum_{n=0}^{\infty} \frac{(\alpha)_n \tau^n}{(\beta)_n} = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\alpha \tau \beta q/\beta)_n \beta^n \tau^n q^{n^2-n} (1 - \alpha \tau q^{2n})}{(\beta)_n (\tau)_n+1}.
\]

We shall be using the transformation of Andrews and Warnaar [1]

\[
3\phi_2 \left[ \frac{a, -a, aq}{a^2, -aq^2}; q^2; q \right] = \frac{(-q; q)_\infty}{(-aq; q)_\infty} \sum_{r=0}^{\infty} a^r q^{r^2}. \quad (1)
\]
3. Main theorems

We prove two theorems.

**Theorem 1.** One has

\[
\sum_{n=0}^{\infty} \frac{(-zq;q^2)_n(-z^{-1}q;q^2)_nq^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{1 - z^{2n+1}}{1 - z} z^{-n} q^{n(n+1)} = \sum_{n=0}^{\infty} \frac{1 + z + z^2 + \ldots + z^{2n}}{z^n} q^{n(n+1)}.
\]

**Proof.** This theorem was proved in [1] by Andrews and Warnaar using Bailey transform. We give a simpler proof using \(q\)-series. In fact this theorem is a special case of our general theorem [4, Theorem 1, p. 120]. Taking \(\alpha_1 = q, \alpha_2 = q^2, \alpha_3 = -q, \beta_1 = -q^2, t = 1\) and \(\lambda = 1\) in this theorem, we get

\[
\sum_{n=0}^{\infty} (-aq;q^2)_n(-a^{-1}q;q^2)_n(q;q^2)_n(q^2;q^2)_n(-q;q^2)_n q^n = \sum_{n=0}^{\infty} a^n q^{N^2} \sum_{n=0}^{\infty} \frac{(q;q^2)_n(q^2;q^2)_n(-q;q^2)_n}{(q^2;q^2)_{2n}(q^2;q^2)_{2n}-q^2} q^n.
\]

The left side of (3) is

\[
\sum_{n=0}^{\infty} (-aq;q^2)_n(-a^{-1}q;q^2)_n(q;q^2)_n(q^2;q^2)_n(-q;q^2)_n q^n = (1 + q) \sum_{n=0}^{\infty} \frac{(-aq;q^2)_n(-a^{-1}q;q^2)_n q^n}{(-q;q)_{2n+1}}.
\]

Using (1), the right side of (3) may be calculated as follows:

\[
(1 + q) \sum_{r=0}^{\infty} q^{r^2+r} + \sum_{N=1}^{\infty} (a^N + a^{-N})q^{N^2} \times \sum_{n=0}^{\infty} \frac{(q^2;q^2)_n(q^2;q^2)_n(-q^2;q^2)_n q^{n+N}}{(q^2;q^2)_{n+2N}(q^2;q^2)_{n+2N}(-q^2;q^2)_{n+2N}} = (1 + q) \sum_{r=0}^{\infty} q^{r^2+r} + (1 + q) \sum_{N=1}^{\infty} \frac{(a^N + a^{-N})q^{N^2+N}}{(-q;q)_{2N+1}} \times \sum_{n=0}^{\infty} \frac{(q^{2N+1};q^2)_n(q^{2N+1};q^2)_n(-q^{2N+1};q^2)_n q^{n}}{(q^2;q^2)_n(q^{4N+2};q^2)_n(-q^{2N+2};q^2)_n q^n} = (1 + q) \left[ \sum_{r=0}^{\infty} q^{r^2+r} + \sum_{N=1}^{\infty} \frac{(a^N + a^{-N})q^{N^2+N}}{(-q;q)_{2N+1}} \frac{(-q;q)_{\infty}}{(-q^{2N+2};q)_{\infty}} \sum_{r=0}^{\infty} q^{(2N+1)r} q^{r^2} \right] \]
By (4) and (5) we have (2).

\[ \frac{q}{(1 + q) \sum_{r=0}^{\infty} q^2 + r + (1 + q) \sum_{N=1}^{\infty} (a^N + a^{-N}) q^{N^2 + N} \sum_{r=0}^{\infty} (2N+1) r q^2} \]

\[ = (1 + q) \sum_{r=0}^{\infty} q^2 + r + (1 + q) \sum_{N=1}^{\infty} (a^N + a^{-N}) \sum_{r=0}^{\infty} q^2 + r \]

\[ = (1 + q) \sum_{r=0}^{\infty} q^2 + r \sum_{N=1}^{\infty} a^N \]

\[ = (1 + q) \sum_{r=0}^{\infty} \frac{1 - a^2 r + 1}{1 - a} a^{-r} q^2 + r. \]  

(5)

By (4) and (5) we have (2).  

**Theorem 2.** We have the identity

\[ \sum_{n=0}^{\infty} \frac{(-zq^2; q^2)_n}{(-q; q)_{2n+2}} q^n = \sum_{n=0}^{\infty} \frac{1 - z^{2n+2}}{1 - z^2} z^{-n} q^{n(n+2)} \]

\[ = \sum_{n=0}^{\infty} \frac{1 + z^2 + z^4 + \ldots + z^{2n}}{z^n} q^{n(n+2)}. \]  

(6)

**Proof.** In [4, p. 123] we have proved the general equality

\[ \sum_{n=0}^{\infty} \frac{(-\alpha)_{n+1}(-q/\alpha)_n((\alpha r)_n)(tq^\lambda)_n}{(q2n+1)((\beta)_n)_n} = \frac{1 + \alpha}{1 - q} r^{\Theta_{s+1}} \left[ (\alpha r); q^3, tq^\lambda \right] \]

\[ + \sum_{N=0}^{\infty} \frac{(\alpha 1 + \alpha - N)q^{(N^2 + N)/2}((\alpha r)_N)(tq^\lambda)_N}{(q)_{2N+1}((\beta)_N)_N} \]

\[ \times \sum_{n=0}^{\infty} \frac{((\alpha r) q^N)_n}{(q)_n(q2N+2)_n((\beta)_n)_{q_N}_n} (tq^\lambda)_n, \]

where \((\alpha r)\) denotes the sequence \(\alpha_1, \alpha_2, \ldots, \alpha_r\) and \(\lambda\) is a suitable constant.

Setting here \(q = q^2, \alpha = z^{-1}, \alpha_1 = q^2, \alpha_2 = q^3, \alpha_3 = -q^2, \beta_1 = -q^4, t = 1,\) and \(\lambda = 1/2,\) we get

\[ (1 + z^{-1}) \sum_{n=0}^{\infty} \frac{(-z^{-1}q^2; q^2)_n(-zq^2; q^2)_n(q^2; q^2)_n(q^3; q^2)_n(-q^2; q^2)_n q^n}{(q^2; q^2)_{2n+1}(-q^4; q^2)_n} \]

\[ = \sum_{N=0}^{\infty} \frac{(z^{-N-1} + z^N)q^{N^2 + N}(q^2; q^2)_N(q^3; q^2)_N(-q^2; q^2)_N q^N}{(q^2; q^2)_{2N+1}(-q^4; q^2)_N} \]

\[ \times \sum_{n=0}^{\infty} \frac{(q^{2N+2}; q^2)_n(q^{2N+3}; q^2)_n(-q^{2N+2}; q^2)_n q^n}{(q^2; q^2)_n(q^{2N+4}; q^2)_n(-q^{2N+4}; q^2)_n}. \]  

(7)
The left side of (7) is equal to
\[ \frac{(1 + q^2)(1 + z^{-1})}{1 - q} \sum_{n=0}^{\infty} \frac{(-z^{-1}q^2;q^2)_n(-zq^2;q^2)_n}{(-q;q)_{2n+2}}q^n. \] (8)

In view of (1), the right side of (7) becomes
\[ \frac{(1 + q^2)}{1 - q} \sum_{N=0}^{\infty} \frac{(z^{-N+1} + z^N)q^{N^2+2N}}{(-q;q)_{2N+2}} \times \sum_{n=0}^{\infty} \frac{(q^{2N+2};q^2)_n(q^{2N+3};q^2)_n(-q^{2N+2};q^2)_n}{(q^2;q^2)_n(q^{4N+4};q^2)_n(-q^{2N+4};q^2)_n}q^n. \]

\[ = \frac{(1 + q^2)}{1 - q} \sum_{N=0}^{\infty} (z^{-N+1} + z^N)q^{N^2+2N}(-q;q)_{\infty} \sum_{r=0}^{\infty} q^{2(N+2)r}q^{r^2}. \]

\[ = \frac{(1 + q^2)}{1 - q} \sum_{r=0}^{\infty} q^{r^2+2r} \sum_{N=0}^{\infty} (z^{-N+1} + z^N). \]

\[ = \frac{(1 + q^2)}{1 - q} \sum_{r=0}^{\infty} q^{r^2+2r} \left[ (z^{-1} + z^{-2} + ... + z^{-r-1}) + (1 + z + ... + z^r) \right]. \]

\[ = \frac{(1 + q^2)}{1 - q} \sum_{r=0}^{\infty} q^{r^2+2r} \left[ (1 + z + ... + z^r) \left( 1 + \frac{1}{z^{r+1}} \right) \right]. \]

\[ = \frac{(1 + q^2)}{1 - q} \sum_{r=0}^{\infty} q^{r^2+2r} \left[ \frac{1 - z^{r+1}}{1 - z} \cdot \frac{(1 + z^{r+1})}{z^{r+1}} \right]. \]

\[ = \frac{(1 + q^2)}{1 - q} \sum_{r=0}^{\infty} q^{r^2+2r} \frac{(1 - z^{2r+2})}{1 - z} z^{-r-1}. \] (9)

Hence by (8) and (9), after dividing by (1 + z^{-1}), we have (6). \[ \square \]

4. Two identities of Ramanujan

We give simple proofs of two identities of Ramanujan found in the “Lost” Notebook [2]. The first identity of Ramanujan is
\[ \sum_{n=0}^{\infty} \frac{(q;q^2)_n^2}{(-q;q)_{2n+1}} q^n = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)}. \]

We give two proofs which are simple deductions of Theorem 1. The first is simply put \( z = -1 \) in Theorem 1. The second is taking \( z = -e^{i\pi} \) and then \( z = \pi \) in Theorem 1.
The second identity of Ramanujan is
\[ \sum_{n=0}^{\infty} \frac{(q; -q)_n}{(-q; q)_{2n+1}} q^n = \sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)}. \] (10)

Simple calculation shows that the left side of (10) is
\[ \sum_{n=0}^{\infty} \frac{(q; q^2)_n(-q^2; q^2)_n}{(-q; q)_{n+1}(-q^2; q^2)_n} q^n = \sum_{n=0}^{\infty} \frac{(q; q^2)_n}{(-q; q)_{n+1}} q^n. \]

Applying Rogers–Fine identity [3] in the above, we have the right side of (10).

5. Some more identities

The following identities come as special cases of the above two theorems.

We have the identities
\[ \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(-q^2; q^2)_n(1 + q^{2n+1})} q^n = \sum_{n=0}^{\infty} (2n + 1)q^{n(n+1)} \]

\( (z = 1 \text{ in Theorem 1}), \)

\[ \sum_{n=1}^{\infty} \frac{(-q^2; q^2)_n}{(-q; q^2)_n(1 + q^{2n})} q^{n-1} = \sum_{n=1}^{\infty} nq^{n^2-1} \]

\( (z = 1 \text{ in Theorem 2}), \)

\[ \sum_{n=1}^{\infty} \frac{(q^2; q^2)_n}{(-q; q)_{2n}} q^{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1}nq^{n^2-1} \]

\( (z = -1 \text{ in Theorem 2}). \)

6. False and partial theta function identities

By specializing \( z \), we get the following identities for false and partial theta functions:
\[ \frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q; q)_{2n+1}} \theta_{3,n}(z, q) = \sum_{n=0}^{\infty} \frac{\sin(2n + 1)z}{\sin z} q^{n(n+1)} \]

\( (e^{2z} \text{ for } z \text{ in Theorem 1}), \)

\[ \frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q; q)_{2n+1}} \theta_{3,n}(\pi/2, q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \]

\( (z = \pi/2 \text{ in (11)}), \)
\[-\frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q; q)_{2n+1}} \theta_{4; n}(z, q) = \sum_{n=0}^{\infty} (-1)^n \cos(2n + 1)z \theta_{4; n}(n+1) \quad (12)\]

\[-e^{2iz} \text{ for } z \text{ in Theorem 1},\]

\[-\frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q; q)_{2n+1}} \theta_{4; n}(\pi, q) = \sum_{n=0}^{\infty} (-1)^n q^n(n+1) \quad (z = \pi \text{ in (12)}),\]

\[-\frac{q^{-\frac{1}{4}}}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q; q)_{2n+2}} \theta_{2; n}(z, q) = \sum_{r=0}^{\infty} \sin(2r + 2)z \theta_{2; n}(r+2) \quad (r = 2z \text{ for } z \text{ in Theorem 2}),\]

\[-\frac{q^{-\frac{1}{4}}}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q; q)_{2n+2}} \theta_{2; n}(\pi, q) = -\sum_{r=0}^{\infty} (2r + 2)q^{r+2} \quad (z = \pi \text{ in (13)}),\]

\[-\frac{q^{-\frac{1}{4}}}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q; q)_{2n+2}} \theta_{1; n}(z, q) = 2 \sum_{r=0}^{\infty} (-1)^r \sin(2r + 2)z \theta_{1; n}(r+2) \quad (14)\]

\[-e^{2iz} \text{ for } z \text{ in Theorem 2},\]

\[-\frac{q^{-\frac{1}{4}}}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q; q)_{2n+2}} \theta_{1; n}(\pi/2, q) = 2 \sum_{r=0}^{\infty} (r + 1)q^{r+2} \quad (z = \pi/2 \text{ in (14)}).\]

References


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