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On the identities of Ramanujan – a q-series approach

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Dedicated to my parents Professor H. M. Srivastava and Pushpa Srivastava

Abstract. The idea of this paper is to prove two general theorems using q-series and deduce some partial theta identities and false theta identities. Some of the identities of Ramanujan come as special cases.

1. Introduction

Andrews and Warnaar in [1] used Bailey transform to prove some identities found on page 13 of Ramanujan's Lost Notebook [2] and some more identities. On an empirical exploration of these identities, Andrews and Warnaar using MACSYMA got the identity

$$\sum_{n=0}^{\infty} \frac{(-zq;q^2)_n(-z^{-1}q;q^2)_n}{(-q;q)_{2n+1}} q^n = \sum_{n=0}^{\infty} \frac{1-z^{2n+1}}{1-z} z^{-n} q^{n(n+1)}$$
$$= \sum_{n=0}^{\infty} \frac{1+z+z^2+...+z^{2n}}{z^n} q^{n(n+1)}.$$

They proved this theorem using Bailey transform. They put the query: "Where does it fit in the classical theory of q-series"? This paper is an answer to this query. It is to be seen that Theorems 1 and 7 of [1] rely on Symmetric Bilateral Bailey Transform [1, Lemma 2.1, p. 175]

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n,$$

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and my (3.1) of [4] can also be deduced from the above transform by a suitable choice of α_n, β_n and γ_n. This implies that (3.1) of [4] is in fact a precursor of some of the results of [1].

The aim in writing this paper is to give a simple proof of this theorem using q-series, and deduce some partial theta identities, false theta identities and identities of Ramanujan.

2. The q-notations and Jacobi theta functions

Throughout the paper we shall be employing the standard q-hypergeometric notation. For $|q| < 1$,

$$(a;q^k)_n = \prod_{r=1}^{n}(1-aq^{k(r-1)}), \quad n \geq 1,$$
$$(a;q^k)_0 = 1,$$
$$(a;q^k)_\infty = \prod_{r=0}^{\infty}(1-aq^{kr}).$$

The partial products for the four classical Jacobi's theta functions are

$$\theta_{1;N}(z,q) = 2q^{\frac{1}{4}}\sin z\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1-2q^{2m}\cos 2z+q^{4m}),$$
$$\theta_{2;N}(z,q) = 2q^{\frac{1}{4}}\cos z\prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1+2q^{2m}\cos 2z+q^{4m}),$$
$$\theta_{3;N}(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1+2q^{2m-1}\cos 2z+q^{4m-2}),$$
$$\theta_{4;N}(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=1}^{N}(1-2q^{2m-1}\cos 2z+q^{4m-2}).$$

We suppose throughout that $q = exp(2\pi\iota\tau)$, $\mathrm{Im}(\tau) > 0$. Rogers–Fine identity [3, p. 334] is

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n\tau^n}{(\beta)_n} = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha\tau q/\beta)_n\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta)_n(\tau)_{n+1}}.$$

We shall be using the transformation of Andrews and Warnaar [1]

$${}_3\phi_2\left[\begin{matrix}a,-a,aq\\a^2,-aq^2\end{matrix};q^2;q\right] = \frac{(-q;q)_\infty}{(-aq;q)_\infty}\sum_{r=0}^{\infty}a^r q^{r^2}. \tag{1}$$

3. Main theorems

We prove two theorems.

Theorem 1. *One has*

$$\sum_{n=0}^{\infty}\frac{(-zq;q^2)_n(-z^{-1}q;q^2)_n}{(-q;q)_{2n+1}}q^n=\sum_{n=0}^{\infty}\frac{1-z^{2n+1}}{1-z}z^{-n}q^{n(n+1)}$$
$$=\sum_{n=0}^{\infty}\frac{1+z+z^2+...+z^{2n}}{z^n}q^{n(n+1)}. \tag{2}$$

Proof. This theorem was proved in [1] by Andrews and Warnaar using Bailey transform. We give a simpler proof using q-series. In fact this theorem is a special case of our general theorem [4, Theorem 1, p. 120]. Taking $\alpha_1=q,\alpha_2=q^2,\alpha_3=-q,\beta_1=-q^3,t=1$ and $\lambda=1$ in this theorem, we get

$$\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n(-a^{-1}q;q^2)_n(q;q^2)_n(q^2;q^2)_n(-q;q^2)_n}{(q^2;q^2)_{2n}(-q^3;q^2)_n}q^n$$
$$=\sum_{N=-\infty}^{\infty}a^Nq^{N^2}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(q^2;q^2)_n(-q;q^2)_n}{(q^2;q^2)_{n+N}(q^2;q^2)_{n-N}(-q^3;q^2)_n}q^n. \tag{3}$$

The left side of (3) is

$$\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n(-a^{-1}q;q^2)_n(q;q^2)_n(q^2;q^2)_n(-q;q^2)_n}{(q;q)_{2n}(-q;q)_{2n}(-q^3;q^2)_n}q^n$$
$$=(1+q)\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n(-a^{-1}q;q^2)_n}{(-q;q)_{2n+1}}q^n. \tag{4}$$

Using (1), the right side of (3) may be calculated as follows:

$$(1+q)\sum_{r=0}^{\infty}q^{r^2+r}+\sum_{N=1}^{\infty}(a^N+a^{-N})q^{N^2}$$
$$\times\sum_{n=0}^{\infty}\frac{(q;q^2)_{n+N}(q^2;q^2)_{n+N}(-q;q^2)_{n+N}}{(q^2;q^2)_n(q^2;q^2)_{n+2N}(-q^3;q^2)_{n+N}}q^{n+N}$$
$$=(1+q)\sum_{r=0}^{\infty}q^{r^2+r}+(1+q)\sum_{N=1}^{\infty}\frac{(a^N+a^{-N})q^{N^2+N}}{(-q;q)_{2N+1}}$$
$$\times\sum_{n=0}^{\infty}\frac{(q^{2N+1};q^2)_n(q^{2N+2};q^2)_n(-q^{2N+1};q^2)_n}{(q^2;q^2)_n(q^{4N+2};q^2)_n(-q^{2N+3};q^2)_n}q^n$$
$$=(1+q)\left[\sum_{r=0}^{\infty}q^{r^2+r}+\sum_{N=1}^{\infty}\frac{(a^N+a^{-N})q^{N^2+N}}{(-q;q)_{2N+1}}\frac{(-q;q)_\infty}{(-q^{2N+2};q)_\infty}\sum_{r=0}^{\infty}q^{(2N+1)r}q^{r^2}\right]$$

$$= (1+q)\sum_{r=0}^{\infty} q^{r^2+r} + (1+q)\sum_{N=1}^{\infty}(a^N + a^{-N})q^{N^2+N}\sum_{r=0}^{\infty} q^{(2N+1)r}q^{r^2}$$

$$= (1+q)\sum_{r=0}^{\infty} q^{r^2+r} + (1+q)\sum_{N=1}^{\infty}(a^N + a^{-N})\sum_{r=N}^{\infty} q^{r^2+r}$$

$$= (1+q)\sum_{r=0}^{\infty} q^{r^2+r}\sum_{N=-r}^{r} a^N$$

$$= (1+q)\sum_{r=0}^{\infty}\frac{1-a^{2r+1}}{1-a}a^{-r}q^{r^2+r}. \tag{5}$$

By (4) and (5) we have (2). □

Theorem 2. *We have the identity*

$$\sum_{n=0}^{\infty}\frac{(-zq^2;q^2)_n(-z^{-1}q^2;q^2)_n}{(-q;q)_{2n+2}}q^n = \sum_{n=0}^{\infty}\frac{1-z^{2n+2}}{1-z^2}z^{-n}q^{n(n+2)}$$
$$= \sum_{n=0}^{\infty}\frac{1+z^2+z^4+...+z^{2n}}{z^n}q^{n(n+2)}. \tag{6}$$

Proof. In [4, p. 123] we have proved the general equality

$$\sum_{n=0}^{\infty}\frac{(-\alpha)_{n+1}(-q/\alpha)_n((\alpha_r))_n}{(q)_{2n+1}((\beta_s))_n}(tq^\lambda)^n = \frac{1+\alpha}{1-q}\ {}_r\phi_{s+1}\left[\begin{matrix}(\alpha_r);\\(\beta_s),q^2;tq^\lambda\end{matrix}\right]$$
$$+\sum_{N=0}^{\infty}\frac{(\alpha^{1+N}+\alpha^{-N})q^{(N^2+N)/2}((\alpha_r))_N}{(q)_{2N+1}((\beta_s))_N}(tq^\lambda)^N$$
$$\times\sum_{n=0}^{\infty}\frac{((\alpha_r)q^N)_n}{(q)_n(q^{2N+2})_n((\beta_s)q^N)_n}(tq^\lambda)^n,$$

where (α_r) denotes the sequence $\alpha_1,\alpha_2,...,\alpha_r$ and λ is a suitable constant. Setting here $q\to q^2, \alpha = z^{-1}, \alpha_1 = q^2, \alpha_2 = q^3, \alpha_3 = -q^2, \beta_1 = -q^4, t = 1$, and $\lambda = 1/2$, we get

$$(1+z^{-1})\sum_{n=0}^{\infty}\frac{(-z^{-1}q^2;q^2)_n(-zq^2;q^2)_n(q^2;q^2)_n(q^3;q^2)_n(-q^2;q^2)_n}{(q^2;q^2)_{2n+1}(-q^4;q^2)_n}q^n$$
$$= \sum_{N=0}^{\infty}\frac{(z^{-N-1}+z^N)q^{N^2+N}(q^2;q^2)_N(q^3;q^2)_N(-q^2;q^2)_N}{(q^2;q^2)_{2N+1}(-q^4;q^2)_N}q^N$$
$$\times\sum_{n=0}^{\infty}\frac{(q^{2N+2};q^2)_n(q^{2N+3};q^2)_n(-q^{2N+2};q^2)_n}{(q^2;q^2)_n(q^{4N+4};q^2)_n(-q^{2N+4};q^2)_n}q^n. \tag{7}$$

The left side of (7) is equal to

$$\frac{(1+q^2)(1+z^{-1})}{1-q}\sum_{n=0}^{\infty}\frac{(-z^{-1}q^2;q^2)_n(-zq^2;q^2)_n}{(-q;q)_{2n+2}}q^n. \tag{8}$$

In view of (1), the right side of (7) becomes

$$\begin{aligned}
&\frac{(1+q^2)}{1-q}\sum_{N=0}^{\infty}\frac{(z^{-N-1}+z^N)q^{N^2+2N}}{(-q;q)_{2N+2}}\\
&\qquad\times\sum_{n=0}^{\infty}\frac{(q^{2N+2};q^2)_n(q^{2N+3};q^2)_n(-q^{2N+2};q^2)_n}{(q^2;q^2)_n(q^{4N+4};q^2)_n(-q^{2N+4};q^2)_n}q^n\\
&=\frac{(1+q^2)}{1-q}\sum_{N=0}^{\infty}\frac{(z^{-N-1}+z^N)q^{N^2+2N}(-q;q)_\infty}{(-q;q)_{2N+2}(-q^{2N+3};q)_\infty}\sum_{r=0}^{\infty}q^{(2N+2)r}q^{r^2}\\
&=\frac{(1+q^2)}{1-q}\sum_{N=0}^{\infty}(z^{-N-1}+z^N)q^{N^2+2N}\sum_{r=0}^{\infty}q^{(2N+2)r}q^{r^2}\\
&=\frac{(1+q^2)}{1-q}\sum_{r=0}^{\infty}q^{r^2+2r}\sum_{N=0}^{r}(z^{-N-1}+z^N)\\
&=\frac{(1+q^2)}{1-q}\sum_{r=0}^{\infty}q^{r^2+2r}\left[(z^{-1}+z^{-2}+\ldots+z^{-r-1})+(1+z+\ldots+z^r)\right]\\
&=\frac{(1+q^2)}{1-q}\sum_{r=0}^{\infty}q^{r^2+2r}\left[(1+z+\ldots+z^r)\left(1+\frac{1}{z^{r+1}}\right)\right]\\
&=\frac{(1+q^2)}{1-q}\sum_{r=0}^{\infty}q^{r^2+2r}\left[\left(\frac{1-z^{r+1}}{1-z}\right)\frac{(1+z^{r+1})}{z^{r+1}}\right]\\
&=\frac{(1+q^2)}{1-q}\sum_{r=0}^{\infty}q^{r^2+2r}\frac{(1-z^{2r+2})}{1-z}z^{-r-1}.
\end{aligned} \tag{9}$$

Hence by (8) and (9), after dividing by $(1+z^{-1})$, we have (6). □

4. Two identities of Ramanujan

We give simple proofs of two identities of Ramanujan found in the "Lost" Notebook [2]. The first identity of Ramanujan is

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n^2}{(-q;q)_{2n+1}}q^n=\sum_{n=0}^{\infty}(-1)^nq^{n(n+1)}.$$

We give two proofs which are simple deductions of Theorem 1. The first is simply put $z=-1$ in Theorem 1. The second is taking $z=-e^{2\iota z}$ and then $z=\pi$ in Theorem 1.

The second identity of Ramanujan is

$$\sum_{n=0}^{\infty} \frac{(q;-q)_n}{(-q;q)_{2n+1}} q^n = \sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)}. \tag{10}$$

Simple calculation shows that the left side of (10) is

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q^2;q^2)_n}{(-q;q)_{n+1}(-q^2;q^2)_n} q^n = \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(-q;q)_{n+1}} q^n.$$

Applying Rogers–Fine identity [3] in the above, we have the right side of (10).

5. Some more identities

The following identities come as special cases of the above two theorems. We have the identities

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(-q^2;q^2)_n(1+q^{2n+1})} q^n = \sum_{n=0}^{\infty} (2n+1) q^{n(n+1)}$$

($z=1$ in Theorem 1),

$$\sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}}{(-q;q^2)_n(1+q^{2n})} q^{n-1} = \sum_{n=1}^{\infty} n q^{n^2-1}$$

($z=1$ in Theorem 2),

$$\sum_{n=1}^{\infty} \frac{(q^2;q^2)_{n-1}^2}{(-q;q)_{2n}} q^{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} n q^{n^2-1}$$

($z=-1$ in Theorem 2).

6. False and partial theta function identities

By specializing z, we get the following identities for false and partial theta functions:

$$\frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n+1}} \theta_{3;n}(z,q) = \sum_{n=0}^{\infty} \frac{\sin(2n+1)z}{\sin z} q^{n(n+1)} \tag{11}$$

($e^{2\iota z}$ for z in Theorem 1),

$$\frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n+1}} \theta_{3;n}(\pi/2,q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)}$$

($z=\pi/2$ in (11)),

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_{2n+1}}\theta_{4;n}(z,q)=\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)z}{\cos z}q^{n(n+1)} \tag{12}$$

$(-e^{2\iota z}$ for z in Theorem 1),

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_{2n+1}}\theta_{4;n}(\pi,q)=\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}$$

($z=\pi$ in (12)),

$$\frac{q^{-\frac{1}{4}}}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_{2n+2}}\theta_{2;n}(z,q)=\sum_{r=0}^{\infty}\frac{\sin(2r+2)z}{\sin z}q^{r(r+2)} \tag{13}$$

$(e^{2\iota z}$ for z in Theorem 2),

$$\frac{q^{-\frac{1}{4}}}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_{2n+2}}\theta_{2;n}(\pi,q)=-\sum_{r=0}^{\infty}(2r+2)q^{r(r+2)}$$

($z=\pi$ in (13)),

$$\frac{q^{-\frac{1}{4}}}{(q^2;q^2)_\infty\sin z}\sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_{2n+2}}\theta_{1;n}(z,q)=2\sum_{r=0}^{\infty}\frac{(-1)^r\sin(2r+2)z}{\sin 2z}q^{r(r+2)} \tag{14}$$

$(-e^{2\iota z}$ for z in Theorem 2),

$$\frac{q^{-\frac{1}{4}}}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_{2n+2}}\theta_{1;n}(\pi/2,q)=2\sum_{r=0}^{\infty}(r+1)q^{r(r+2)}$$

($z=\pi/2$ in (14)).

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