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Further remarks on permissible covariance structures for simultaneous retention of BLUEs in linear models

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Abstract. We consider the partitioned linear model $\mathscr{M}_{12}(\mathbf{V}_0) = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}_0\}$ and the corresponding small model $\mathscr{M}_1(\mathbf{V}_0) = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{V}_0\}$. We define the set $\mathscr{V}_{1/12}$ of nonnegative definite matrices $\mathbf{V}$ such that every representation of the best linear unbiased estimator, BLUE, of $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ under $\mathscr{M}_{12}(\mathbf{V}_0)$ remains BLUE under $\mathscr{M}_{12}(\mathbf{V})$. Correspondingly, we can characterize the set $\mathscr{V}_1$ of matrices $\mathbf{V}$ such that every BLUE of $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ under $\mathscr{M}_1(\mathbf{V}_0)$ remains BLUE under $\mathscr{M}_1(\mathbf{V})$. In this paper we focus on the mutual relations between the sets $\mathscr{V}_1$ and $\mathscr{V}_{1/12}$.

1. Introduction and preliminaries

In this paper we consider the partitioned linear model $\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}$, sometimes called big or full model, shortly denoted

$$\mathscr{M}_{12}(\mathbf{V}) = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\} = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}\},$$

and the corresponding small model $\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}$, denoted as

$$\mathscr{M}_1(\mathbf{V}) = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{V}\}.$$

In the above linear models, $\mathbf{y}$ is an n-dimensional observable random vector, and $\boldsymbol{\varepsilon}$ is an unobservable random error vector with a known covariance matrix $\operatorname{cov}(\boldsymbol{\varepsilon}) = \mathbf{V} = \operatorname{cov}(\mathbf{y})$ and expectation $\operatorname{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$. The matrix $\mathbf{X}$ is a known $n \times p$ matrix, i.e., $\mathbf{X} \in \mathbb{R}^{n\times p}$, partitioned columnwise as $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$, $\mathbf{X}_i \in \mathbb{R}^{n\times p_i}$, $i = 1, 2$. Vector $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')' \in \mathbb{R}^p$ is a vector

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of fixed (but unknown) parameters; here symbol $'$ stands for the transpose. We will also denote $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$, $\boldsymbol{\mu}_i = \mathbf{X}_i\boldsymbol{\beta}_i$, $i = 1, 2$.

As regards notations, the symbols $\mathrm{r}(\mathbf{A})$, $\mathbf{A}^-$, $\mathbf{A}^+$ and $\mathscr{C}(\mathbf{A})$, denote, respectively, the rank, a generalized inverse, the Moore–Penrose inverse, and the column space of the matrix $\mathbf{A}$. Furthermore, we will write $\mathbf{P_A} = \mathbf{AA}^+ = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ to denote the orthogonal projector onto $\mathscr{C}(\mathbf{A})$ and $\mathbf{Q_A} = \mathbf{I}_a - \mathbf{P_A}$ where $\mathbf{I}_a$ is the identity matrix of order a with a being the number of rows in $\mathbf{A}$. In particular, we denote

$$\mathbf{M} = \mathbf{I}_n - \mathbf{P_X}\,, \quad \mathbf{M}_i = \mathbf{I}_n - \mathbf{P}_i\,, \quad \mathbf{P}_i = \mathbf{P}_{\mathbf{X}_i}\,,\ i = 1, 2.$$

Under the model $\mathscr{M}_{12}(\mathbf{V}) = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$, the statistic $\mathbf{Gy}$, where $\mathbf{G}$ is an $n \times n$ matrix, is the best linear unbiased estimator, BLUE, of $\mathbf{X}\boldsymbol{\beta}$ if $\mathbf{Gy}$ is unbiased, i.e., $\mathbf{GX} = \mathbf{X}$, and it has the smallest covariance matrix in the Löwner sense among all unbiased linear estimators of $\mathbf{X}\boldsymbol{\beta}$; shortly denoted

$$\mathrm{cov}(\mathbf{Gy}) \leq_{\mathrm{L}} \mathrm{cov}(\mathbf{Cy}) \quad \text{for all } \mathbf{C} \in \mathbb{R}^{n\times n}\colon\ \mathbf{CX} = \mathbf{X}\,,$$

i.e., for all $\mathbf{C} \in \mathbb{R}^{n\times n}$ such that $\mathbf{CX} = \mathbf{X}$ we have

$$\mathrm{cov}(\mathbf{Cy}) - \mathrm{cov}(\mathbf{Gy}) = \mathbf{AA}' \quad \text{for some } \mathbf{A}\,.$$

The BLUE of an estimable parametric function $\mathbf{K}\boldsymbol{\beta}$, where $\mathbf{K} \in \mathbb{R}^{k\times p}$, is defined in the corresponding way. Recall that $\mathbf{K}\boldsymbol{\beta}$ is said to be estimable if it has a linear unbiased estimator which happens if and only if $\mathscr{C}(\mathbf{K}') \subseteq \mathscr{C}(\mathbf{X}')$. In particular, $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is estimable in the partitioned model if and only if

$$\mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2) = \{\mathbf{0}\}\,. \tag{1}$$

For the proof of the following fundamental lemma, see, e.g., Rao [13, p. 282].

Lemma 1.1. *Consider the partitioned linear model $\mathscr{M}_{12}(\mathbf{V}) = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$. Then the statistic $\mathbf{Gy}$ is the* BLUE *for $\mathbf{X}\boldsymbol{\beta}$ if and only if $\mathbf{G}$ satisfies the equation*

$$\mathbf{G}(\mathbf{X} : \mathbf{VM}) = (\mathbf{X} : \mathbf{0})\,. \tag{2}$$

The corresponding condition for $\mathbf{By}$ to be the BLUE *of an estimable parametric function $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is*

$$\mathbf{B}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{X}_1 : \mathbf{0} : \mathbf{0})\,. \tag{3}$$

Equation (2) has always a solution for $\mathbf{G}$ while (3) has a solution for $\mathbf{B}$ if and only if $\boldsymbol{\mu}_1$ is estimable in $\mathscr{M}_{12}(\mathbf{V})$. Solutions are unique if and only if $\mathscr{C}(\mathbf{X} : \mathbf{VM}) = \mathbb{R}^n$.

For later considerations, we collect some useful results into the following lemma. As references, we may mention [14, Lemma 2.1] and [6, Cor. 6.2].

Lemma 1.2. *Consider the partitioned model* $\mathscr{M}_{12}(\mathbf{V}) = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$*, and let "*$\oplus$*" refer to the direct sum and "*$\boxplus$*" to the direct sum of orthogonal subspaces. Then*

(a) $\mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2) = \mathscr{C}(\mathbf{X}_1 : \mathbf{M}_1\mathbf{X}_2)$, *i.e.*, $\mathscr{C}(\mathbf{X}) = \mathscr{C}(\mathbf{X}_1) \boxplus \mathscr{C}(\mathbf{M}_1\mathbf{X}_2)$,
(b) $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{X} : \mathbf{VM}) = \mathscr{C}(\mathbf{X}) \oplus \mathscr{C}(\mathbf{VM}) = \mathscr{C}(\mathbf{X}) \boxplus \mathscr{C}(\mathbf{MV})$,
(c) $\mathbf{M} = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}} = \mathbf{I}_n - (\mathbf{P}_{\mathbf{X}_1} + \mathbf{P}_{\mathbf{M}_1\mathbf{X}_2}) = \mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2} = \mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2}\mathbf{M}_1$,
(d) $\mathrm{r}(\mathbf{M}_1\mathbf{X}_2) = \mathrm{r}(\mathbf{X}_2) - \dim \mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2)$.

Consider now the linear model $\mathscr{M}_1(\mathbf{V}_0) = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{V}_0\}$. Then it is well known, see, e.g., Rao [12, Sec. 4], that one solution for $\mathbf{G}$ in $\mathbf{G}(\mathbf{X}_1 : \mathbf{V}_0\mathbf{M}_1) = (\mathbf{X}_1 : \mathbf{0})$ is

$$\mathbf{G}_1 = \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^-\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{W}_1^-, \tag{4}$$

where $\mathbf{W}_1$ is any matrix of the form

$$\mathbf{W}_1 = \mathbf{V}_0 + \mathbf{X}_1\mathbf{U}_1\mathbf{U}_1'\mathbf{X}_1' \quad \text{such that } \mathscr{C}(\mathbf{W}_1) = \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_0).$$

The choice of $\mathbf{U}_1$ is free subject to $\mathscr{C}(\mathbf{W}_1) = \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_0)$. In particular, if $\mathscr{C}(\mathbf{X}_1) \subseteq \mathscr{C}(\mathbf{V}_0)$, in which case we say that $\mathscr{M}_1(\mathbf{V}_0)$ is a weakly singular linear model, then we can choose $\mathbf{U}_1 = \mathbf{0}$ and one solution for $\mathbf{G}$ in $\mathbf{G}(\mathbf{X}_1 : \mathbf{V}_0\mathbf{M}_1) = (\mathbf{X}_1 : \mathbf{0})$ is

$$\mathbf{G}_2 = \mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_0^-\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{V}_0^-. \tag{5}$$

Correspondingly, one solution for $\mathbf{B}$ in (3) concerning the partitioned model $\mathscr{M}_{12}(\mathbf{V}_0) = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}_0\}$ is

$$\mathbf{B}_1 = (\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-,$$

where $\mathbf{W}$ is any matrix of the form

$$\mathbf{W} = \mathbf{V}_0 + \mathbf{XUU}'\mathbf{X}' \quad \text{such that } \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V}_0). \tag{6}$$

Now the *general* solution for $\mathbf{G}$ in $\mathbf{G}(\mathbf{X}_1 : \mathbf{V}_0\mathbf{M}_1) = (\mathbf{X}_1 : \mathbf{0})$ is

$$\mathbf{G}_0 = \mathbf{G}_1 + \mathbf{N}(\mathbf{I}_n - \mathbf{P}_{(\mathbf{X}_1 : \mathbf{V}_0)}) = \mathbf{G}_1 + \mathbf{NQ}_{(\mathbf{X}_1 : \mathbf{V}_0)}, \tag{7}$$

where $\mathbf{G}_1$ is as in (4) and $\mathbf{N} \in \mathbb{R}^{n\times n}$ is free to vary. Thus, if we want that *every* representation of the BLUE of $\boldsymbol{\mu}_1$ under $\mathscr{M}_1(\mathbf{V}_0)$ provides also the BLUE of $\boldsymbol{\mu}_1$ under $\mathscr{M}_1(\mathbf{V}) = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{V}\}$, then by Lemma 1.1, the matrix $\mathbf{G}_0$ in (7) has to satisfy

$$\mathbf{G}_0(\mathbf{X}_1 : \mathbf{VM}_1) = (\mathbf{G}_1 + \mathbf{NQ}_{(\mathbf{X}_1 : \mathbf{V}_0)})(\mathbf{X}_1 : \mathbf{VM}_1) = (\mathbf{X}_1 : \mathbf{0}) \text{ for all } \mathbf{N}. \tag{8}$$

As concluded by Rao [12, Th. 5.2] and [13, Th. 4.2] the statement (8) holds if and only if

$$\mathscr{C}(\mathbf{VM}_1) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1). \tag{9}$$

Notice that in the *consistent* model the numerical value of $\mathbf{G}_0\mathbf{y}$ is unique once the response vector $\mathbf{y}$ has been observed; the model $\mathscr{M}_1(\mathbf{V}_0)$ is called consistent if $\mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{V}_0)$ with probability 1.

Next question: for a given $\mathbf{V}_0$, how can we characterize the set, say $\mathscr{V}_1$, of nonnegative definite matrices $\mathbf{V}$ that satisfy (9)? By Rao [12, Th. 5.3], (9) is equivalent to the fact that $\mathbf{V}$ can be expressed as $\mathbf{V} = \mathbf{X}_1\mathbf{A}\mathbf{A}'\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\mathbf{B}\mathbf{B}'\mathbf{M}_1\mathbf{V}_0$ for some $\mathbf{A}$ and $\mathbf{B}$, i.e.,

$$\mathbf{V} \in \mathscr{V}_1 \iff \mathbf{V} = \mathbf{X}_1\mathbf{A}\mathbf{A}'\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\mathbf{B}\mathbf{B}'\mathbf{M}_1\mathbf{V}_0 \quad \text{for some } \mathbf{A} \text{ and } \mathbf{B}.$$

Further equivalent conditions appear in Lemma 1.3 below. It may be mentioned, as noted by Mitra and Moore [8, p. 148] and Rao [13, p. 289], that Rao in Theorem 5.3 of his 1971 paper had an unnecessary condition $\mathscr{C}(\mathbf{X}_1 : \mathbf{V}_0\mathbf{M}_1) = \mathbb{R}^n$ which was not used in the proofs.

For the property that every representation of the BLUE of $\boldsymbol{\mu}_1$ under $\mathscr{M}_1(\mathbf{V}_0)$ remains BLUE of $\boldsymbol{\mu}_1$ under $\mathscr{M}_1(\mathbf{V})$ we will use the notation

$$\mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_1(\mathbf{V}_0)) \subseteq \mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_1(\mathbf{V}))\,. \tag{10}$$

It is worth emphasizing that the notation of the above type is merely symbolic; we are interested in the multipliers of the response vector $\mathbf{y}$ which have specific properties. The notation $\mathcal{B}(\boldsymbol{\eta} \,|\, \mathscr{A})$ refers to the set of all representations of the BLUE of parametric function $\boldsymbol{\eta}$ under the model $\mathscr{A}$.

Some equivalent statements to (10) are given as follows.

Lemma 1.3. *Consider the linear models* $\mathscr{M}_1(\mathbf{V}_0) = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{V}_0\}$ *and* $\mathscr{M}_1(\mathbf{V}) = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{V}\}$. *Then the following statements are equivalent:*

(a) $\mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_1(\mathbf{V}_0)) \subseteq \mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_1(\mathbf{V}))$, *i.e.,* $\mathbf{V} \in \mathscr{V}_1$.
(b) $\mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)$.
(c) $\mathbf{V} = \mathbf{X}_1\mathbf{A}\mathbf{A}'\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\mathbf{B}\mathbf{B}'\mathbf{M}_1\mathbf{V}_0$, *for some* $\mathbf{A}$ *and* $\mathbf{B}$.
(d) $\mathbf{V} = \mathbf{V}_0 + \mathbf{X}_1\mathbf{C}\mathbf{C}'\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\mathbf{D}\mathbf{D}'\mathbf{M}_1\mathbf{V}_0$, *for some* $\mathbf{C}$ *and* $\mathbf{D}$.
(e) $\mathbf{V} = \mathbf{V}_0 + \mathbf{X}_1\boldsymbol{\Lambda}\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\boldsymbol{\Delta}\mathbf{M}_1\mathbf{V}_0$, *for some matrices* $\boldsymbol{\Lambda}$ *and* $\boldsymbol{\Delta}$ *such that* $\mathbf{V}$ *is nonnegative definite.*
(f) $\mathbf{V} = \mathbf{X}_1\boldsymbol{\Sigma}\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\boldsymbol{\Gamma}\mathbf{M}_1\mathbf{V}_0$, *for some matrices* $\boldsymbol{\Sigma}$ *and* $\boldsymbol{\Gamma}$ *such that* $\mathbf{V}$ *is nonnegative definite.*

Later we shall utilise the representation (c) which is somewhat simpler than those in (d), (e) and (f). For the proof of Lemma 1.3 and related discussion, see, e.g., Mitra and Moore [8, Th. 4.1–4.2], Rao [11, Lemma 5], [12, Th. 5.2, Th. 5.5], [13, p. 289], and Baksalary and Mathew [1, Th. 3].

Let us next consider the estimation of $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ in the partitioned model; we of course assume that the disjointness (1) holds so that $\boldsymbol{\mu}_1$ is estimable. Let $\mathscr{V}_{1/12}$ denote the set of nonnegative definite matrices $\mathbf{V}$ such that every representation of the BLUE of $\boldsymbol{\mu}_1$ under $\mathscr{M}_{12}(\mathbf{V}_0)$ remains BLUE under $\mathscr{M}_{12}(\mathbf{V})$, i.e.,

$$\mathbf{V} \in \mathscr{V}_{1/12} \iff \mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_{12}(\mathbf{V}_0)) \subseteq \mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_{12}(\mathbf{V}))\,.$$

In view of Haslett and Puntanen [3, Th. 2.1], see also Mathew and Bhimasankaram [7, Th. 2.4], the following holds:

Lemma 1.4. *Consider the partitioned linear models*

$$\mathscr{M}_{12}(\mathbf{V}_0) = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}_0\} \text{ and } \mathscr{M}_{12}(\mathbf{V}) = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}\}\,,$$

where $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is estimable. Then the following statements are equivalent:

(a) $\mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_1(\mathbf{V}_0)) \subseteq \mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_1(\mathbf{V}))$, *i.e.,* $\mathbf{V} \in \mathscr{V}_{1/12}$.
(b) $\mathscr{C}(\mathbf{M}_2\mathbf{V}\mathbf{M}) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{V}_0\mathbf{M})$.
(c) $\mathscr{C}(\mathbf{V}\mathbf{M}) \subseteq \mathscr{C}(\mathbf{X}_2 : \mathbf{V}_0\mathbf{M})$.

In this paper we focus on the mutual relations between the sets $\mathscr{V}_1$ and $\mathscr{V}_{1/12}$.

Remark 1.1. Theorem 2.4 in the paper of Mathew and Bhimasankaram [7] says the following: Every linear representation of the BLUE of estimable parametric function $\mathbf{K}\boldsymbol{\beta}$ under $\mathscr{M}_{12}(\mathbf{V}_0)$ continues to be its BLUE under $\mathscr{M}_{12}(\mathbf{V})$ if and only if

$$\mathscr{C}(\mathbf{V}\mathbf{M}) \subseteq \mathscr{C}(\mathbf{X}_0 : \mathbf{V}_0\mathbf{M})\,, \tag{11}$$

where $\mathbf{X}_0 = \mathbf{X}(\mathbf{I}_p - \mathbf{K}^-\mathbf{K})$. Thus using $\mathbf{K}^+$, we have $\mathbf{X}_0 = \mathbf{X}(\mathbf{I}_p - \mathbf{P}_{\mathbf{K}'}) = \mathbf{X}\mathbf{Q}_{\mathbf{K}'}$. In particular, if $\mathbf{K} = (\mathbf{X}_1 : \mathbf{0})$, then $\mathbf{K}\boldsymbol{\beta} = \boldsymbol{\mu}_1$ and

$$\mathbf{Q}_{\mathbf{K}'} = \mathbf{I}_p - \mathbf{P}_{\mathbf{K}'} = \begin{pmatrix} \mathbf{I}_{p_1} - \mathbf{P}_{\mathbf{X}_1'} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p_2} \end{pmatrix},$$

and so $\mathbf{X}_0 = (\mathbf{X}_1 : \mathbf{X}_2)\mathbf{Q}_{\mathbf{K}'} = (\mathbf{0} : \mathbf{X}_2)$. This confirms the equivalence of (c) of Lemma 1.4 and (11) when $\mathbf{K}\boldsymbol{\beta} = \boldsymbol{\mu}_1$. □

Remark 1.2. If $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M})$, then by the equivalence of (a) and (b) of Lemma 1.4,

$$\mathscr{C}(\mathbf{M}_2\mathbf{V}\mathbf{M}) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{V}_0\mathbf{M}) \iff \mathscr{C}(\mathbf{V}\mathbf{M}) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M})\,,$$

which further means an interesting relation:

$$\text{If } \mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}) \text{ then } \quad \mathbf{V} \in \mathscr{V}_{1/12} \iff \mathbf{V} \in \mathscr{V}_{12}\,,$$

where $\mathscr{V}_{12}$ refers to the set of $\mathbf{V}$ satisfying

$$\mathcal{B}(\boldsymbol{\mu} \,|\, \mathscr{M}_{12}(\mathbf{V}_0)) \subseteq \mathcal{B}(\boldsymbol{\mu} \,|\, \mathscr{M}_{12}(\mathbf{V}))\,,$$

i.e.,

$$\mathbf{V} \in \mathscr{V}_{12} \iff \mathbf{V} = \mathbf{X}\mathbf{C}\mathbf{C}'\mathbf{X}' + \mathbf{V}_0\mathbf{M}\mathbf{D}\mathbf{D}'\mathbf{M}\mathbf{V}_0 \quad \text{for some } \mathbf{C} \text{ and } \mathbf{D}.$$

The relations between the sets $\mathscr{V}_1$ and $\mathscr{V}_{12}$ were studied by Haslett et al. [2] and Haslett and Puntanen [4]. □

Remark 1.3. Before leaving this section, let us consider the model

$$\mathscr{M}_{12}(\mathbf{W}) = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{W}\},$$

where $\mathbf{W}$ is defined as in (6), i.e.,

$$\mathbf{W} = \mathbf{V}_0 + \mathbf{XUU}'\mathbf{X}' \quad \text{such that } \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V}_0). \tag{12}$$

Now it is clear that

$$\mathbf{G}(\mathbf{X} : \mathbf{V}_0\mathbf{M}) = (\mathbf{X} : \mathbf{0}) \iff \mathbf{G}(\mathbf{X} : \mathbf{WM}) = (\mathbf{X} : \mathbf{0})$$

so that the BLUE of $\boldsymbol{\mu}$ under $\mathscr{M}_{12}(\mathbf{V}_0)$ remains BLUE under $\mathscr{M}_{12}(\mathbf{W})$, i.e., $\mathbf{W} \in \mathscr{V}_{12}$. By (12) the model $\mathscr{M}_{12}(\mathbf{W})$ is a weakly singular linear model since $\mathscr{C}(\mathbf{X}) \subseteq \mathscr{C}(\mathbf{W})$. The weakly singular linear model $\mathscr{M}_{12}(\mathbf{W})$ has such a property, cf. (5) regarding a weakly singular $\mathscr{M}_1(\mathbf{V}_0)$, that the BLUE of $\boldsymbol{\mu}$ can be expressed as

$$\mathrm{BLUE}(\boldsymbol{\mu} \mid \mathscr{M}_{12}(\mathbf{W})) = \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\mathbf{y},$$

which of course is one representation for the BLUE of $\boldsymbol{\mu}$ under $\mathscr{M}_{12}(\mathbf{V}_0)$; see the corresponding representation (4) for the BLUE of $\boldsymbol{\mu}_1$ in $\mathscr{M}_1(\mathbf{V}_0)$.

For weakly singular linear models, see, e.g., Mitra and Rao [9] and Zyskind and Martin [16]. □

2. Main results

Haslett et al. [2] showed the following result (Theorem 4.1 therein):

Lemma 2.1. *The matrix* $\mathbf{V}$ *belongs to* $\mathscr{V}_{1/12}$ *if and only if it can be expressed as*

$$\begin{aligned}\mathbf{V} &= \mathbf{X}_1\mathbf{L}_{11}\mathbf{X}_1' + \mathbf{X}_2\mathbf{L}_{22}\mathbf{X}_2' + \mathbf{V}_0\mathbf{M}\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0 + \mathbf{Z} + \mathbf{Z}' \\ &= \mathbf{XLL}'\mathbf{X}' + \mathbf{V}_0\mathbf{M}\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0 + \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0 + \mathbf{V}_0\mathbf{M}\mathbf{L}_{32}\mathbf{X}_2'\end{aligned} \tag{13}$$

for some $\mathbf{L}' = (\mathbf{L}_1' : \mathbf{L}_2')$, $\mathbf{L}_3$, $\mathbf{L}_{ij} = \mathbf{L}_i\mathbf{L}_j'$, $i, j = 1, 2, 3,$ *and*

$$\mathbf{Z} = \mathbf{X}_1\mathbf{L}_{12}\mathbf{X}_2' + \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0\,.$$

Remark 2.1. According to Mathew and Bhimasankaram [7, Th. 2.4], every representation of the BLUE of estimable parametric function $\mathbf{K}\boldsymbol{\beta}$ under $\mathscr{M}_{12}(\mathbf{V}_0)$ continues to be its BLUE under $\mathscr{M}_{12}(\mathbf{V})$ if and only if $\mathbf{V}$ can be expressed as

$$\mathbf{V} = \mathbf{X}\mathbf{U}_1\mathbf{X}' + \mathbf{V}_0\mathbf{M}\mathbf{U}_2\mathbf{M}\mathbf{V}_0 + \mathbf{X}_0\mathbf{U}_3\mathbf{M}\mathbf{V}_0 + \mathbf{V}_0\mathbf{M}\mathbf{U}_3'\mathbf{X}_0'\,, \tag{14}$$

where $\mathbf{X}_0$ is defined as in Remark 1.1, i.e., $\mathbf{X}_0 = \mathbf{X}\mathbf{Q}_{\mathbf{K}'}$, and $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3$ are arbitrary subject to the condition that $\mathbf{V}$ is nonnegative definite. Noting that if $\mathbf{K} = (\mathbf{X}_1 : \mathbf{0})$, we have $\mathbf{X}_0 = (\mathbf{0} : \mathbf{X}_2)$, and (13) is essentially the same as (14), the representation (13) yielding "automatically" to a nonnegative definite $\mathbf{V}$. □

Let us next study when does an arbitrary $\mathbf{V} \in \mathscr{V}_{1/12}$ belong also to $\mathscr{V}_1$. In other words, if we have an arbitrary $\mathbf{V}$ of the form (13), what is needed that $\mathbf{V}$ can be expressed in the form

$$\mathbf{V} = \mathbf{X}_1\mathbf{A}\mathbf{A}'\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\mathbf{B}\mathbf{B}'\mathbf{M}_1\mathbf{V}_0$$

for some $\mathbf{A}$ and $\mathbf{B}$? Shortly said, we want to find a necessary and sufficient condition for the inclusion $\mathscr{V}_{1/12} \subseteq \mathscr{V}_1$. We can also express our aim so that we want to characterize the set of nonnegative definite matrices $\mathbf{V}$ which satisfy the following implication:

$$\mathscr{C}(\mathbf{M}_2\mathbf{V}\mathbf{M}) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{V}_0\mathbf{M}) \implies \mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,.$$

Let $\mathbf{V} \in \mathscr{V}_{1/12}$ be of the form (13). Then $\mathbf{V} \in \mathscr{V}_1$ if and only if $\mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)$, i.e.,

$$\begin{aligned}\mathscr{C}(\mathbf{V}\mathbf{M}_1) &= \mathscr{C}\big[(\mathbf{X}\mathbf{L}\mathbf{L}'\mathbf{X}' + \mathbf{V}_0\mathbf{M}\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0 + \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0 + \mathbf{V}_0\mathbf{M}\mathbf{L}_{32}\mathbf{X}_2')\mathbf{M}_1\big]\\ &= \mathscr{C}\big[(\mathbf{X}\mathbf{L}\mathbf{L}'\mathbf{X}' + \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0)\mathbf{M}_1 + \mathbf{V}_0\mathbf{M}\mathbf{R}\big]\\ &\subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,, \end{aligned} \tag{15}$$

where $\mathbf{R} = (\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0 + \mathbf{L}_{32}\mathbf{X}_2')\mathbf{M}_1$. Now in light of

$$\mathscr{C}(\mathbf{V}_0\mathbf{M}) = \mathscr{C}(\mathbf{V}_0\mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2}) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,,$$

the inclusion (15) holds if and only if

$$\mathscr{C}\big[(\mathbf{X}\mathbf{L}\mathbf{L}'\mathbf{X}' + \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0)\mathbf{M}_1\big] \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,,$$

i.e.,

$$\mathscr{C}\big[(\mathbf{X}_1\mathbf{L}_{12}\mathbf{X}_2' + \mathbf{X}_2\mathbf{L}_{22}\mathbf{X}_2' + \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0)\mathbf{M}_1\big] \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,.$$

Thus we have proved the following.

Theorem 2.1. *The inclusion $\mathscr{V}_{1/12} \subseteq \mathscr{V}_1$, i.e., the implication*

$$\mathscr{C}(\mathbf{M}_2\mathbf{V}\mathbf{M}) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{V}_0\mathbf{M}) \implies \mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,,$$

holds if and only if the matrix $\mathbf{V} \in \mathscr{V}_{1/12}$ can be expressed as

$$\begin{aligned}\mathbf{V} &= \mathbf{X}_1\mathbf{L}_{11}\mathbf{X}_1' + \mathbf{X}_2\mathbf{L}_{22}\mathbf{X}_2' + \mathbf{V}_0\mathbf{M}\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0 + \mathbf{Z} + \mathbf{Z}'\\ &= \mathbf{X}\mathbf{L}\mathbf{L}'\mathbf{X}' + \mathbf{V}_0\mathbf{M}\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0 + \mathbf{F} + \mathbf{F}'\,, \end{aligned} \tag{16}$$

with

$$\mathbf{Z} = \mathbf{X}_1\mathbf{L}_{12}\mathbf{X}_2' + \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0\,, \quad \mathbf{F} = \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0\,,$$

for some $\mathbf{L}' = (\mathbf{L}_1' : \mathbf{L}_2')$, $\mathbf{L}_3$, $\mathbf{L}_{ij} = \mathbf{L}_i\mathbf{L}_j'$, $i,j = 1,2,3$, where

$$\mathscr{C}\big[(\mathbf{X}_1\mathbf{L}_{12}\mathbf{X}_2' + \mathbf{X}_2\mathbf{L}_{22}\mathbf{X}_2' + \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0)\mathbf{M}_1\big] \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,. \tag{17}$$

Consider then the special case when

$$\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,. \tag{18}$$

Then, under (18), obviously

$$\mathscr{C}\big[(\mathbf{X}_2\mathbf{L}_{22}\mathbf{X}_2' + \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0)\mathbf{M}_1\big] \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,,$$

and thereby the inclusion (17) holds if and only if

$$\mathscr{C}(\mathbf{X}_1\mathbf{L}_{12}\mathbf{X}_2'\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,,$$

which, in view of the disjointness of $\mathscr{C}(\mathbf{X}_1)$ and $\mathscr{C}(\mathbf{V}_0\mathbf{M}_1)$, holds if and only if $\mathbf{X}_1\mathbf{L}_{12}\mathbf{X}_2'\mathbf{M}_1 = \mathbf{0}$ which further, by the disjointness of $\mathscr{C}(\mathbf{X}_1)$ and $\mathscr{C}(\mathbf{X}_2)$, is equivalent to

$$\mathbf{X}_1\mathbf{L}_{12}\mathbf{X}_2' = \mathbf{0}\,. \tag{19}$$

In light of (19) the expression (16) becomes as (20) in the following theorem.

Theorem 2.2. *If $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)$, then $\mathscr{V}_{1/12} \subseteq \mathscr{V}_1$ if and only if matrix $\mathbf{V} \in \mathscr{V}_{1/12}$ can be expressed in the form*

$$\mathbf{V} = \mathbf{X}_1\mathbf{L}_{11}\mathbf{X}_1' + \mathbf{X}_2\mathbf{L}_{22}\mathbf{X}_2' + \mathbf{V}_0\mathbf{M}\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0 + \mathbf{F} + \mathbf{F}'\,, \tag{20}$$

where $\mathbf{F} = \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0$, for some $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$, $\mathbf{L}_{ij} = \mathbf{L}_i\mathbf{L}_j'$.

What about the reverse inclusion $\mathscr{V}_1 \subseteq \mathscr{V}_{1/12}$? Take an arbitrary $\mathbf{V} \in \mathscr{V}_1$ so that

$$\mathbf{V} = \mathbf{X}_1\mathbf{A}\mathbf{A}'\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\mathbf{B}\mathbf{B}'\mathbf{M}_1\mathbf{V}_0$$

for some $\mathbf{A}$ and $\mathbf{B}$. Now $\mathbf{V} \in \mathscr{V}_{1/12}$ if and only if $\mathscr{C}(\mathbf{M}_2\mathbf{V}\mathbf{M}) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{V}_0\mathbf{M})$, which obviously holds if and only if

$$\mathscr{C}(\mathbf{M}_2\mathbf{V}_0\mathbf{M}_1\mathbf{B}\mathbf{B}'\mathbf{M}_1\mathbf{V}_0\mathbf{M}) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{V}_0\mathbf{M})\,. \tag{21}$$

Thus we have the following.

Theorem 2.3. *The inclusion $\mathscr{V}_1 \subseteq \mathscr{V}_{1/12}$, i.e., the implication*

$$\mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1) \implies \mathscr{C}(\mathbf{M}_2\mathbf{V}\mathbf{M}) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{V}_0\mathbf{M})\,,$$

holds if and only if an arbitrary $\mathbf{V} \in \mathscr{V}_1$ can be expressed as

$$\mathbf{V} = \mathbf{X}_1\mathbf{A}\mathbf{A}'\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\mathbf{B}\mathbf{B}'\mathbf{M}_1\mathbf{V}_0\,,$$

for some $\mathbf{A}$ and $\mathbf{B}$, where (21) holds.

What about the equality $\mathscr{V}_1 = \mathscr{V}_{1/12}$? Consider such a situation when

$$\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,, \quad \text{i.e.,} \quad \mathbf{X}_2 = \mathbf{V}_0\mathbf{M}_1\mathbf{S} \;\text{ for some } \mathbf{S} \in \mathbb{R}^{n\times p_2}. \tag{22}$$

In other words, we should combine Theorems 2.2 and 2.3, i.e., our request is that the following two expressions are both holding:

(i) $\mathbf{V} = \mathbf{X}_1\mathbf{L}_{11}\mathbf{X}_1' + \mathbf{X}_2\mathbf{L}_{22}\mathbf{X}_2' + \mathbf{V}_0\mathbf{M}\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0 + \mathbf{F} + \mathbf{F}'$,
where $\mathbf{F} = \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0$,

(ii) $\mathbf{V} = \mathbf{X}_1\mathbf{A}\mathbf{A}'\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\mathbf{B}\mathbf{B}'\mathbf{M}_1\mathbf{V}_0$, subject to (21).

Thus our aim is to characterize the set of matrices $\mathbf{V}$ belonging to $\mathscr{V}_1 \cap \mathscr{V}_{1/12}$.

To clarify the situation: for any choice of $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$, the matrix $\mathbf{V}$ in (i) should be possible to express as in (ii) for some $\mathbf{A}$ and $\mathbf{B}$; correspondingly, as a reverse relation, for any choice of $\mathbf{A}$ and $\mathbf{B}$ the matrix $\mathbf{V}$ in (ii) should be possible to express as in (i) for some $\mathbf{L}_1, \mathbf{L}_2$ and $\mathbf{L}_3$.

In the comparison described above, it is clear that the matrix $\mathbf{X}_2$ may be causing problems: it has no role in (ii) at all. To overcome this problem we have made the assumption (22).

Let us denote $\mathbf{V}$ in (i) as

$$\mathbf{V} = \mathbf{X}_1\mathbf{L}_{11}\mathbf{X}_1' + \mathbf{V}_* \,.$$

Then using (22) and denoting $\mathbf{M} = \mathbf{M}_1\mathbf{Q}$, where $\mathbf{Q} = \mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2}$, the matrix $\mathbf{V}_*$ can be expressed as

$$\begin{aligned}
\mathbf{V}_* &= \mathbf{X}_2\mathbf{L}_{22}\mathbf{X}_2' + \mathbf{V}_0\mathbf{M}\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0 + \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0 + \mathbf{V}_0\mathbf{M}\mathbf{L}_{32}\mathbf{X}_2' \\
&= \mathbf{V}_0\mathbf{M}_1(\mathbf{S}\mathbf{L}_{22}\mathbf{S}' + \mathbf{Q}\mathbf{L}_{33}\mathbf{Q})\mathbf{M}_1\mathbf{V}_0 + \mathbf{V}_0\mathbf{M}_1(\mathbf{S}\mathbf{L}_{23}\mathbf{Q} + \mathbf{Q}\mathbf{L}_{32}\mathbf{S}')\mathbf{M}_1\mathbf{V}_0 \\
&= \mathbf{V}_0\mathbf{M}_1(\mathbf{S}\mathbf{L}_{22}\mathbf{S}' + \mathbf{Q}\mathbf{L}_{33}\mathbf{Q} + \mathbf{S}\mathbf{L}_{23}\mathbf{Q} + \mathbf{Q}\mathbf{L}_{32}\mathbf{S}')\mathbf{M}_1\mathbf{V}_0 \\
&= \mathbf{V}_0\mathbf{M}_1\mathbf{T}\mathbf{M}_1\mathbf{V}_0 \,, \qquad (23)
\end{aligned}$$

where $\mathbf{T} = (\mathbf{S}\mathbf{L}_2 + \mathbf{Q}\mathbf{L}_3)(\mathbf{S}\mathbf{L}_2 + \mathbf{Q}\mathbf{L}_3)'$. Thus by (23) the matrix $\mathbf{V}$ given in (i) can be expressed as in (ii) but we still need to check that the condition corresponding to (21) is holding. This is indeed true since

$$\mathscr{C}(\mathbf{M}_2\mathbf{V}_*\mathbf{M}) = \mathscr{C}(\mathbf{M}_2\mathbf{V}_0\mathbf{M}\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0\mathbf{M}) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{V}_0\mathbf{M}) \,.$$

As a matter of fact, the development in (23) is not necessary as we know that the matrix $\mathbf{V}$ in (i) belongs to $\mathscr{V}_1$, i.e., $\mathbf{V}$ has a representation of the type $\mathbf{V} = \mathbf{X}_1\mathbf{A}\mathbf{A}'\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\mathbf{B}\mathbf{B}'\mathbf{M}_1\mathbf{V}_0$ for some $\mathbf{A}$ and $\mathbf{B}$. However, we find it instructive to go through (23).

Thus we can conclude that the set of matrices $\mathbf{V}$ satisfying both (i) and (ii) is the set defined by (i) and so the following holds.

Theorem 2.4. *Suppose that $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)$. Then $\mathscr{V}_{1/12} = \mathscr{V}_1$ holds, i.e., $\mathbf{V} \in \mathscr{V}_{1/12} \cap \mathscr{V}_1$ if and only if $\mathbf{V}$ is of the form*

$$\mathbf{V} = \mathbf{X}_1\mathbf{L}_{11}\mathbf{X}_1' + \mathbf{X}_2\mathbf{L}_{22}\mathbf{X}_2' + \mathbf{V}_0\mathbf{M}\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0 + \mathbf{F} + \mathbf{F}' \,, \qquad (24)$$

where $\mathbf{F} = \mathbf{X}_2\mathbf{L}_{23}\mathbf{M}\mathbf{V}_0$, for some $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$, $\mathbf{L}_{ij} = \mathbf{L}_i\mathbf{L}_j'$. In other words, $\mathscr{V}_{1/12} \subseteq \mathscr{V}_1 \implies \mathscr{V}_{1/12} = \mathscr{V}_1$.

Remark 2.2. What happens if $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)$ is not holding? For example, is it then possible that

$$\mathscr{V}_1 \cap \mathscr{V}_{1/12} = \{\emptyset\}?$$

Putting $\mathbf{L}_2 = \mathbf{0}$ in (16) yields the following representation:

$$\mathbf{V}_a = \mathbf{X}_1\mathbf{L}_{11}\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}\mathbf{L}_{33}\mathbf{M}\mathbf{V}_0 \in \mathscr{V}_{1/12}\,.$$

Now it is clear that $\mathbf{V}_a$ belongs also to $\mathscr{V}_1$, so that $\mathscr{V}_1 \cap \mathscr{V}_{1/12} \neq \{\emptyset\}$.

Let $\mathbf{U}$ be a matrix satisfying

$$\mathscr{C}(\mathbf{U}) = \mathscr{C}(\mathbf{X}_2) \cap \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,.$$

Then there exist $\mathbf{L}_2$ and $\mathbf{H}$ such that

$$\mathscr{C}(\mathbf{X}_2\mathbf{L}_2) = \mathscr{C}(\mathbf{U}) = \mathscr{C}(\mathbf{V}_0\mathbf{M}_1\mathbf{H})$$

and thereby there exists $\mathbf{J}$ so that

$$\mathbf{X}_2\mathbf{L}_2 = \mathbf{V}_0\mathbf{M}_1\mathbf{H}\mathbf{J}\,.$$

It can be shown that substituting the above expression into (20) provides one expression for $\mathbf{V}$ which belongs to $\mathscr{V}_1 \cap \mathscr{V}_{1/12}$. □

3. Conclusions

In this article we consider the partitioned linear model

$$\mathscr{M}_{12}(\mathbf{V}_0) = \{\mathbf{y},\, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2,\, \mathbf{V}_0\}$$

and the corresponding small model

$$\mathscr{M}_1(\mathbf{V}_0) = \{\mathbf{y},\, \mathbf{X}_1\boldsymbol{\beta}_1,\, \mathbf{V}_0\}\,.$$

Following Rao [12, Sec. 5], we can characterize the sets $\mathscr{V}_1$, $\mathscr{V}_{1/12}$ and $\mathscr{V}_{12}$ of nonnegative definite matrices $\mathbf{V}$ so that

$$\begin{aligned}
\mathbf{V} \in \mathscr{V}_1 &\iff \mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_1(\mathbf{V}_0)) \subseteq \mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_1(\mathbf{V}))\,,\\
\mathbf{V} \in \mathscr{V}_{1/12} &\iff \mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_{12}(\mathbf{V}_0)) \subseteq \mathcal{B}(\boldsymbol{\mu}_1 \,|\, \mathscr{M}_{12}(\mathbf{V}))\,,\\
\mathbf{V} \in \mathscr{V}_{12} &\iff \mathcal{B}(\boldsymbol{\mu} \,|\, \mathscr{M}_{12}(\mathbf{V}_0)) \subseteq \mathcal{B}(\boldsymbol{\mu} \,|\, \mathscr{M}_{12}(\mathbf{V}))\,,
\end{aligned}$$

where $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$, $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$, and

$$\mathscr{M}_{12}(\mathbf{V}) = \{\mathbf{y},\, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2,\, \mathbf{V}\} \text{ and } \mathscr{M}_1(\mathbf{V}) = \{\mathbf{y},\, \mathbf{X}_1\boldsymbol{\beta}_1,\, \mathbf{V}\}\,.$$

The notation $\mathcal{B}(\boldsymbol{\eta} \,|\, \mathscr{A}) \subseteq \mathcal{B}(\boldsymbol{\eta} \,|\, \mathscr{B})$ means that every representation of the BLUE for parametric vector $\boldsymbol{\eta}$ under model $\mathscr{A}$ remains BLUE for $\boldsymbol{\eta}$ under $\mathscr{B}$.

It appears that $\mathbf{V}$ belongs to the class $\mathscr{V}_1$ if and only if

$$\mathscr{C}(\mathbf{V}\mathbf{M}_1) \subseteq \mathscr{C}(\mathbf{V}_0\mathbf{M}_1)\,,$$

which further holds if and only if $\mathbf{V}$ can be expressed in form

$$\mathbf{V} = \mathbf{X}_1\mathbf{A}\mathbf{A}'\mathbf{X}_1' + \mathbf{V}_0\mathbf{M}_1\mathbf{B}\mathbf{B}'\mathbf{M}_1\mathbf{V}_0\,,$$

for some matrices $\mathbf{A}$ and $\mathbf{B}$; above $\mathbf{M}_1 = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}_1}$. Corresponding characterizations can be done for $\mathscr{V}_{12}$ and $\mathscr{V}_{1/12}$. In this article we focus on characterizing the mutual relations between the sets $\mathscr{V}_1$ and $\mathscr{V}_{1/12}$. The relations between the sets $\mathscr{V}_1$ and $\mathscr{V}_{12}$ were studied by Haslett et al. [2] and Haslett and Puntanen [4].

We may complete this paper by mentioning briefly the special case when $\mathbf{V}_0 = \mathbf{I}$, the identity matrix of order n. This means that we have the model $\mathscr{M}_{12}(\mathbf{I}) = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{I}\}$ so that the BLUEs under $\mathscr{M}_{12}(\mathbf{I})$ are ordinary least squares estimators, OLSEs. For example,

$$\mathcal{B}(\boldsymbol{\mu} \mid \mathscr{M}_{12}(\mathbf{I})) = \big\{\mathbf{By} : \mathbf{B}(\mathbf{X} : \mathbf{M}) = (\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{0})\big\} = \big\{\mathbf{P_X y}\big\}\,,$$
$$\mathcal{B}(\boldsymbol{\mu}_1 \mid \mathscr{M}_{12}(\mathbf{I})) = \big\{\mathbf{Cy} : \mathbf{C}(\mathbf{X} : \mathbf{M}) = (\mathbf{X}_1 : \mathbf{0} : \mathbf{0})\big\} = \big\{\mathbf{P}_{\mathbf{X}_1 \cdot \mathbf{X}_2}\mathbf{y}\big\}\,,$$

where $\mathbf{P}_{\mathbf{X}_1 \cdot \mathbf{X}_2} = \mathbf{X}_1(\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-}\mathbf{X}_1'\mathbf{M}_2$. Thus the inclusion

$$\mathcal{B}(\boldsymbol{\mu} \mid \mathscr{M}_{12}(\mathbf{I})) \subseteq \mathcal{B}(\boldsymbol{\mu} \mid \mathscr{M}_{12}(\mathbf{V}))$$

can be interpreted as the equality $\mathrm{OLSE}(\boldsymbol{\mu}) = \mathrm{BLUE}(\boldsymbol{\mu})$ under $\mathscr{M}_{12}(\mathbf{V})$. For the extended review of this problem area, see Markiewicz et al. [5] and the references therein like Rao [10] and Zyskind [15].

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