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PH approximation of two-barrier ruin probability for Lévy risk having two-sided PH jumps

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Abstract. In this paper, we study a Lévy risk process consisting of Brownian component together with premiums and claims that are phase-type with many phases. Our aim is to approximate the probability of ruin without touching an upper barrier a. In line with this, the study demonstrates that the described Lévy risk process can essentially be replaced with a simpler risk process in which both premiums and claims are phase-type with just few phases.

1. Introduction

A real-valued stochastic process $\{R_t : t \geq 0\}$ is said to be a Lévy process if: (i) $R_0 = 0$ almost surely (a.s.), (ii) the increments are independent, i.e., for any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \infty$ the random variables $R_{t_2} - R_{t_1}, R_{t_3} - R_{t_2}, \cdots, R_{t_n} - R_{t_{n-1}}$ are independent, (iii) the increments are stationary, i.e., for any $0 < s < t$, the distribution of $R_t - R_s$ only depends on $t - s$, and (iv) it has D-paths or cádlág (French: "continue á droite, limite á gauche") trajectories. That is, paths are right continuous with left limits [2].

Assume that the reserve of an insurer with initial capital $u \in [b, a]$, where a and b are the upper and lower boundaries, respectively, and having both sided jumps, can be expressed by the following equation:

$$R_t = u + \sum_{i=1}^{N_t^1} p_i - \sum_{i=1}^{N_t^2} c_i + \mu t + \sigma W_t \quad \text{with} \quad R_0 = u, \tag{1}$$

where positive jumps $\{p_n\}_{n\geq 1}$ are a family of i.i.d. random variables having distribution F_p and occur at the epochs of the Poisson(λ_p) process N_t^1, also independent of N_t^1, and are of phase-type with representation $(\boldsymbol{\alpha}_p, \mathbf{T}_p)$;

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and negative jumps $\{c_n\}_{n\geq 1}$ are a family of i.i.d.random variables having distribution F_c and occur at the epochs of the Poisson(λ_c) process N_t^2, also independent of N_t^2, and are of phase-type with representation $(\boldsymbol{\alpha}_c, \mathbf{T}_c)$. The parameters μ and $\sigma > 0$ are respectively the drift and the variability of the Brownian motion, and W_t is a standard Brownian motion. The term $\mu t + \sigma W_t$ represents the other random fluctuations in the money flow of the company, for example number of clients may change etc. It is also assumed that all component processes in (1) are mutually independent. Then the process (1) satisfies all four conditions of a Lévy process.

If the distributions F_p and F_c of premiums and claims in (1) are PH with a big number of phases, then technically it is hard to handle the process of calculating the probability of up-crossing before down-crossing (ruin in case of $b = 0$) and vice-versa. To reduce the work load, we propose to replace the reserve process (1) with a modified reserve process below in such a way that the first four moments of aggregate premiums and aggregate claims of both processes (1) and (2) are correspondingly equal. The modified reserve process is defined by

$$\tilde{R}_t = u + \sum_{i=1}^{\tilde{N}_t^1} \tilde{p}_i - \sum_{i=1}^{\tilde{N}_t^2} \tilde{c}_i + \mu t + \sigma W_t, \tag{2}$$

where the notation is similar to (1) except the premiums and claims in (2) are PH with just two phases instead of many phases. For two phases PH distribution, we will use some well known PH distributions, like hyper-exponential, Coxian and Erlang.

The reason for considering only first four moments is that in order to represent two phases PH distribution, we need maximum four parameters. For example, in case of two phases hyper-exponential, we need two transition rate parameters, $\tilde{\mu}_{*1}$ and $\tilde{\mu}_{*2}$, Poisson intensity $\tilde{\lambda}_*$, and initial probabilities of the transient states, $\tilde{\alpha}_{*1}$ and $\tilde{\alpha}_{*2}$. However, since $\tilde{\alpha}_{*1} + \tilde{\alpha}_{*2} = 1$, it is enough to know only one of the initial probabilities. For two phases Coxian, we need two transition rate probabilities, Poisson intensity, and the transition probability from state 1 to state 2, i.e. four parameters in total. In case of Erlang, we need only two parameters: the transition intensity $\tilde{\mu}_*$ and the Poisson intensity $\tilde{\lambda}_*$.

Let us define the stopping times as follows:
$\tau_a = \inf\{t \geq 0 : R_t \geq a\}$, $\tau_b = \inf\{t \geq 0 : R_t \leq b\}$ and $\tau = \tau_a \wedge \tau_b$.

Let us consider a random walk defined by the risk process (1): $R_n = u + S_n$, where $S_n = \sum_{i=1}^{n}(R_i - R_{i-1})$. Then we can use the following theorem [6].

Theorem 1. *For a random walk on* $\mathbb{R}$*, there are only four possibilities, one of which has probability* 1*: (i)* $S_n = 0$ *for all* n*, (ii)* $S_n \to \infty$*, (iii)* $S_n \to -\infty$*, (iv)* $-\infty = \liminf S_n < \limsup S_n = \infty$.

We are not interested in case (i) but the other cases ensure that (if there are two boundaries) the process will almost surely attain either one or both boundaries, i.e. $\mathbb{P}_u(\tau < \infty) = 1$.

Throughout the paper we use $\mathbb{P}_u$ to denote the law of R_t such that $R_0 = u$, and $\mathbb{E}_u$ for corresponding expectation. To evaluate the probability of crossing the upper barrier a before crossing the lower barrier b, we will use an exact formula given by Asmussen & Albrecher ([3], p.353). By assuming $b = 0$, we can obtain the probability of up-crossing before ruin, and the probability of ruin before up-crossing.

A number of scholars have put forward various approaches to approximate the initial process by another, usually a simpler process. Several important ideas for approximation of ruin probabilities have been proposed already in case of classical risk process – the process (1) without Brownian component and without premium jumps. In his well-known work De Vylder [5] showed that matching first three moments it is possible to replace the initial process with a modified process with exponentially distributed claims, resulting in a surprisingly good approximation of the ruin probability. Burnecki et al. [4] proposed a generalized form of De Vylder approximation, using gamma distributed claims. In Grandell's papers [7] and [8] different approximations of ruin probabilities for the classical risk process are described in details. More recently, Mircea and Covrig [9], in addition to well-known approximation methods like De Vylder, Cramer-Lundburg, Beekman-Bowers, Grandell, Renyi, Tijms, Willmot, also studied diffusion approximations which can be obtained approximating the classical risk process by a Brownian motion with drift. In the same direction, Vatamidou et al. [11] studied evaluation of ruin probabilities for classical risk processes with heavy tailed claims by proposing approximation of the claim sizes with a phase-type distribution. Their goal was to investigate the number of phases required for achieving a prespecified accuracy for the ruin probability and to provide error bounds. Differently from the classical risk process, Stanford et al. [10] obtained recursive formulas of the ruin probability for non-Poisson risk processes where both claim sizes and the inter-claim revenue follow selected phase type distributions.

In this paper, our idea here is to use phase-type distributions (with a possibly small number of phases) in approximating ruin probabilities for much more general risk processes than classical ones, namely for Lévy risks processes (1). In the context of the two-barriers problem (which is our interest), there is an important difference which adds further difficulty in handling Lévy risk. Namely, in classical risk processes, where the premiums inflow is a deterministic linear process μt, the reserve attains the upper barrier a without overshoot ($R_{\tau_a} = a$), while in Lévy processes (1) the up-crossing can easily be the result of an up-jump with a random overshoot ($R_{\tau_a} > a$). However, assuming PH distributed claims and premiums, the situation is still analytically tractable.

This paper is arranged as follows:

- Section 2 includes some preliminaries on phase-type distributions, for example representation, density, moment generating function, moments, Lévy exponent, and the exact formula of probability of up-crossing before down-crossing;
- Section 3 describes the techniques of replacing the original reserve process where premiums and claims are PH with many phases with a modified process where premiums and claims are PH with just two phases;
- Section 4 puts forward a numerical example to justify and illustrate our methods. There is also a concluding part.

2. Preliminaries on the PH distribution

Here we bring some important definitions, corollaries and theorems related to the phase-type distribution. Some of those are repeated from Ali and Pärna [1]. Nevertheless, the terminology and notation are based on Asmussen & Albrecher [3].

2.1. Phase type distribution. Let $\{X_t\}_{t\geq 0}$ be a continuous time Markov chain with finitely many states denoted by $1, 2, \ldots, n, \Delta$. The state Δ is assumed to be absorbing, and all other states are transient. The transition probability matrix of X_t is denoted by $\mathbf{P}$, the i^{th} row being the conditional distribution of the next state given the current state i. Let $\mathbf{T}$ denote the transition intensity matrix for the states $1, \ldots, n$. Then the intensity matrix (transition rate matrix, infinitesimal generator) for the whole Markov chain can be written in block-partitioned form as

$$\left(\begin{array}{c|c} \mathbf{T} & \mathbf{t} \\ \hline \mathbf{0} & 0 \end{array}\right),$$

where

$$\mathbf{t} = -\mathbf{T}\mathbf{e}$$

and $\mathbf{e} = (1, 1, \cdots, 1)'$. The vector $\mathbf{t}$ represents the *exit rate vector* with its i-th component t_i being the intensity of leaving the state i for the absorbing state Δ.

Definition 1. The distribution of the absorption time in the Markov chain described above is called the phase type distribution.

Let $\boldsymbol{\alpha}$ be a row vector representing the initial distribution of states $1, 2, \ldots, n$. The couple $(\boldsymbol{\alpha}, \mathbf{T})$ is called the *representation* of the phase type distribution. The density of the phase type distribution can be written as

$$f(x) = \boldsymbol{\alpha} e^{\mathbf{T}x}\mathbf{t}, \quad x \geq 0, \tag{3}$$

and the moment generating function (m.g.f.) is

$$M(\gamma) = \boldsymbol{\alpha}(-\gamma \mathbf{I} - \mathbf{T})^{-1}\mathbf{t}, \quad \gamma \in \mathbb{C}, \tag{4}$$

where $\mathbf{I}$ is the identity matrix.

It is seen that the phase type distribution is a generalization of the exponential distribution which corresponds to the case $n = 1$. An important theoretical fact is that any probability distribution can arbitrarily well be approximated with a properly chosen PH distribution [3].

It is well-known that if Z_t is a compound Poisson process, $Z_t = \sum_{i=1}^{N_t} X_i$, where the Poisson process N_t has rate λ, and $\{X_n\}$ is a family of i.i.d. random variables which is independent from N_t, then for $\gamma \in \mathbb{C}$ the m.g.f of Z_t is given by

$$M_{Z_t}(\gamma) = e^{\lambda t (M_{X_1}(\gamma) - 1)}, \tag{5}$$

where $M_{X_1}(\gamma)$ is m.g.f. of X_1:

$$M_{X_1}(\gamma) = \mathbb{E}[e^{\gamma X_1}]. \tag{6}$$

Let the premiums and claims of the risk process (1) be of PH with representation $(\boldsymbol{\alpha}_p, \mathbf{T}_p)$ and $(\boldsymbol{\alpha}_c, \mathbf{T}_c)$ respectively. Then, for $\gamma \in \mathbb{C}$, by finding cumulant generating function (c.g.f) of the process (1), we obtain its Lévy exponent

$$\mathcal{K}(\gamma) = \gamma\mu + \frac{\gamma^2\sigma^2}{2} + \lambda_p(\boldsymbol{\alpha}_p(-\gamma \mathbf{I}_p - \mathbf{T}_p)^{-1}\mathbf{t}_p - 1) + \lambda_c(\boldsymbol{\alpha}_c(\gamma \mathbf{I}_c - \mathbf{T}_c)^{-1}\mathbf{t}_c - 1) \tag{7}$$

where $\mathbf{I}_p$ and $\mathbf{I}_c$ are identity matrices of the same order as $\mathbf{T}_p$ and $\mathbf{T}_c$, respectively.

2.2. Moments of a phase-type distribution. The following corollary can be derived from the definition of PH distribution [3].

Corollary 1. *The n^{th} moment of a phase-type distributed random variable X is given by*

$$\mathbb{E}[X^n] = (-1)^n n! \boldsymbol{\alpha} \mathbf{T}^{-n} \mathbf{e}, \tag{8}$$

where $\boldsymbol{\alpha}$ is the initial distribution, $\mathbf{T}$ is the transition rate matrix and $\mathbf{e}$ is the column vector of $1's$.

We now apply the formula (8) in case of premiums and claims both having two phase hyper-exponential distribution – one of the distributions we use later in our main application. Let their respective representations be $(\tilde{\boldsymbol{\alpha}}_p, \tilde{\mathbf{T}}_p)$ and $(\tilde{\boldsymbol{\alpha}}_c, \tilde{\mathbf{T}}_c)$, where

$$\tilde{\mathbf{T}}_p = \begin{pmatrix} -\tilde{\mu}_{p_1} & 0 \\ 0 & -\tilde{\mu}_{p_2} \end{pmatrix}, \quad \tilde{\boldsymbol{\alpha}}_p = \begin{pmatrix} \tilde{\alpha}_{p_1} & 1 - \tilde{\alpha}_{p_1} \end{pmatrix}, \quad \tilde{\mathbf{t}}_p = \begin{pmatrix} \tilde{\mu}_{p_1} \\ \tilde{\mu}_{p_2} \end{pmatrix}$$

and

$$\tilde{\mathbf{T}}_c = \begin{pmatrix} -\tilde{\mu}_{c_1} & 0 \\ 0 & -\tilde{\mu}_{c_2} \end{pmatrix}, \quad \tilde{\boldsymbol{\alpha}}_p = (\tilde{\alpha}_{c_1} \quad 1 - \tilde{\alpha}_{c_1}), \quad \tilde{\mathbf{t}}_c = \begin{pmatrix} \tilde{\mu}_{c_1} \\ \tilde{\mu}_{c_2} \end{pmatrix}.$$

Hence, using (8), we can calculate the moments of premiums:

$$\mathbb{E}(\tilde{p}) = (-1)1!\,(\tilde{\alpha}_{p_1} 1 - \tilde{\alpha}_{p_1}) \begin{pmatrix} -\tilde{\mu}_{p_1} & 0 \\ 0 & -\tilde{\mu}_{p_2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_1}} + \frac{1 - \tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_2}}$$

and

$$\mathbb{E}(\tilde{p}^2) = (-1)^2 2!\,(\tilde{\alpha}_{p_1} 1 - \tilde{\alpha}_{p_1}) \begin{pmatrix} -\tilde{\mu}_{p_1} & 0 \\ 0 & -\tilde{\mu}_{p_2} \end{pmatrix}^{-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2!\Big(\frac{\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_1}^2} + \frac{1 - \tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_2}^2}\Big).$$

The other necessary moments are obtained as follows:

$$\mathbb{E}(\tilde{p}^3) = 3!\Big(\frac{\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_1}^3} + \frac{1 - \tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_2}^3}\Big), \quad \mathbb{E}(\tilde{p}^4) = 4!\Big(\frac{\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_1}^4} + \frac{1 - \tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_2}^4}\Big).$$

In a similar way, we get the first 4 moments of claims:

$$\mathbb{E}(\tilde{c}) = \frac{\tilde{\alpha}_{c_1}}{\tilde{\mu}_{c_1}} + \frac{1 - \tilde{\alpha}_{c_1}}{\tilde{\mu}_{c_2}}, \qquad \mathbb{E}(\tilde{c}^2) = 2!\Big(\frac{\tilde{\alpha}_{c_1}}{\tilde{\mu}_{c_1}^2} + \frac{1 - \tilde{\alpha}_{c_1}}{\tilde{\mu}_{c_2}^2}\Big),$$

$$\mathbb{E}(\tilde{c}^3) = 3!\Big(\frac{\tilde{\alpha}_{c_1}}{\tilde{\mu}_{c_1}^3} + \frac{1 - \tilde{\alpha}_{c_1}}{\tilde{\mu}_{c_2}^3}\Big), \qquad \mathbb{E}(\tilde{c}^4) = 4!\Big(\frac{\tilde{\alpha}_{c_1}}{\tilde{\mu}_{c_1}^4} + \frac{1 - \tilde{\alpha}_{c_1}}{\tilde{\mu}_{c_2}^4}\Big).$$

In the following, in order to avoid tedious repetitions, if the same formula is valid for both premiums and claims – like the last group of formulas – it will be presented only once (the reader has just to replace symbol p by c to get the parallel formulas for claims).

2.3. Probability of up-crossing before ruin. The following theorem is taken into consideration from Asmussen & Albrecher ([3], p. 353). It provides the exact probability of up-crossing before down-crossing (if $b = 0$, then ruin). We denote by $\mathbf{e}_i$ the column vector of zeros except 1 at the position i.

Theorem 2. *Assume that there exist $n = n_p + n_c + 2$ distinct complex numbers γ_k such that $\mathcal{K}(\gamma_k) = 0, k = 1, 2, \cdots, n$. Define $\eta_0^p(\gamma) = \eta_0^c(\gamma) = 1$ and $\eta_i^p(\gamma) = \mathbf{e}_i'(-\gamma \mathbf{I}_p - \mathbf{T}_p)^{-1}\mathbf{t}_p$, $\eta_i^c(\gamma) = \mathbf{e}_i'(-\gamma \mathbf{I}_c - \mathbf{T}_c)^{-1}\mathbf{t}_c$, and denote by $\zeta_0^p, \zeta_1^p, \cdots, \zeta_{n_p}^p$, $\zeta_0^c, \zeta_1^c, \cdots, \zeta_{n_c}^c$ the solutions of the system of n linear equations*

$$e^{\gamma_k u} = e^{\gamma_k a} \sum_{i=0}^{n_p} \eta_i^p(\gamma_k)\zeta_i^p + e^{\gamma_k b} \sum_{i=0}^{n_c} \eta_i^c(\gamma_k)\zeta_i^c. \tag{9}$$

Then

$$\mathbb{P}_u[\tau_a < \tau_b] = \sum_{i=0}^{n_p} \zeta_i^p\ . \tag{10}$$

Comment: On the right hand side of the last formula, ζ_0^p is the probability that a is up-crossed before b is down-crossed and that the up-crossing results from the Brownian motion and not a jump; and $\zeta_i^p,\ i > 0$, is the probability that a is up-crossed by a jump being in phase i at the up-crossing.

3. Replacing the original process

3.1. The original risk process: an example. Let us assume that both the premiums and the claims of the Lévy risk reserve given in (1) are PH with many phases, possibly a mixture of Erlang distribution, hyper-exponential distribution, hypo-exponential distribution and Coxian distribution.

Let the phase diagram of premiums be as follows:

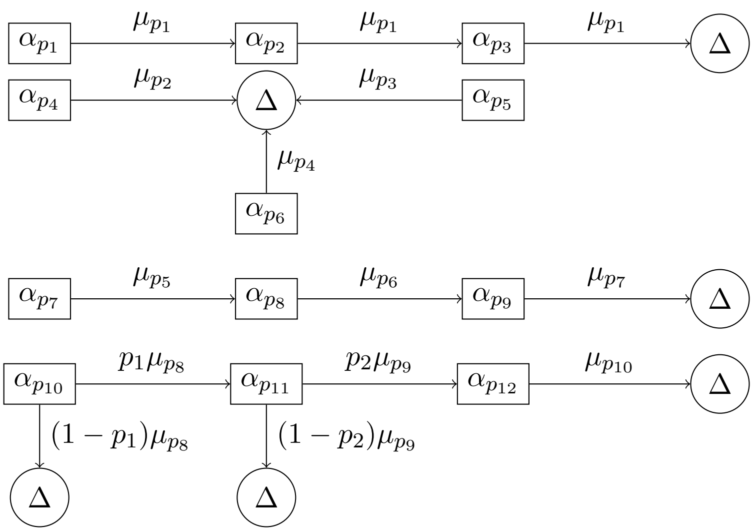


FIGURE 1. Phase-diagram of premium distribution

Corresponding intensity matrix, initial probabilities and the exit rate vector are:

$$\mathbf{T}_p = \begin{bmatrix} -\mu_{p_1} & \mu_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu_{p_1} & \mu_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_{p_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_{p_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu_{p_4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{p_5} & \mu_{p_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{p_6} & \mu_{p_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{p_7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{p_8} & p_1\mu_{p_8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{p_9} & p_2\mu_{p_9} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{p_{10}} \end{bmatrix},$$

$\boldsymbol{\alpha}_p = \big(\alpha_{p_1}, 0, 0, \alpha_{p_4}, \alpha_{p_5}, \alpha_{p_6}, \alpha_{p_7}, 0, 0, \alpha_{p_{10}} =, 0, 0\big),$
where $\alpha_{p_1} + \alpha_{p_4} + \alpha_{p_5} + \alpha_{p_6} + \alpha_{p_7} + \alpha_{p_{10}} = 1,$ and
$\mathbf{t}_p = -\mathbf{T}_p\mathbf{e} = \big(0, 0, \mu_{p_1}, \mu_{p_2}, \mu_{p_3}, \mu_{p_4}, 0, 0, \mu_{p_7}, (1-p_1)\mu_{p_8}, (1-p_2)\mu_{p_9}, \mu_{p_{10}}\big)'.$

At the same time, let the phase diagram of claims be as follows:

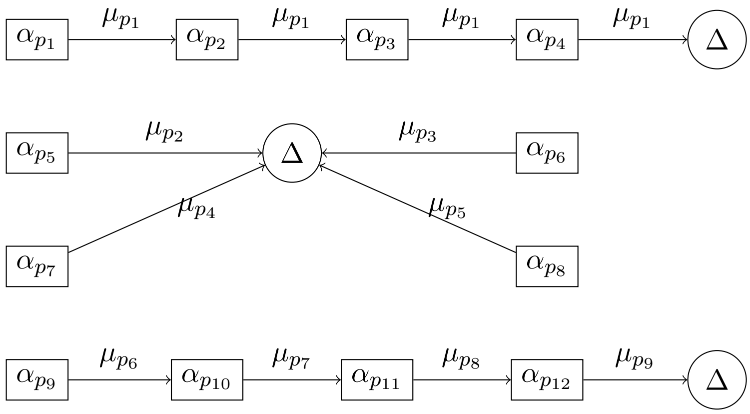


FIGURE 2. Phase-diagram of claim distribution

Corresponding intensity matrix, initial probabilities and the exit rate vector are:

$$\mathbf{T}_c = \begin{bmatrix} -\mu_{c_1} & \mu_{c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu_{c_1} & \mu_{c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu_{c_1} & \mu_{c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_{c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_{c_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu_{c_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{c_4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{c_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{c_6} & \mu_{c_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{c_7} & \mu_{c_7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{c_8} & \mu_{c_8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_{c_9} \end{bmatrix},$$

$$\boldsymbol{\alpha}_c = \big(\alpha_{c_1}, 0, 0, 0, \alpha_{c_5}, \alpha_{c_6}, \alpha_{c_7}, \alpha_{c_8}, \alpha_{c_9}, 0, 0, 0\big),$$
where $\alpha_{c_1} + \alpha_{c_5} + \alpha_{c_6} + \alpha_{c_7} + \alpha_{c_8} + \alpha_{c_9} = 1,$ and
$$\mathbf{t}_c = -\mathbf{T}_c\mathbf{e} = \big(0, 0, 0, \mu_{c_1}, \mu_{c_2}, \mu_{c_3}, \mu_{c_4}, \mu_{c_5}, 0, 0, 0, \mu_{c_9}\big)'.$$

For such a risk process, in principle, Theorem 2 applies directly, although in practice a special software is needed – in our case *Maxima* computer algebra system was used – to calculate the Lévy exponent $\mathcal{K}(\gamma)$ and its $n = n_p + n_c + 2 = 26$ roots, and solve the linear system of equations of order 26. Our idea is to replace the risk process (1) with a much simpler risk process (2), where both premiums and claims are of PH with only two phases, making the order of the system as small as $n = 6$. In doing this, we require that the first four moments of the aggregate premiums (claims) of the initial process (1) are equal to respective moments of the replaced process (2). Therefore we need the moments of aggregate premiums (claims) of the initial process (1). In fact, the derivation below is valid for an arbitrary compound Poisson process.

For $\gamma \in \mathbb{C}$, denoting the k^{th} moment of premiums by ξ_k^p, i.e. $\mathbb{E}[p_1^k] = \xi_k^p$, we can express the cumulant generating function of aggregate premiums, $P_t = \sum_{i=1}^{N_t^1} p_i$, as follows:

$$\begin{aligned}\log \mathbb{E}[e^{\gamma \sum_{i=1}^{N_t^1} p_i}] &= \lambda_p t \Big\{ M_{p_1}(\gamma) - 1] \Big\} \\ &= \lambda_p t \Big\{ \mathbb{E}[1 + p_1\gamma + \frac{p_1^2\gamma^2}{2!} + \frac{p_1^3\gamma^3}{3!} + \frac{p_1^4\gamma^4}{4!} + o(\gamma^4)] - 1 \Big\} \\ &= \lambda_p t \Big(\xi_1^p \gamma + \xi_2^p \frac{\gamma^2}{2!} + \xi_3^p \frac{\gamma^3}{3!} + \xi_4^p \frac{\gamma^4}{4!} + o(\gamma^4) \Big).\end{aligned}$$

Hence, the moments of aggregate premiums of the initial process (1) (according to the relations between moments and cumulants) are:

$$\begin{aligned}\mathbb{E}[P_t] &= \lambda_p t \xi_1^p \\ \mathbb{E}[P_t^2] &= \lambda_p t \xi_2^p + (\lambda_p t \xi_1^p)^2 \\ \mathbb{E}[P_t^3] &= \lambda_p t \xi_3^p + 3(\lambda_p t \xi_2^p)(\lambda_p t \xi_1^p) + (\lambda_p t \xi_1^p)^3 \\ \mathbb{E}[P_t^4] &= \lambda_p t \xi_4^p + 4(\lambda_p t \xi_3^p)(\lambda_p t \xi_1^p) + 3(\lambda_p t \xi_2^p)^2 + 6(\lambda_p t \xi_2^p)(\lambda_p t \xi_1^p)^2 + \\ &\quad (\lambda_p t \xi_1^p)^4.\end{aligned} \tag{11}$$

Similar formulas (with index p replaced by c) can be obtained for aggregated claims $C_t = \sum_{i=1}^{N_t^2} c_i$.

After finding the moments of aggregate premiums and claims of the replaced process (2), and equating the first four moments, one can express the parameters of the replaced process in terms of the moments of the initial process. In the next three subsections, we will do this step by step.

3.2. Parameters of replaced hyper-exponential distribution. Let us replace the PH premiums of the initial process (1) with a two phases hyper-exponential distribution with representation $(\tilde{\boldsymbol{\alpha}}_p, \tilde{\mathbf{T}}_p)$. The phase diagram, intensity matrix and exit rate vector of two phases hyper-exponential distribution are as follows:

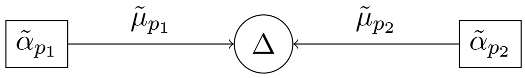


Figure 3. Phase-diagram of two phases hyper-exponential distribution

$$\tilde{\mathbf{T}}_p = \begin{bmatrix} -\tilde{\mu}_{p_1} & 0 \\ 0 & -\tilde{\mu}_{p_2} \end{bmatrix}, \quad \tilde{\mathbf{t}}_p = \begin{bmatrix} \tilde{\mu}_{p_1} \\ \tilde{\mu}_{p_2} \end{bmatrix}$$

with $\tilde{\alpha}_{p_1} + \tilde{\alpha}_{p_2} = 1$. After finding c.g.f. of the replaced aggregate premiums, $\tilde{P}_t = \sum_{i=1}^{\tilde{N}_t^1} \tilde{p}_i$ and using (8), the parameters of the replaced premiums, $(\tilde{\lambda}_p, \tilde{\alpha}_{p_1}, \tilde{\mu}_{p_1}, \tilde{\mu}_{p_2})$ must satisfy the following system of equations:

$$\begin{cases} \lambda_p \xi_1^p = \tilde{\lambda}_p \Big(\frac{\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_1}} + \frac{1-\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_2}} \Big), \quad \lambda_p \xi_2^p = 2! \tilde{\lambda}_p \Big(\frac{\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_1}^2} + \frac{1-\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_2}^2} \Big), \\ \lambda_p \xi_3^p = 3! \tilde{\lambda}_p \Big(\frac{\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_1}^3} + \frac{1-\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_2}^3} \Big), \quad \lambda_p \xi_4^p = 4! \tilde{\lambda}_p \Big(\frac{\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_1}^4} + \frac{1-\tilde{\alpha}_{p_1}}{\tilde{\mu}_{p_2}^4} \Big). \end{cases} \tag{12}$$

To solve the system above, we used the computer algebra system *Maxima* which resulted in:

$$\begin{cases} \tilde{\lambda}_p & = \frac{(6\lambda_p (\xi_1^p)^2 \xi_4^p - 24\lambda_p \xi_1^p \xi_2^p \xi_3^p + 18\lambda_p (\xi_2^p)^3)}{(3\xi_2^p \xi_4^p - 4(\xi_3^p)^2)} \\ \tilde{\alpha}_{p_1} & = \frac{N_1}{D_1} \\ \tilde{\mu}_{p_1} & = \frac{24\xi_1^p \xi_3^p - 36(\xi_2^p)^2}{3\xi_1^p \xi_4^p - 6\xi_2^p \xi_3^p + D_2} \\ \tilde{\mu}_{p_2} & = \frac{24\xi_1^p \xi_3^p - 36(\xi_2^p)^2}{3\xi_1^p \xi_4^p - 6\xi_2^p \xi_3^p - D_2}, \end{cases} \tag{13}$$

where

$$N_1 = \Big[9(\xi_1^p)^4 (\xi_4^p)^3 - \sqrt{3(\xi_1^p)^2 (\xi_4^p)^2 + (36(\xi_2^p)^3 - 36\xi_1^p \xi_2^p \xi_3^p)\xi_4^p + 32\xi_1^p (\xi_3^p)^3 - 36(\xi_2^p)^2 (\xi_3^p)^2} * \big\{ \big(3^{3/2} (\xi_1^p)^3 (\xi_4^p)^2 + (3^{7/2} \xi_1^p (\xi_2^p)^3 - 10 * 3^{3/2} (\xi_1^p)^2 \xi_2^p \xi_3^p)\xi_4^p + 16\sqrt{3} (\xi_1^p)^2 (\xi_3^p)^3 - 2 * 3^{5/2} (\xi_2^p)^4 \xi_3^p \big) \big\} - \big\{ 144(\xi_1^p)^3 \xi_2^p \xi_3^p - 135(\xi_1^p)^2 (\xi_2^p)^3 \big\} (\xi_4^p)^2 + \big\{ 96(\xi_1^p)^3 (\xi_3^p)^3 + 324(\xi_1^p)^2 (\xi_2^p)^2 (\xi_3^p)^2 - 756\xi_1^p (\xi_2^p)^4 \xi_3^p + 324(\xi_2^p)^6 \big\} \xi_4^p - 384(\xi_1^p)^2 \xi_2^p (\xi_3^p)^4 + 720\xi_1^p (\xi_2^p)^3 (\xi_3^p)^3 - 324(\xi_2^p)^5 (\xi_3^p)^2 \Big],$$

$$D_1 = \Big\{ (18(\xi_1^p)^4 (\xi_4^p)^3 + \big(270(\xi_1^p)^2 (\xi_2^p)^3 - 288(\xi_1^p)^3 \xi_2^p \xi_3^p \big)(\xi_4^p)^2 + \big(192(\xi_1^p)^3 (\xi_3^p)^3 + 648(\xi_1^p)^2 (\xi_2^p)^2 (\xi_3^p)^2 - 1512\xi_1^p (\xi_2^p)^4 \xi_3^p + 648(\xi_2^p)^6 \big)\xi_4^p - 768(\xi_1^p)^2 \xi_2^p (\xi_3^p)^4 + 1440\xi_1^p (\xi_2^p)^3 (\xi_3^p)^3 - 648(\xi_2^p)^5 (\xi_3^p)^2 \Big\},$$

$$D_2 = \sqrt{3}\sqrt{3(\xi_1^p)^2 (\xi_4^p)^2 + \big(36(\xi_2^p)^3 - 36\xi_1^p \xi_2^p \xi_3^p \big)\xi_4^p + 32\xi_1^p (\xi_3^p)^3 - 36(\xi_2^p)^2 (\xi_3^p)^2}.$$

A fairly identical procedure can be carried out on the claims side, taking to the system equivalent to (12) and solutions as in (13). All these formulas will later be used in the numerical example.

3.3. Parameters of a replaced Coxian distribution. Let us replace the premiums and claims of the initial process (1) with two phases Coxian distribution with representation $(\tilde{\boldsymbol{\alpha}}_*, \tilde{\mathbf{T}}_*)$. The phase diagram, intensity matrix and exit rate vector for two phases Coxian distribution are as follows:

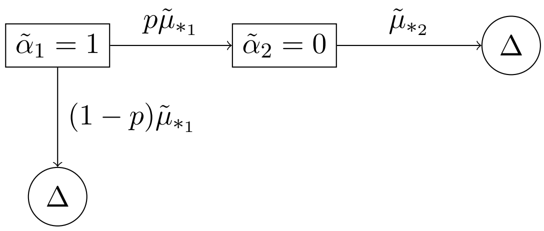


FIGURE 4. Phase-diagram of two phases Coxian distribution

$$\tilde{\mathbf{T}}_* = \begin{bmatrix} -\tilde{\mu}_{*1} & p\tilde{\mu}_{*1} \\ 0 & -\tilde{\mu}_{*2} \end{bmatrix}, \quad \tilde{\mathbf{t}}_* = \begin{bmatrix} (1-p)\tilde{\mu}_{*1} \\ \tilde{\mu}_{*2} \end{bmatrix}.$$

After finding c.g.f. of the replaced aggregate premiums, $\tilde{P}_t = \sum_{i=1}^{\tilde{N}_t^1} \tilde{p}_i$ and using for moments the notation of the replaced premiums given in (8), the parameters $(\tilde{\lambda}_p, p, \tilde{\mu}_{p1}, \tilde{\mu}_{p2})$ must satisfy the following system of equations:

$$\begin{cases} \lambda_p \xi_1^p &= \tilde{\lambda}_p \left(\frac{1}{\tilde{\mu}_{p_1}} + \frac{p}{\tilde{\mu}_{p_2}}\right), \\ \lambda_p \xi_2^p &= 2\tilde{\lambda}_p \left(\frac{1}{\tilde{\mu}_{p_1}^2} + p\left(\frac{1}{\tilde{\mu}_{p_2}^2} + \frac{1}{\tilde{\mu}_{p1}\tilde{\mu}_{p2}}\right)\right), \\ \lambda_p \xi_3^p &= 6\tilde{\lambda}_p \left(\frac{1}{\tilde{\mu}_{p_1}^3} + p\left(\frac{1}{\tilde{\mu}_{p_1}^2\tilde{\mu}_{p2}} + \frac{1}{\tilde{\mu}_{p1}\tilde{\mu}_{p_2}^2} + \frac{1}{\tilde{\mu}_{p_2}^3}\right)\right), \\ \lambda_p \xi_4^p &= 24\tilde{\lambda}_p \left(\frac{1}{\tilde{\mu}_{p_1}^4} + p\left(\frac{1}{\tilde{\mu}_{p_1}^3\tilde{\mu}_{p2}} + \frac{1}{\tilde{\mu}_{p_1}^2\tilde{\mu}_{p_2}^2} + \frac{1}{\tilde{\mu}_{p1}\tilde{\mu}_{p_2}^3} + \frac{1}{\tilde{\mu}_{p_2}^4}\right)\right). \end{cases} \tag{14}$$

As always, an identical system of equations can be obtained for claims. The analytical solutions of the above system have been obtained but, unfortunately, the expressions for $\tilde{\lambda}_p$, p, $\tilde{\mu}_{p_1}$, and $\tilde{\mu}_{p2}$ are too long to be presented here (several pages are needed). However, these formulas will be applied, and the results will be shown in our numerical example section.

3.4. Parameters of a replaced Erlang distribution. Let us replace the premiums and claims of the initial process (1) with the two phases Erlang distribution with representation $(\tilde{\boldsymbol{\alpha}}_*, \tilde{\mathbf{T}}_*)$. The phase diagram, intensity matrix, and exit rate vector for the two phases Erlang distribution are as follows:

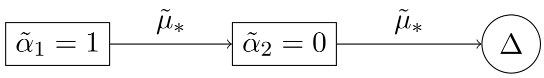


FIGURE 5. Phase-diagram of two phases Erlang distribution

$$\tilde{\mathbf{T}}_* = \begin{bmatrix} -\tilde{\mu}_* & \tilde{\mu}_* \\ 0 & -\tilde{\mu}_* \end{bmatrix}, \quad \tilde{\mathbf{t}}_* = \begin{bmatrix} 0 \\ \tilde{\mu}_* \end{bmatrix}$$

In case of Erlang approximation, we need to find only two parameters, the intensity of the exponential waiting time $\tilde{\mu}_p$, and the rate of the Poisson process, $\tilde{\lambda}_p$. Hence, we only need the first two moments of premiums (or claims) of the initial process. For premiums, according to (8), we have the following system of equations:

$$\begin{cases} \lambda_p \xi_1^p &= \frac{2\tilde{\lambda}_p}{\tilde{\mu}_p}, \\ \lambda_p \xi_2^p &= \frac{6\tilde{\lambda}_p}{\tilde{\mu}_p^2}. \end{cases}$$

After solving the above system we have:

$$\tilde{\lambda}_p = \frac{3\lambda_p (\xi_1^p)^2}{2\xi_2^p}, \quad \tilde{\mu}_p = \frac{3\lambda_p \xi_1^p}{\xi_2^p}. \tag{15}$$

Similar formulas are valid for claims. In the next section, we will use the formulas obtained for numerical calculation of Erlang parameters.

4. Numerical example

Here we apply the relationships obtained in the previous section to handle numerically the example of the risk process described in 3.1. Recall that in this example, both premiums and claims are of PH distribution with as many as 12 phases. Next a detailed numerical description of the process is given. The two barriers are set to $a = 5$ and $b = 0$ meaning that, when calculating probabilities of crossing barriers, the initial capital u can only be taken within the interval $[0, 5]$.

Let the transition rates of the premiums of the initial process be $\mu_{p_1} = 10, \mu_{p_2} = 7, \mu_{p_3} = 8, \mu_{p_4} = 6, \mu_{p_5} = 14, \mu_{p_6} = 3, \mu_{p_7} = 9, \mu_{p_8} = 13, \mu_{p_9} = 12, \mu_{p_{10}} = 11$, hence the exit rate vector for premiums is
$\mathbf{t}_p = \big(0, 0, 10, 7, 8, 6, 0, 0, 9, 6.5, 8, 11\big)'$. Let the initial probability vector be $\boldsymbol{\alpha}_p = (0.25, 0, 0, 0.1, 0.05, 0.15, 0.2, 0, 0, 0.25, 0, 0)$, and let the intensity of the Poisson process of premiums be $\lambda_p = 2.5$.

Similarly, let the transition rates of claims be $\mu_{c_1} = 17, \mu_{c_2} = 12, \mu_{c_3} = 13, \mu_{c_4} = 11, \mu_{c_5} = 18, \mu_{c_6} = 19, \mu_{c_7} = 16, \mu_{c_8} = 4, \mu_{c_9} = 20$, therefore the exit rate vector is $\mathbf{t}_c = \big(0, 0, 0, 17, 12, 13, 11, 18, 0, 0, 0, 20\big)$. Let the initial probability vector be $\boldsymbol{\alpha}_c = \big(0.4, 0, 0, 0, 0.05, 0.075, 0.1, 0.075, 0.3, 0, 0, 0\big)$, and let the intensity of the Poisson process of claims be $\lambda_c = 2$.

According to (7), Lévy exponent for the initial process is $\mathcal{K}(\gamma) = \frac{N_1}{D_1}$, where

$$\begin{aligned} N_1 = {} & 40\gamma^{26} + 2720.0\gamma^{25} + 9800.0\gamma^{24} - 3.0337 \times 10^6\gamma^{23} - 4.6665 \times 10^7\gamma^{22} \\ & + 1.5047 \times 10^9\gamma^{21} + 3.3016 \times 10^{10}\gamma^{20} - 4.4542 \times 10^{11}\gamma^{19} \\ & - 1.2354 \times 10^{13}\gamma^{18} + 9.121 \times 10^{13}\gamma^{17} + 2.9516 \times 10^{15}\gamma^{16} \\ & - 1.466 \times 10^{16}\gamma^{15} - 4.8046 \times 10^{17}\gamma^{14} + 2.0305 \times 10^{18}\gamma^{13} \\ & + 5.3956 \times 10^{19}\gamma^{12} - 2.3888 \times 10^{20}\gamma^{11} - 4.0542 \times 10^{21}\gamma^{10} \\ & + 2.1285 \times 10^{22}\gamma^{9} + 1.8553 \times 10^{23}\gamma^{8} - 1.2416 \times 10^{24}\gamma^{7} \\ & - 3.9428 \times 10^{24}\gamma^{6} + 3.9187 \times 10^{25}\gamma^{5} - 6.591 \times 10^{24}\gamma^{4} - 4.3881 \times 10^{26}\gamma^{3} \end{aligned}$$

$$+ 7.4586 \times 10^{26}\gamma^2 + 1.7411 \times 10^{26}\gamma$$

and

$$D_1 = 80\gamma^{24} + 5440.0\gamma^{23} + 2.032 \times 10^4\gamma^{22} - 6.0182 \times 10^6\gamma^{21} - 9.311 \times 10^7\gamma^{20}$$
$$+ 2.957 \times 10^9\gamma^{19} + 6.5177 \times 10^{10}\gamma^{18} - 8.659 \times 10^{11}\gamma^{17} - 2.4115 \times 10^{13}\gamma^{16}$$
$$+ 1.7536 \times 10^{14}\gamma^{15} + 5.6814 \times 10^{15}\gamma^{14} - 2.7936 \times 10^{16}\gamma^{13} -$$
$$- 9.0795 \times 10^{17}\gamma^{12} + 3.8391 \times 10^{18}\gamma^{11} + 9.9435 \times 10^{19}\gamma^{10}$$
$$- 4.4529 \times 10^{20}\gamma^9 - 7.2096 \times 10^{21}\gamma^8 + 3.862 \times 10^{22}\gamma^7 + 3.1201 \times 10^{23}\gamma^6$$
$$- 2.1565 \times 10^{24}\gamma^5 - 5.8615 \times 10^{24}\gamma^4 + 6.3312 \times 10^{25}\gamma^3 - 3.1088 \times 10^{25}\gamma^2$$
$$- 5.9555 \times 10^{26}\gamma + 1.0939 \times 10^{27}.$$

The roots of the Lévy exponent, found by means of Maxima software, are the following:

$$\gamma_1 = 0.0, \gamma_2 = -0.20813, \gamma_3 = 3.4367, \gamma_4 = -4.5359, \gamma_5 = 6.0828, \gamma_6 = 7.0399,$$
$$\gamma_7 = 8.0096, \gamma_8 = 10.901, \gamma_9 = 8.9903, \gamma_{10} = -11.021, \gamma_{11} = -12.007,$$
$$\gamma_{12} = 9.2545 + 2.3176i, \gamma_{13} = \bar{\gamma_{12}}, \gamma_{14} = -13.006, \gamma_{15} = -15.9985,$$
$$\gamma_{16} = -18.0000, \gamma_{17} = 13.4275, \gamma_{18} = 11.8296 + 0.86840i, \gamma_{19} = \bar{\gamma_{18}},$$
$$\gamma_{20} = -19.0946, \gamma_{21} = -13.9247 + 4.02944i, \gamma_{22} = \bar{\gamma_{21}}, \gamma_{23} = 14.0495,$$
$$\gamma_{24} = -19.7581, \gamma_{25} = -20.3134 + 2.8034i, \gamma_{26} = \bar{\gamma_{25}}$$

Now, using (10) for different values of initial capital u, we can calculate the exact probability of up-crossing before ruin. We will show the obtained results in the results subsection.

Next we replace the premiums and claims of the initial process with two phases premiums and claims. We will use, one after another, hyper-exponential, Coxian and Erlang distributions, and each time calculate probability of up-crossing before ruin.

4.1. Hyper-exponential approximation. If we replace the initial premiums and claims with two phases hyper-exponential premiums and claims, then, according to (13), the proper parameters for premiums are: $\tilde{\mu}_{p_1} = 2.456953$, $\tilde{\mu}_{p_2} = 4.4277$, $\tilde{\lambda}_p = 2.642963$, initial probabilities $\tilde{\boldsymbol{\alpha}}_p = (0.09596735, 0.90403265)$; and proper parameters for claims are: $\tilde{\mu}_{c_1} = 0.938091$, $\tilde{\mu}_{c_2} = 4.638382$, $\tilde{\lambda}_c = 2.242284$, initial probabilities $\tilde{\boldsymbol{\alpha}}_c = (0.0001477795, 0.9998522205)$. According to (7), the Lévy exponent of the replaced process is $\mathcal{K}(\gamma) = \frac{N}{D}$, where $N = 5.6446 \times 10^{54}\gamma^6 - 7.3842 \times 10^{54}\gamma^5 - 1.8589 \times 10^{56}\gamma^4 + 2.3641 \times 10^{56}\gamma^3 + 4.6583 \times 10^{56}\gamma^2 + 8.5053 \times 10^{55}\gamma - 2.6306 \times 10^{42}$, $D = 1.1289 \times 10^{55}\gamma^4 - 1.4768 \times 10^{55}\gamma^3 - 2.6148 \times 10^{56}\gamma^2 + 3.4667 \times 10^{56}\gamma + 5.3438 \times 10^{56}$. The roots of the Lévy exponents are $\gamma_1 = 3.0929 \times 10^{-14}, \gamma_2 = -0.20813, \gamma_3 = -0.93869, \gamma_4 = 2.5692, \gamma_5 = 5.4222$ and $\gamma_6 = -5.5364$. Now using (10), one can calculate the probability of up-crossing before ruin for different values of u.

4.2. Coxian approximation. After substituting the first four moments of premiums of the initial process in (14) and solving, we obtain the values $p = 0.042715$, $\tilde{\mu}_{p_1} = 4.4277, \tilde{\mu}_{p_2} = 2.4570$ and $\tilde{\lambda}_p = 2.6430$. Similarly, for claims we get $p = 1.179 \times 10^{-4}$, $\tilde{\mu}_{c_1} = 4.6384, \tilde{\mu}_{c_2} = 0.93811$ and $\tilde{\lambda}_c = 2.2423$. According to (7), the Lévy exponent of the replaced process is $\mathcal{K}(\gamma) = \frac{N}{D}$, where $N = 4.2334 \times 10^{56}\gamma^6 - 5.538 \times 10^{56}\gamma^5 - 1.3942 \times 10^{58}\gamma^4 + 1.7731 \times 10^{58}\gamma^3 + 3.4937 \times 10^{58}\gamma^2 + 6.379 \times 10^{57}\gamma - 1.9393 \times 10^{44}$ and $D = 8.4668 \times 10^{56}\gamma^4 - 1.1076 \times 10^{57}\gamma^3 - 1.9611 \times 10^{58}\gamma^2 + 2.5999 \times 10^{58}\gamma + 4.0079 \times 10^{58}$. The roots of the Lévy exponent are as follows: $\gamma_1 = 3.0401 \times 10^{-14}, \gamma_2 = -0.20813, \gamma_3 = -0.9387, \gamma_4 = 2.5692, \gamma_5 = 5.4222$ and $\gamma_6 = -5.5364$. Now using (10), one can calculate probability of up-crossing before ruin for different values of u.

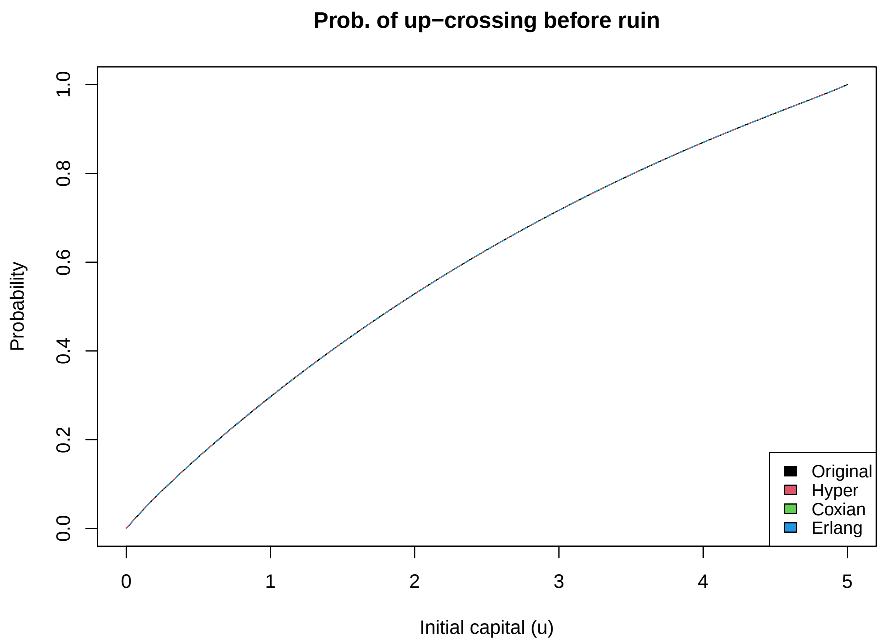
4.3. Erlang approximation. After replacing both premiums and claims of the initial process with two phases Erlang distribution according to (15), we obtain the values $\tilde{\mu}_{p_1} = 5.8837$, $\tilde{\lambda}_p = 1.8912$, $\tilde{\mu}_{c_1} = 6.9376$, and $\tilde{\lambda}_c = 1.6779$. Now, according to (7), the Lévy exponent of the replaced process is $\mathcal{K}(\gamma) = (6.3611 * 10^{11}\gamma^6 + 1.3408 * 10^{12}\gamma^5 - 5.5764 * 10^{13}\gamma^4 - 6.4299 * 10^{13}\gamma^3 + 1.6115 * 10^{15}\gamma^2 + 3.3738 * 10^{14}\gamma + 1.2047)/(1.2722 * 10^{12}\gamma^4 + 2.6815 * 10^{12}\gamma^3 - 1.0245 * 10^{14}\gamma^2 - 1.0946 * 10^{14}\gamma + 2.1197 * 10^{15})$.

The roots of the Lévy exponent are $\gamma_1 = -3.5707 * 10^{-15}, \gamma_2 = -0.20794, \gamma_3 = -7.3815 + 1.712i, \gamma_4 = -7.3815 - 1.712i, \gamma_5 = 6.4316 + 1.7484i$ and $\gamma_6 = 6.4316 - 1.7484i$. Finally, using (10), we can calculate the probability of up-crossing before ruin for different values of u.

4.4. Results. The following table (a fragment) shows the probabilities of up-crossing before ruin for different values of initial capital u:

u	Original	Hyper	Coxian	Erlang
0.00	3.893033e-14	-3.628825e-16	5.540273e-17	1.479499e-16
0.05	1.852855e-02	1.852938e-02	1.852938e-02	1.852455e-02
0.10	3.620403e-02	3.621841e-02	3.621841e-02	3.618799e-02
0.15	5.319133e-02	5.323253e-02	5.323253e-02	5.314816e-02
0.20	6.962394e-02	6.969755e-02	6.969755e-02	6.953576e-02
0.25	8.560616e-02	8.570927e-02	8.570927e-02	8.545553e-02
0.30	1.012167e-01	1.013407e-01	1.013407e-01	1.009895e-01
......				
......				
4.90	9.863957e-01	9.863917e-01	9.863917e-01	9.863169e-01
4.95	9.930806e-01	9.930798e-01	9.930798e-01	9.930352e-01
5.00	1.0000000	1.0000000	1.0000000	1.0000000

Graphical representation of the above probabilities is as follows:

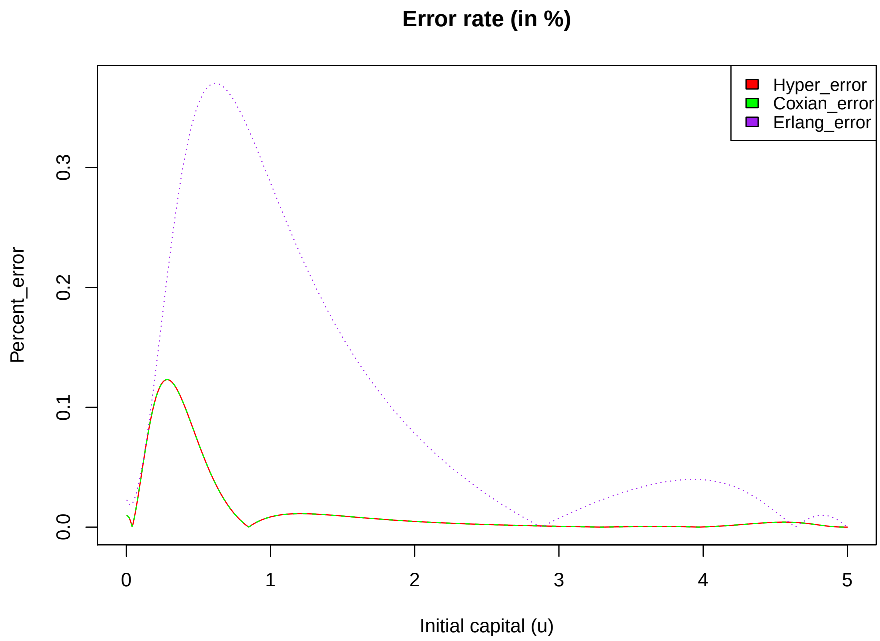


One can see that the four graphs practically coincide.

We can also calculate the relative percentage error ($\frac{True-Replaced}{True} * 100\%$) for the replaced processes. The following table (a fragment) represents the corresponding percentage errors:

u	Hyper_error	Coxian_error	Erlang_error
0.00	0.000000e+00	0.000000e+00	0.000000e+00
0.05	-4.485713e-03	-4.483786e-03	2.156812e-02
0.10	-3.973565e-02	-3.973371e-02	4.428843e-02
0.15	-7.744282e-02	-7.744085e-02	8.116038e-02
0.20	-1.057259e-01	-1.057239e-01	1.266563e-01
0.25	-1.204482e-01	-1.204462e-01	1.759647e-01
0.30	-1.225649e-01	-1.225629e-01	2.244462e-01
0.35	-1.152970e-01	-1.152950e-01	2.682891e-01
0.40	-1.022442e-01	-1.022422e-01	3.049698e-01
....			
.			
4.95	7.792619e-05	7.796427e-05	4.575202e-03
5.00	1.497691e-11	1.496581e-11	1.496581e-11

Graphical representation of the errors table looks like this:



One can see that the error graphs of hyper-exponential and Coxian approximations practically coincide, showing that these two distributions provide equally good solutions. The quality of Erlang approximation is, however, somewhat lower than in two other cases.

5. Conclusion

From the results, we see that it is possible to replace PH premiums and claims with a big number of phases in the initial process by modified PH premiums and claims having only two phases. While the errors obtained from the replaced Erlang distribution are not so big, nevertheless, the error graph suggests that hyper-exponential and Coxian distributions suit perfectly. Considering that the formulas for calculation of modified parameters for Coxian distribution are significantly more involved as compared to hyper-exponential distribution, we conclude that approximation with hyper-exponential distribution is more appropriate.

Since, in theory, each distribution can be approximated by a PH distribution, which in turn (as our analysis has shown) can be approximated by a PH distribution with just few phases, the paper provides a general way for simple approximate calculations of ruin probabilities of Lévy processes in the two barriers setup.

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