Weyl's Conception of the Continuum in a Husserlian Transcendental Perspective

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This article attempts to broaden the phenomenologically motivated perspective of H. Weyl's *Das Kontinuum* (1918) in the hope of elucidating the differences between the intuitive and mathematical continuum and further providing a deeper phenomenological interpretation. It is known that Weyl sought to develop an arithmetically based theory of continuum with the reasoning that one should be based on the naturally accessible domain of natural numbers and on the classical first-order predicate calculus to found a theory of mathematical continuum free of impredicative circularities (such as the standard definition of the least upper bound of a set of real numbers) only to stumble, to cite a key question, in the evident lack of intuitive support for the notion of points of the continuum. In this motivation, I set out to deal from a Husserlian viewpoint with the general notion of points as appearances reducible to individuals of pre-predicative experience in contrast with the notion of an interval of real numbers taken as an abstraction based on the intuition of time-flowing experience. I argue that the notions of points and of real intervals in the above sense are not by essence related to objective temporality and thus their incompatibility in mathematical terms is ultimately due to deeper constituting reasons independently of any causal and spatio-temporal constraints.

Keywords: actual infinity, completed totality, continuous unity, intuitive continuum, individual, iteration principle, mathematical continuum, restriction principle, temporal consciousness, time-point

1. Introduction

This phenomenologically motivated article is primarily based on Hermann Weyl's famous monograph *Das Kontinuum* (henceforth *DK*), first published in 1918, and still remaining the broadest known mathematical text in foundations taking into account Husserl's phenomenology of inner time-consciousness in the problematic of mathematical continuum. I note here that

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Husserl's student Oscar Becker had also published mathematical-philosophical texts influenced by the Husserlian phenomenology (Becker 1914; Becker 1927) and has been lately found to have had an interesting correspondence on these matters with Weyl himself.[1]

As J. da Silva has put it, the strict adherence of Weyl's arithmetical theory to what had been disclosed in the phenomenological analysis of intuitive continuum, most notably the acceptance of everything originally given as given in its "bodily" form in the modes and the bounds presented therein, makes it possible that phenomenological ideas can help us achieve a deeper insight into the mathematical theory of *DK* (Da Silva 1997, 281). Not to weigh less in the overall evaluation of the content of *DK* is H. Weyl's general scientific formation and his fundamental work in the theory of relativity[2] which made him especially apt in grasping the fundamental questions that a deeper analysis of spatiotemporal continuum generates, especially in the face of certain theoretical deficiencies with regard to a predicative foundation of continuum in taking into account the nature of the system of real numbers. It should be added that interest in Weyl's program for the arithmetical foundation of mathematics initiated with *Das Kontinuum*, originally overshadowed by Hilbert's finitistic foundations program in mathematics and L.E.J. Brouwer's intuitionistic theory, has been to a certain extent revived thanks to a considerable research work that has helped underline its significance for a viable predicative theory in mathematics (Feferman 1998, 249).[3] It is notable, though, that in the years following the first publication of *DK* and possibly due to insurmountable difficulties in eliminating, among others, an impredicative definition of the least upper bound principle for reals as well as certain measurability results, Weyl abandoned the phenomenological motivation of the original system of *DK* to espouse Brouwer's intuitionistic model of choice sequences. Later on he even went so far as to deny the relevance of phenomenological analysis altogether with regard to classical mathematics, especially in taking the latter as an objectifying metatheory for physics, on the assumption that Hilbert's formalism prevails over intuitionism. Of course it is known that Hilbert's formalism and his finitistic consistency program for mathematics suffered a serious blow just a few years

[1] On the correspondence between Weyl and Becker see the work of T. Ryckman and P. Mancosu in (Ryckman and Mancosu 2002; Ryckman and Mancosu 2005). The reader is also referred to (Ryckman and Mancosu 2002) for details on Weyl's first exposure to phenomenological analysis dating back to his graduate years between 1904 and 1908. In Weyl's own confession it was around 1912 in Göttingen that he "owed to Husserl's influence a liberation from his previous positivistic allegiances" (Ryckman and Mancosu 2002, 132).

[2] See, for instance, his *Space-Time-Matter* (1922), first published in the same year as *DK*.

[3] Among the recent literature on the subject one can cite (Feferman 2000), (Feist 2002) and (Feist 2004).

ahead with Gödel's incompleteness results.[4] However, I will leave out of discussion Weyl's subsequent estrangement from Husserlian phenomenology and try instead to offer a deeper phenomenological perspective to certain key concepts of the *DK* system with regard to the mathematical vs intuitive continuum.

Generally Weyl's approach in *DK* may be viewed in the wake of the theoretical discussions provoked by Cantor's introduction in the late 19th century of the theory of infinite ordinals and cardinals as well as the notion of transfinite sets in the foundations of mathematics, and the subsequent attempts to secure a sound foundation of set theory excluding apparent antinomies such as the set of all sets. In this general setting Weyl was largely influenced by Poincaré's definitionist approach of mathematics which implied the acceptance of the following general positions:

(i) the concept of a natural number sequence as an irreducible minimum of any abstract mathematical thought and the ensuing acceptance of the principle of induction as a key proof-theoretic tool; (ii) the acceptance that all other mathematical concepts, e.g., those of sets and functions, are to be introduced by an explicit definition founded on the acceptance of position (i); (iii) the thesis that there are not in principle completed infinite totalities; and (iv) the claim that definitions which single out a definite mathematical object by implicit or explicit reference to an assumed complete totality of which the defined object is a part should be excluded altogether from mathematics. Position (iv) is apparently the predicativist position in mathematics which in later years provided for many theoretical discussions and also for alternative approaches in the mathematical foundations (Feferman 1998, 254). Except for those properly espousing the definitionist-predicativist approach of mathematical theories, the acceptance of the domain of natural numbers as the most fundamental domain of mathematical intuition is to be found also in Kurt Gödel's writings suggesting that arithmetic in mathematics is the domain of elementary indisputable evidence that may be most fittingly compared with sense perception in the case of sensory knowledge (Tieszen 2011, 176). In a certain sense this view is reflected in Husserl's writings too with reference, for instance, to the determination of thematic objects (including perceptual ones) in the form of "*and so on*" in which every object-substrate of determination is always passively pregiven with an original horizon of indeterminate determinability "beyond the succession of actually constituted determinations and open to new properties which must be expected" (Husserl 1939, 217–218).

[4] A discussion of Weyl's later objections to phenomenological analysis in relation to mathematics, as presented by him in the mathematics seminar of the university of Hamburg in 1927, is undertaken by J. Toader in (Toader 2013a) and (Toader 2013b).

On this account, Weyl's three major theoretical positions can be summed up as follows: (i) definitionism as the conditioning of the introduction of any ideal entity on the explicit relations linking its constituent elements; (ii) intuitionism implying a mathematical universe in which all entities are assumed to be generated by logical principles from a basic category of objects given intuitively as primitive objects and relations irreducible to anything semantically more fundamental. In contrast to Brouwerian intuitionism, Weyl's original version admits of a definite truth-value for every well-built proposition involving only primitive objects and consequently does not contradict the Excluded Middle Principle; (iii) predicativism as the rejection of every impredicative definition, that is, of every definition that presupposes the knowledge of the totality of objects among which the object to be defined is taken. This position is materialized by restricting the scope of quantifiers to primitive entities alone which are taken in the terminology of the ramified type theory of *Principia Mathematica* as objects of Level 1. Weyl was particularly attentive to avoid the *circulus vitiosus* of impredicative definitions by restricting the use of quantifiers in the set-theoretic comprehension scheme and was attentive as well to avoid the pitfalls of set-theoretical antinomies by subjecting the $\in$ relation to type restrictions.

In view of the three positions above implying the existence of a basic category of primitive objects and relations and the additional condition of completeness of the number system involved, Weyl conceived of natural numbers as intuitively the most proper such category. The completeness of the basic category involved here is taken in the sense of existence of a definite totality generated by an indefinite iteration, this latter thought in turn as a homogeneous relation of its elements, i.e., of the successor relation starting from the element named 1 (Bernard 2009, 161). The intuition of iteration makes it possible to apply the notion of completeness of natural numbers to every totality of ideal objects isomorphic to it.

What is more, this structure not only guarantees a well-defined meaning to the logical expressions "there is a natural number n such that…" or "for every natural number n such that…" to define a new entity, but it also helps found the possible definition of a new entity by extending these expressions to include a sequence of sets iterated by a homogeneous functional relation $\Phi(X_i, X_j)$, where the $X_i s$ are sets of the same category. One may then define a new entity, e.g., by virtue of the existential formula "there is some set among the sequence $< X_1, X_2, \ldots, X_n, \ldots >$ such that…" As we will see in the next sections in spite of the intuitive clarity due mainly to the natural intuition of natural numbers as an ontological-categorial domain, Weyl was sooner or later to be faced with the difficulty of applying his arithmetical-predicative scheme to capture the inherent impredicativity of mathematical continuum.

The present paper as rather philosophically oriented will not enter into the mathematical technicalities of Weyl's proposed arithmetical theory of continuum, at least no more so than what is thought to be enough to comprehend the mathematical content of the issues raised in relation to the nature of continuum. Given Weyl's orientation toward interpreting the incompatibility between the intuitive continuum as associated with a continuous flow of time and the durationless character of (natural) numbers (meant as temporal points) in terms of the phenomenology of internal time, I will be mostly focused on elucidating the character of points as appearances in their most abstract form of "empty-of-content" individuals-objects of intentionality and as noematical[5] objects within a constituted, immanent[6] continuous unity. This unity is not, in principle, identifiable with the real-world spatio-temporal continuity. In this level of discourse it will be seen that Weyl's undertaking to describe the continuum of mathematical analysis based on the basic ontological-relational domain of natural numbers is bound to reach the impasse of an inherent incompatibility whose origin is well beyond the formal-mathematical realm, which is to say that it may be taken as ultimately founded on the subjective origin of the self-constituting inner temporality. To this goal an in-depth analysis will be undertaken of points as intentional individuals and of actual infinity (in the sense of a completed totality) as rooted in the continuous immanent unity of each one's consciousness.

2. A critical review of Weyl's treatment of the dialectical opposition of points vs. continuum

In a first reading of Weyl's foundation of mathematical continuum in *DK* (Weyl 1994, Ch. 2, §6) one may note a distinct view between, on the one

[5] A noematical object, a phenomenological term, is constituted by certain modes as a well-defined object immanent to the temporal flux of a subject's consciousness possibly transformed, in the sense of a formal-ontological object, to a syntactical object of a formal theory. It can then be said to be given apodictically in experience inasmuch as: (1) it can be recognized by a perceiver directly as a manifested essence in any perceptual judgement (2) it can be predicated as existing according to the descriptive norms of a language and (3) it can be verified as such (as a re-identifying object) in multiple acts more or less at will. For more details on this concept and the general meaning of noema the reader may consult Husserl's *Ideas I* (1976, §87–§94).

[6] The term immanent which is widely used in Husserlian and generally phenomenological texts can be roughly explained as referring to what is or has become correlative (or "co-substantial") to the being of the flux of consciousness in contrast to what is "external" or transcendental to it. For instance, a tree is external as such to the consciousness of an observer while its appearance as the image of it within his consciousness is immanent to it.

hand, real numbers in terms of an abstract scheme representing ever embedding parts within a definite whole (with continuous functions as uniquely defined dependencies of overlapping "continua") and, on the other, time- and space-points as "non-existent" individuals within an ever in-act temporal flow which are eventually considered as abstractions of what is immediately given in experience. In Weyl's view, "the exhibition of a single point is impossible" and further "as points are not individuals they cannot be characterized by their properties," in contrast to the formal treatment of the elements of the continuum of real numbers considered as genuine individuals. In fact, there is a clear distinction in the way points may be treated as "individuals" in a theory of synthetic (without co-ordinates) homogenous space and the way they are treated in terms of the arithmetico-analytic concept of a real number which belongs to the purely formal sphere where "…those ideas thoroughly crystallize into full definiteness."

Further, in a next level, there is little evidence that time- or space-points as individuals of an arithmetized theory of time and space correspond to what is immediately given in (intentional) experience, in other words, there is a separation *in rem* of mathematical and intuitive continuum (Weyl 1994, 93). In Weyl's approach if we have to talk about continuity, e.g., in the geometry of straight lines, then space-points may be only meant as *functions of*, referring to a coordinate system which unavoidably "fulfills" the underlying act of a pure sense creating ego (Weyl 1994, 94). This way, a space-point is taken as associated with a "motion" corresponding to a coordinate system in terms of which it "moves," its motion being actually an abstract scheme to represent its constitution as "being-in-the-flow" within an immanent self-constituting unity.

On this account, the principle that to each (temporal) point belonging, e.g., to a unit time-span interval, corresponds a real number and vice versa is merely a kind of conventional continuity axiom that stands in contrast to the intuitively evident "being-of-the-flow." Yet it is established as a cornerstone of a pure arithmetical analysis which is necessitated by the need of an extra axiomatical principle, extraneous to first-order logic, to accede to a description of continuous processes originating in a time or space theory. As it may be already seen Weyl shifted his theoretical focus to the discrepancy between an intuitively given continuum and the concept of number as a time or space point, primarily turning his attention to the temporal continuum as the most fundamental one. He claimed that this discrepancy would cease to exist if the following conditions were met:

(1) Insofar as a certain intuition is associated with temporal duration, a temporal expression of the kind "during a certain period *L*, I see this man

crossing the street," might be replaced by the expression "in every time-point which falls within the time span *L*, I see this man crossing the street."

(2) If *P* is a time-point and *OE* is a certain time span, supposing that we have constructed an elementary theory of time on the assumption of a time-point as a basic category and the relations "earlier" and "equal" defined accordingly in a natural way, then the domain of rational numbers to which the number ϵ belongs so that $OL = \epsilon OE$ for a time-point *L* arbitrarily "earlier" than *P*, can be constructed arithmetically in pure number theory based on certain principles of definition set by Weyl to define a real number. Further, on this assumption and taking *OE* as a unit time span to every time-point *P* corresponds a definite real number and conversely to every real number corresponds a definite time-point; [*Continuity Principle*] (Weyl 1994, 89).

On the supposition that a pure theory of time can be founded by the postulation of a time-point as a basic category and the relations of "earlier" and "equal" defined in a natural way compatible with the intuition of time, Weyl claimed in the first place that the very intuition of time might suffice to determine the one-to-one correspondence between time-points and real numbers. Obviously such a correspondence not only cannot be demonstrated, it is even intuitively unthinkable and Weyl considered this talk as utterly nonsensical to the extent that a "theoretical clarification of the essence of time's continuous flow is not forthcoming" inasmuch as continuum fails to satisfy certain features of first-order calculus described in the first chapter of *DK*. Moreover the notion of "a point in the continuum lacks the required support in intuition" (Weyl 1994, 90). Further, pointing implicitly to the Husserlian description of the inner time-consciousness by the claim that to every temporal point corresponds a definite experiential whole, he tried to strengthen his argument on the discrepancy of mathematical and intuitive continuum by referring also to the retentional schemes of the temporal flux of consciousness. On this account, having an original impression at some instant A necessarily implies that one has already in place and in an a priori mode except for the primary memory[7] of A the primary memories corresponding to the original impressions for all instants B_i occurring

[7] The primary memory is well-known to phenomenologists to be the a priori associated "imprint" within consciousness of each original impression which should not be confused with the actively re-produced memory (rememoration) in the actual now of consciousness. This sentence is difficult to understand. Also, maybe we should aim for avoiding word repetition here? "known […] to phenomenologists," "well known to phenomenologists." Also, here and elsewhere, maybe "a priori" should be in italics for the sake of clarity?

an arbitrarily short time earlier than A. These instants B_i are constituted in consciousness as a descending continuous degradation of retentions.[8]

At this point already came to Weyl's attention the inherent circularity in the description of the intuitive continuum in terms of the flux of temporal consciousness. The infinite sequence of point-like moments of experience fitting endlessly into one another in the progression of time and in the form of a continuous unity apprehended as such at any moment of reflection ultimately seemed to him as an absurdity. He was intuitive enough to note that abstracting each temporal now from its being-in-the-flux as part of a changing experiential content and treating it as an object of reflection, it would then instantly become a being-in-the-flux in its own right in which one could place new points possibly associated with new original impressions. In fact, this may be taken as a first step in facing the tantalizing question of an endlessly regressing sequence of reflections which the ego of temporal consciousness may bear on its objectified self. Ultimately Weyl landed in questioning the nature of continuity, in other words the flowing from point to point as ever eluding us, "in other words, the secret of how the continually enduring present can continually slip away into the receding past" (Weyl 1994, 92).

Of course the original Husserlian approach to the modes of constitution of temporal consciousness is far more articulate than Weyl's brief reference to the absurdity of passing from an infinite series of moments of perceptual experience to the apprehension of a continuously progressing experiential whole. In his main treatise on the phenomenology of time consciousness *Vorlesungen zur Phänomenologie des inneren Zeibewusstseins* (*Lessons on the Phenomenology of Inner-Time Consciousness*) (1966), Husserl dealt with the issue of the apparent incompatibility of the point-like character of each time perceptual experiences registered by an intentionally oriented consciousness as original impressions and the constitution of a temporal duration as a definite immanent whole by primarily appealing to two diverse intentional forms, the transversal (*Querintentionalität*) and the longitudinal intentionality (*Längstintentionalität*) of consciousness (Husserl 1966, §38, §39).

It is notable, though, that in the Husserlian phenomenology the ultimate step to reach the *prima causa* of the continuous unity of the temporal flux in the present now of reflection, involves a quite obscure notion which is the absolute (or pure) ego of consciousness, a notion Husserl was trying to clarify till his latest years only with limited success almost certainly due to the essentially transcendental character of this kind of subjectivity. It is

[8] The reader interested in a more detailed description of the intentional forms of the flux of consciousness (namely the transversal and the longitudinal intentionality) should consult the original Husserlian texts in (Husserl 1966, §11, §12, §38, §39).

noteworthy that although Weyl did not enter into the deep waters of the transcendence of the absolute ego, yet he was reserved enough to note that

> Each one of us, at every moment, directly experiences the true character of this temporal continuity. But, because of the genuine primitiveness of phenomenal time, we cannot put our experiences into words. (Weyl 1994, 92)

One may apply similar argumentation with regard to any intuitively given continuum, in particular to the continuum of spatial extension (Weyl 1994, 92). In fact, in original Husserlian view spatial continuum is considered as an objective form of temporal fulfillment. More specifically, Husserl referred to temporal extension as "fraternal" (*verschwistert*) to the spatial one. He pointed out that

> Like temporality, spatiality pertains to the essence of the appearing thing. The appearing thing, whether changing or unchanging, endures and fills a time; furthermore it fulfills a space, its space, even if this may be different at different points of time. If we abstract from time and extract a point of the thing's duration, then to the time-filling content of the thing there belongs the thing's spatial expanse. (Husserl 1973, 55)

As a matter of fact in view of the phenomenologically founded incompatibility between mathematical and intuitive continuum Weyl was eventually totally dismissive of the validity of the notion of an individual point as an independent, self-standing object. Concerning objectively constituted time he came to conclude that:

> (1) An individual point in it [constituted inner temporality] is non-independent, i.e., it is pure nothingness when taken by itself and exists only as a "point of transition" (which, of course, can in no way be understood mathematically); (2) it is due to the essence of time (and not to contingent imperfections in our medium) that a fixed time-point cannot be exhibited in any way, that always only an approximate, never an exact determination is possible. (Weyl 1994, 92)

On these grounds one may not be entitled to a formalization of the continuum based on the exact concept of a real number in that, ontologically speaking, there exists a kind of redundancy which is left "untreated" in the mathematical definition of real numbers by rational or non-standard approximations e.g. by approximating rational sequences or by ad hoc non-conventional entities in the form of infinitesimals. At this point Weyl seemed to resort to some kind of ultimate subjectivity referred to as "*Logos dwelling in reality*" (Weyl 1994, 93), which is reminiscent of the phenomenological pure ego, yet avoided any further descent into these murky waters.

In the bottom line, in Weyl's view, one should settle for a theory of continuum that establishes its reasonableness the way a physical theory does. One could cite, for instance, quantum-mechanical theory which establishes its authority by seeking a rational justification provided by its interpretative and predictive power in the derived "exact" physical world subjected to certain idealizations with regard to its pre-predicative experiential origins. Consequently the abstract scheme of real numbers associated with the possibility of infinite embeddings of possible parts into a presupposed completed whole and that of functions as uniquely-valued correspondences of "overlapping" continua are rationally justified in the context of an objective reality which is of a constitutive sense; meaning one that is "meaningful" only in the presence of at least one phenomenological reduction performing consciousness.[9] In this sense a real analysis, as we know it, can give an exact account of various physical phenomena within an idealized objective world e.g., in the description of the motion of a point along a surface. Yet, it is unreasonable to construct a space-time theory as a formal-mathematical discipline without conceding to the impossibility of existence of a single time- or space-point as a self-standing individual of experience solely characterized by its properties insofar as time-space points are apprehended solely within the continuous flow of immanent temporality. This is a major impediment to the possibility of a space-time theory construed as a formal-axiomatical one by postulating a continuity axiom which guarantees that given a unit time span OE to every point P of OE corresponds a real number and vice versa. This impredicativity-generating incongruence is also implicitly in place in Weyl's subsequent proposal (which he did not elaborate further) to provide a higher-order analysis, termed by him hyperanalysis, in which real numbers are admitted as a basic category like naturals and where certain sets are introduced in whose definition the existential quantifiers refer to real numbers.[10]

Ultimately Weyl saw the points of a space-time theory only relative to a coordinate system, that is, essentially as functions of a "meaning-providing something" in the understanding that they cannot be construed as self-stand-

[9] In rough terms one can refer to the phenomenological attitude in contrast to the natural attitude as the performance of the phenomenological reduction by an intentionally directed consciousness in which things of the objective world are put into "parentheses" in absolute ontological terms and taken as only valid in their appearance as such and such within consciousness. (Considering that this is a clarificatory footnote, for the sake of clarity I would also consider slicing this sentence up into shorter sentences.) This includes the possibility of consciousness to reflect on itself.

[10] As a matter of fact more sets of real numbers are generated in hyperanalysis than in standard analysis. In this case the totality of points of spaces, surfaces, lines, etc., can be constructed arithmetically as three-dimensional sets of real numbers (Weyl 1994, 95–96).

ing entities but only relative to a coordinate system which is thought of as the "residue" of the objectifying ego after its eradication in the geometrical-physical world. Yet they are thought to be grounded all the same on the world of experience as primordially given and susceptible to the objectification process originating in the sense-giving ego (Weyl 1994, 94).

In short, Weyl admitted even indirectly to insurmountable obstacles, pertaining to the subjective foundation of the continuum, in achieving its formalization in a way that real numbers as well-defined individuals would represent time- or space points while staying clear of any sort of circularity in the construction. This led him to point out that

> both contemporary analysis and its principles are left hanging in a nebulous limbo half-way between intuition and the world of formal concepts - even though, under the mask of its vague presentations of set and function, analysis is able to pass itself off as a science operating in the formal-conceptual sphere. (Weyl 1994, 96)

This is an argumentation that may also apply against his proposed version of hyperanalysis to the extent that it claims to introduce sets by quantifying over real numbers in the place of basic-category objects. More specifically, the "filling-in" and the "there is" principles[11] as principles arising from the three major theses of Weyl's approach, namely from definitionism, intuitionism and predicativism, essentially refer to natural numbers in the sense of primitive objects (and relations) given straightforward to us by intuition which moreover, taken as a basic category of entities, form a complete system of definite self-existent objects. Accordingly, Weyl's proposed hyperanalysis

[11] The "filling-in" and the "there is" principles are respectively established in (Weyl 1994) as follows:

> If, for instance, $U(xyz)$ is a judgment with three blanks and a is a given object of our [basic] category, then the judgment $U(xya)$, which is produced by the operation of "filling in," is one with only two blanks. In particular a closed judgment (i.e., one without blanks), a judgment in the proper sense, which affirms a state of affairs, arises from a judgment scheme when all its blanks are filled by certain given objects of our category.
>
> Let[...]$U(xyz)$ be a judgment with three blanks. Then we can form the judgment $U(xy*) = V(xy)$, which means "There is an object z of our [basic] category such that the relation $U(xyz)$ obtains." Similarly, we can form $U(*y*)$, meaning "There is an object x and an object z such that $U(xyz)$ is true." The number of blanks of a judgment scheme will be reduced by application of this scheme too. If no blanks at all are left, then here again a judgment in the proper sense arises, about which it is appropriate to ask whether it is true or not. (Weyl 1994, 10–11)

must at least admit to a conception of real numbers as well-defined individuals possibly identifiable with points of a space-time span by the *Continuity Principle*,[12] (even in discarding the possibility of their generation by a homogenous iteration operation), so as to rightfully claim the introduction of new sets by quantifying over real numbers in the place of basic-category objects. In short, one is about to be faced again with the "vague" nature of real numbers as presumed individuals of an axiomatical form of space-time theory by reproducing a circularity on the level of hyperanalysis.

However the constraints of Weyl's *restriction principle*, namely his predicativist rejection of the notion of a set as a definite totality to quantify over, can be partly overcome by the (so-called) arithmetism which is considered a way to legitimize quantification over sets on the condition of a partial application of abstraction on their elements (Bernard 2009, 158, 162–163). More specifically, one can imagine a totality of sets in which at least one variable $n \in N$ is left independent as an enumerator of the sets within the totality, producing in effect a function $R(n)$ which may generate a set as a totality without infringing on Weyl's *restriction principle*. For example, one can define a real number as a set of rationals by leaving as an independent (not abstracted) variable the natural number n enumerating the sequence of its rational approximations. Yet even Weyl's *iteration principle*[13] for partially abstracted functions is not exempt of the possibility of quantifying over objects greater than Level 1 (in the sense of a ramified type theory) in order to introduce new entities or relations. On this account one may refer without entering into more technical details to S. Feferman's introduction of a formal system W, proved to be a conservative extension of the Peano Arithmetic, to accommodate Weyl's mathematics in *DK* where there is no obvious definability model of W in which the class of all total functions of natural numbers $(N \to N)$ consists of arithmetical functions (Feferman 1998, 279).[14] Consequently, and in view of the existing (equivalent) characterizations of hyper-

[12] See, page 105.

[13] In less technical terms the *iteration principle*, which "expands" the iterative structure of natural numbers and in a certain sense the intuitive accessibility of naturals toward totalities of sets, claims that a totality of objects obtained by successive iterations of a homogenous one-to-one set-theoretical operation $\Phi(X)$ between objects of the same category can be regarded as a complete totality available in turn for the definition of new entities.

[14] Feferman, based on Kleene's ${}^2E$ computable functionals over N, has introduced a system W whose definability model makes that the class of all total functions $(N \to N)$ consists of the hyperarithmetic functions and the Π^1_1 partial ones (Feferman 1998, 278–279). There is also a certain controversy regarding the capacity of the theory W to accommodate all scientifically applicable mathematics, in particular its capacity of consistently incorporating a host of mathematical aspects of quantum mechanics. Yet Feferman thought that the matter can be properly treated by pure mathematical-topological means and does not seem to pose a major "ontological" challenge for the theory W (Feferman 1998, 281–283).

arithmetical sets, involving e.g., set-quantification of formulas of second-order arithmetic or transfinite recursion based on so-called Turing jumps, a part of the intuitive clarity of the basic category of natural numbers which would vindicate Weyl's predicativist program seems to be missing.

In sum, Weyl sought to talk about sets as totalities of objects insofar as this is done in the secure stepwise way of a definitionist predicative approach, providing as a basic category the set of natural numbers whose iterative generation lays *eo ipso* the heuristic significance of their inductive character. On this motivation, and mainly on account of the iteration principle, he tried to provide explicit and predicative definitions of subsets and functions within classical logic while seeking to evade impredicative traps by substituting the general definition of sets with that of definable sequences of reals. Yet, he was plainly admitting that

> the continuity given to us immediately by intuition (in the flow of time and in motion) has yet to be grasped mathematically as a totality of discrete "stages" in accordance with that part of its content which can be conceptualized in an "exact" way. More or less arbitrarily axiomatized systems [...] cannot further help us here. We must try to obtain a solution which is based on objective insight. (Weyl 1994, 24)

In other words Weyl's predicativist definitions of sets and functions up to the introduction of ideal elements through the new approach, was ultimately faced with the imposition *in rem* of the incompatible concepts of points and intervals as fundamental structures of a space-time continuum. Further, this means that one had to reconcile the intuitively accessible means provided by a theory founded on the conception of natural numbers as a basic category of objects with the essential nature of intuitive continuum as an ever changing state-of-affairs, associated moreover with an inner temporality and thereby rendering the notion of space-time points as immutable individualities a meaningless notion.

It should be reminded that Weyl's choice to restrict attention to definable sets of the first level, that is, essentially to the category of natural numbers as the absolute operational domain meant giving up the general least upper bound principle for the real number system. This kind of restriction also brought out certain problems concerning Weyl's iteration principle[15] by producing a relation of the type $R *_{\Phi} (\frac{p}{n}, N)$ over the set of natural numbers

[15] Weyl's principle of iteration, presented in three different forms, can be summarily defined in its simplest form as the recursive relation $R_{n+1}(\frac{xx'}{X}) = R_n(\frac{xx'}{F(x)})$, $R_1 = R$, where relations $R_1, R_2, R_3, \ldots$ arise from a single original one $R(n; \frac{xx'}{X})$ which has the blank n affiliated with the category "natural number" and filled in successively by 1, 2, 3,... The relation $R(\frac{xx'}{X})$ gives rise to the function $F(X)$ and has its blanks divided into the dependent variables x, x' and the independent X with the latter affiliated with the category of two-dimensional

N which is not arithmetical in N (due to the diagonalization argument) and hence not definable at level 0; for further details see: (Feferman 1998, 264–265).

3. A phenomenologically motivated approach of the mathematical continuum

Weyl's attempt to do justice to his main positions of definitionism, predicativism and his own version of intuitionism, led him to establish a predicatively founded mathematics which would be ultimately divergent from both Frege's logicist approach and Russerl's approach in the theory of logical types, while preserving fundamental features of respective positions. In at least one crucial point, he distanced himself from Frege, Russell and Poincaré, namely in that the principles of definition must be used to give "a precise account of the sphere of the properties and relations to which the sets and mappings correspond" (Weyl 1994, 47).

He rejected, in particular, the Fregean definition of natural numbers as equivalence classes as well as the reducibility axiom in Russell's *Principia* which he thought were incompatible with his "narrower" procedure associated with the conception of sets and functions founded on the *iteration principle*. In fact his meaning of the concept of a set lent a substantial content to the following assertion: "To every point of a line (given an origin and a unit of length) corresponds a (distance measuring) real number (= a set of rational numbers with the properties a), b) and c)[16] and vice versa" (Weyl 1994, 49). This noteworthy assertion establishes a connection between something given by space intuition (points in space) and something (the set of real numbers) generated by a logical conceptual way. Yet, Weyl conceded to the insufficiency of this assertion with respect to what is given us by intuition, more specifically the assertion above does not offer a morphological description of what is given to our intuition as a constant temporal flow. It rather does so by giving an exact "representation" of an immediately given reality in the actual now of reflection while missing at the same time the grasp of the intuitive evidence of a homogenous flow and further the possibility to describe in exact logical-theoretical terms what is by its nature inexact.

sets whose blanks are filled in with the same categories of objects as the dependent x, x' in R.

[16] In Weyl's definition of a real number as a one-dimensional set α of rational numbers the following properties hold: a) if r is an element of α, then so is every other rational r' such that $r-r'$ is positive; b) for every element r of α there is a rational number r^*, also belonging to α, such that $r^* - r$ is positive c) α is non-empty, but not every rational number is an element of α (Weyl 1994, 31).

This constant temporal process, Weyl stressed, is vital to all exact knowledge of physical reality and through this very process mathematics becomes relevant as a metatheory of natural science. In a definite sense Weyl's reconstruction of the intuitive continuum within a "symbolic universe" ought to establish its reasonableness beyond strict formal consistency in the same way a physical theory does and its direct evidence should formally result as faithfully as possible as presented in intersubjective identity . Ultimately, as discussed already, Weyl was unable to overcome the inherent incompatibility generated by the vague nature of continuum, as subjectively constituted, in contrast with the "exact" nature of formal mathematical objects something that ultimately led him to put in doubt the objective existence of the "points" of continuum (Weyl 1994, 90).

It is noteworthy that Weyl had the insights, well before Gödel's incompleteness results, to sense the restrictions posed by formalism/conventionalism to the meaning-content of mathematics for which certain determinations should be implemented other than those taking its objects/state-of-affairs as simply the result of logical inferences derived from a prescribed set of axioms. In this respect, he argued against the position that mathematical statements as, for instance, Fermat's Last Theorem, are mere consequences of arithmetical axioms by citing an example in which the arithmetical axioms cannot guarantee a sound meaning to the existential predication "there is," therefore putting into doubt the consistency and completeness of Peano's axiomatical system of arithmetic and Dedekind's completeness of the real numbers system. Prompted by his definitionist-predicativist approach and Cantor's proof of the non-denumerability of reals, he had also the mathematical acuteness to foresee the independence of the Axiom of Choice by proposing that "naturally there is no reason at all to assume that every infinite set must contain a denumerable subset - a consequence from which I certainly do not shrink" (Weyl 1994, 28).[17]

In view of these positions one can reasonably argue that Weyl was not only preoccupied with the incompatibility between the spatiotemporal continuum, founded on the intuition of the continuous flow of temporality, and the existing mathematical analysis, but he was also concerned with the sterility of strictly formalist approaches of mathematics. This attitude, to the extent that it recognized a sort of objective reality of mathematical objects originating in the categories of primitive objects and their relations and

[17] Weyl's point in the quoted assertion is that the infinity of a set does not guarantee by itself that there will be a constructive pairing function whose domain is the set of natural numbers and whose range is a subset of the infinite set in question; in other words one has to postulate the possibility of choice as an axiom independently of the meaning-content of infinity in general.

insofar as it refuted the notion of infinite sets as definite totalities by the *restriction* and *iteration principles*, distanced itself as a matter of fact from both later Hilbertian finitistic formalism and Carnap's logical-syntactical program for mathematical theories. More than this, Weyl converged in a certain sense to Gödel's critical attitude toward the respective theoretical positions in recognizing that there is some kind of deficiency of mathematical theories, originating in their formal-axiomatical structure, with respect to the objective existence of mathematical entities. These latter entities taken as evidences knowledgeable by primitive world-experience and consequently conditioned at least partly by the modes of constitution of that experience.

It turns out that a more profound and phenomenologically motivated analysis of intuitive continuum, taken in Weyl's monograph as intimately associated with the notion of space-time continuum, should concentrate on the deeper meaning of points as individuals-objects of intentional experience in contrast with the meaning of intervals as impredicatively defined totalities bearing subtotalities of the same genus and embodying the notion of duration. On this account, it seems quite problematic how one can possibly build a space-time theory by defining space-time points as individuals represented by real numbers when there is no intuition of a durationless time point in the flow of any (within-the-world) experience. Moreover, it seems doubtful whether one can have the intuition of a set of points (these latter taken as set-theoretical individuals) without conceding to the existence of a non-eliminable temporal factor conditioning by the underlying (impredicative) unity of temporal consciousness the very act of colligating an indefinite collection of formal objects in the form of definite wholes in the present now of consciousness.[18]

One faces, in fact, the challenge of building a theory of the most primitive entities and their essential relations while being constrained at the same time by intuitions deeply rooted in a constituted and subjectively generated temporality which cannot be apprehended in reflection but as the "ever-changing" homogenous unity of immanent objects with the modalities of being such and such.

Therefore, one may raise the question of how we can have a proper interpretation of the essential distinctness between points and the intervals of the continuum, indeed of how we can have a proper theory of space-time continuum, without taking into account what is originally given to us as non-mediated experience. That is, without doing justice at the same time

[18] The act of colligating a definite or indefinite collection of formal objects in the form of completed wholes and as such turned to thematic objects "in front of" the intentionality of the ego of consciousness is termed in Husserl's *Experience and Judgment* a retrospective apprehension (*rückgreifendes Erfassen*) (Husserl 1939, 246).

to the intuition of a temporally founded continuum annulling by its own non-eliminability the definability of any object—be it a material or formal one—as a durationless immutable object within the world. At this point it is of primary importance to have in mind the content that may be given to the concept of mathematical objects and their relations in general. Are they to be regarded in the commonly shared view of many platonistic logicians and philosophers of mathematics as atemporal, unalterable objects (and for that reason mind-independent) or rather as subjectively constituted objects of intentionality enjoying "artificially" by virtue of their intersubjective verification within the world of experience the status of immutable transtemporal entities retrievable any time at will in the form of well-meant objects in actuality?

My own position tends to side with the second option on the following grounds: even in regarding mathematical objects as atemporal ones and mind-independent, think for instance of the function $z = tan\pi(x-\frac{1}{2})$ which helps prove the non-denumerability of a real interval, at the very instant they become objects of the intentional directedness[19] of a self-constituting temporal consciousness by this very fact they are being "nullified" as atemporal objects as they instantly become *objects-in-front-of* (a consciousness) adumbrated moreover by the modes of being *objects-in-front-of* a consciousness. This means that an *object-in-front-of* is given as such in original presence together with its "inner" and "outer" horizon in reference to an intentional consciousness which has grasped it as an unambiguous verification of its enactment in the actual now and it is moreover susceptible of every possible modality in having become part of its immanence. This means a further elaboration of the *object's-in-front-of* inner horizon is possible by free variation in imagination, abstractive ideation, comparative reflection, formalization, etc. It follows that mathematical objects as objects of axiomatized theories are each time potentially open to further clarification, further accession to possible hidden properties or relations insofar as they are made objects of the intentional and further explicative regard of at least one subject performing concrete intentional-cognitive acts within-the-world.

A view of mathematical-logical objects as receiving their whole sense of being within the world from each subject's intentionality is given by R. Tieszen in *After Gödel: Platonism and Rationalism in Mathematics and Logic* (2011). More specifically, Tieszen points out that

[19] The key terms intentional or intentionality are fully taken in this article in their phenomenological connotation, that is, roughly as meaning the a priori tendency of consciousness toward "something in general" independently of a material or a general "thingness" content.

> Taking our lead from Husserl's comments in *The Idea of Phenomenology*, *Ideas I*, *Formal and Transcendental Logic*, and elsewhere, we can say that ideal objects are also constituted as such by consciousness, by the monad. [...] whatever things are, mathematical objects or logical included, they are as experienceable things. It is experience alone that prescribes their sense. [...] Nothing exists for me otherwise than by the actual and potential performance of my own consciousness. Whatever is given as an existing object in mathematics or logic is something that has received its whole sense of being from my intentionality. There is no conceivable place where the life of consciousness could be broken through so that we might come upon a transcendent mathematical or logical object that had any other sense than that of an intentional unity (invariant) making its appearance in the subjectivity of consciousness. (Tieszen 2011, 97)

4. Points as empty-of-content substrates of intentionality

Can we have formal "lowest-level" individuals (think possibly of the "points" of the formal space-time theory) as immanent objects independently of an absolute temporal position, the latter state-of-affairs implying their existence as spatio-temporal and therefore "real world" objects? In view of Weyl's inquiry in *DK* discussed in the previous sections about the possibility of interchanging the notion of space-time points with that of real numbers and further the relation of both to spatiotemporal and phenomenological continuum this question is of crucial importance.

In this connection it is important to see that even though Husserl denied in *Erfahrung und Urteil* (*Experience and Judgment*) the status of genuine individuals to the objects of imagination, thus depriving them of the possibility of founding their identification, he nevertheless ascribed an ambivalent status to such individuals; in fact he termed them quasi individuals and the associated identities quasi-identities. He claimed that there is a notion of time that may be associated with these quasi-objects in terms of a broader intuitive unity allotted to them by virtue of being "there" in the uniform stream of consciousness (Husserl 1939, 174–175).

In view of what has been said so far my position concerning mathematical individuals can be summed up as follows: to the extent that mathematical individuals are essentially taken as syntactical atoms of a formal-mathematical theory assigned with a certain ontological sense, for instance those bounded by universal-existential quantifiers in a first-order predicate formula of the kind $(\forall x)(\exists y)Q(x, y)$, they can be logically founded irrespective of their possible spatio-temporal "mirror-image," that means, solely by virtue of the intentionality of a subject's "regard" free of any causality constraints. This means that they can be founded as content-free fulfillments of intentional acts which should not be by necessity causally related with objects-individuals existing in objective spatio-temporal terms. This is a view shared also by Tieszen in (Tieszen 2011), in which he refers to the possibility of the non-existence of an object in real terms even though our (intentional) awareness may be directed to it as if there were indeed such an object. Precisely by virtue of their nature unconstrained by spatio-temporal or causal determinations and also due to their intersubjectively founded identity over time, mathematical-formal individuals can serve as universals of formal-axiomatical theories bearing an ontological sense through bounded predicate formulas.

In these terms the question of the possible existence and the modes of existence of objects as fulfillments of concrete intentional enactments is "freed" from the constraints of their absolute existence as real spatio-temporal objectivities. Husserl had thought of quasi-individuals and generally of quasi-objects, taken as objects of imagination,[20] as united in a most inclusive unity of intuition which cannot be a unity of objectivities in absolute world-time to the extent that objects of imagination and generally objects in the sense of mere intentional enactments have no temporal connection either with objects of perception or among themselves (in real-world terms). This kind of unity is not a unity of objectivities; "it can only be a unity of the lived experiences that constitute objectivities, of lived experiences of perception, of memory, and of imagination" (Husserl 1939, 175).

Consequently and inasmuch as the unity of lived experiences is evidently bound by Husserl to the continuous flow of inner-time consciousness, the connection that might be established with these quasi-objects on the intentional level cannot in principle be causally grounded one.

An individual in purely formal abstraction can therefore be thought of as founded on the a priori enactment of the intentional direction of consciousness, this latter being totally inconceivable without orienting itself toward "anything-whatsoever" (*etwas überhaupt*). In fact an individual in this sense can be brought upon into evidence as intuitively self-given each time of reflection.[21] A "lowest-level" individual as an abstract individual is impene-

[20] Objects of imagination might be thought of as a general category of objects immanent to the consciousness and freed of real-world constraints among which one can think, as a "subspecies" enjoying a special status, mathematical or logical objects. Tieszen thinks of the objects of mathematics and logic as mind-independent in the sense of immanent to the consciousness objects of intentionality which are nonetheless constituted in a rational motivation throughout our experience in mathematical practice (Tieszen 2011, 104–105).

[21] Think, for instance, in extreme introspection as emptying yourself of any contemplative thought, pushing beyond any kind of focusing on whatever imaginable object or state-of-affairs; then you can see that you are by necessity oriented *in extremis* to a vague, indefinite, boundless and yet non-eliminable "something-there."

trable in terms of a thingness content since we cannot bring into reflection something "internal" to it in the form of a new concrete something-there and in the contemplative relation part-of. In other words, the content of each vacuous intentional act, eliminating any possible higher-level distraction, may be taken as founding the notion of an irreducible individual (including quasi-individuals in the sense attributed in *Experience and Judgment*), appropriating a host of categorial properties by virtue of being objectified as a general *Dies-da* (this-here) through the enactment of intentional orientation. This kind of impenetrability in reflection may be thought of, at this level of evidence, as founding individuality *in rem* independently of any reference to a really existing object-counterpart, while the single act of seizing it in the present now of consciousness may be thought of as founding meta-logically its uniqueness.

One may note that Husserl was careful to clarify that "individuation and identity of the individual, as well as the identification founded on it is only possible within the world of actual experience, on the basis of absolute temporal position" (Husserl 1939, 173–174). Consequently imagination, in general, does not generate individuals as irreducible objects associated with a fixed temporal position and spatial content but only quasi-objects and quasi-identities in the broadly conceived unity of intuition. In this approach, such individuals may not serve as ultimate self-evidences in laying the foundation of a theory of judgments, even though in Husserl's earlier texts (in *Ideen I*) "lowest-level" substrates of analytical sentences in the sense of ultimate phenomenological evidences, that is, as noematical nuclei-forms (*Kerngebilde*) deprived of any "inner" analytical content, even of a temporal form, and consequently not necessarily associated with the world of real experience are thought of as establishing the foundation of such objects of mathematical theories, as numbers, sets, classes, functions of sets or classes, Euclidean or non-Euclidean domains, etc., (Husserl 1976, 33–34). Consequently the above Husserlian view in *Ideen I* does not contradict the possibility of founding formal-mathematical individuals as fundamentally intentional objects (of a special status) not necessarily associated with a subject in causal terms within objective spatio-temporality.

Individuals such as those associated with actual experience and quasi-individuals of imaginary intuition (also individuals of memory) can be "embedded" in the unity of intuition only insofar as they are encompassed within the unity of constituted time, this way conditioning the unity of a plurality of various objects brought to the immanence of consciousness on self-constituting inner temporality.[22] Indeed, all immanent objects, together with actual relations (concerning objects of actual experience) or quasi-relations concerning objects of memory or imagination, including mathematical objects as objects of categorial or eidetic intuition are ultimately conditioned on the intuition of the unity of time reduced in turn to the phenomenology of absolute temporal consciousness and its pure ego (Husserl 1939, 182).

Ultimately as the possibility of existence of "lowest-level" individuals of a mathematical theory is not necessarily associated with a spatio-temporal position founded on real-world objectivity one may get as a consequent result the ontological "suspension" of Weyl's proposed continuity principle (Weyl 1994, 6) insofar as it seeks to establish a one-to-one correspondence between the points of a time span and real numbers in taking time-points as individuals in objective time.

5. The possibility of a causality-free infinity

The present approach to the notion of formal-syntactical individuals as independent of spatio-temporal constraints can possibly extend to a notion of infinity also made free of spatio-temporal and consequently causal constraints. This kind of infinity is grounded in the immanence of consciousness and vestiges of this position are recognized from as early on as Husserl's *Philosophie der Arithmetik*. In this early work, generally considered as rather belonging to the psychologistic phase of Husserl's evolving phenomenological formation, Husserl referred to the inner experience as the evident factor for the possibility of representing a multiplicity of objects as an instantaneous lived experience (Husserl 1970, 24).

In later works, Husserl elaborated the conception of an immanent whole of multiplicities of appearances in terms of the continuous unity of a self-constituting temporal consciousness. In fact the foundation of continuous unity, as thematically presented in reflection, was to ultimately rest on the modes of constitution of temporal consciousness and more radically on its self-constituting origin. At the time of *Logische Untersuchungen* (*Logical Investigations*) he conceived of a kind of actual infinity in presentational immediacy in these terms:

> The fact that we freely extend spatial and temporal stretches in imagination, that we can put ourselves in imagination at each fancied boundary of space or time while ever new spaces and times emerge before our inward gaze—all this does not prove the relative foundedness (*Fundierung*) of bits of space and time, and so does not prove

[22] One can construe an extension of relations of actuality subsisting between real individuals to quasi ones and make them appear in the quasi mode "precisely as far as the unity of an intuition of imagination and a world of imagination extends" (Husserl 1939, 187).

> space and time to be really infinite, or even that they could be really infinite, nor even that they really can be so. This can only be proved by a law of causation, which presupposes, and so requires, the possibility of being extended beyond any given boundary. (Husserl 1984, 45)

Later in *Experience and Judgment* (which incorporated most of the key ideas of previous works), after having been engaged in the search for the essential structure of temporal consciousness in the *Phenomenology of Inner-Time Consciousness* and in the *Bernau Manuscripts*, Husserl referred to

> a special kind of constitution of unity which provides the basis for special relations, for the formal relations. It is the formal-ontological unity, which neither rests on the actual connection of the objects united nor is founded on their essential moments or their entire essence. (Husserl 1939, 188)

The objects and the relations referred to here are also thought to include, in the broad sense of formal-ontological ones, such mathematical objects as sets, classes, elements of sets or classes, functions and their domains, Euclidean or non-Euclidean manifolds, etc.

In an apparent reference to his conception of formal ontological objects in *Formal and Transcendental Logic*,[23] Husserl described this formal-ontological unity as a collective form of unity extending to all possible objects individual or not individual. Further the unified "whole" of collection becomes objective, as thematized, if a continuous apprehension of these objects one by one and in their totality takes place through a presentation in the actuality of consciousness. This collective unity is essentially neither founded on real space-time objectivity nor on material elements to the point that even the essence of things is not taken into consideration except insofar as it makes differentiation (between them) possible. By virtue of this unity we may be provided with a subjective foundation to universal-existential predicative forms in such a way that a proposition of the kind: "each and every thing (everything possible and hence everything actual... such that)... is capable of being intuited as actual or possible in the actual present of one consciousness" may be taken as equivalent to the proposition: "each and every thing ... (such that)... is in principle capable of being colligated" (Husserl 1939, 189).

In these terms one may have a notion of actual infinity independent, as it is the case with formal individuals-substrates, of spatiotemporal constraints

[23] For a detailed description of the meaning of formal-ontological objects the reader is referred to Husserl's *Formale und Transcendentale Logik* (*Formal and Transcendental Logic*) (Husserl 1929, §24, §25, §37).

and consequent causal concerns. In fact, this kind of immanent "infinity," objectified as a completed whole in the instantaneous now of reflection, is what makes mathematics such an effective and inexhaustible tool in describing real world processes while by this token and in a certain holistic approach may be seen as de facto establishing the non-decidability of key questions in the foundations of mathematics (Livadas 2013; Livadas 2015). This kind of non-causal infinity to the extent that it may be, according to Gödel, linked with "the psychological fact of the existence of an intuition sufficiently clear to produce an open series of extensions of the axioms of set theory" (Gödel 1990, 268) may pertain also to the conception of transfinite sets in general as definite totalities and further to the ontological status of large cardinals, e.g., of measurable or supercompact cardinals, and consequently underlie the intelligibility of strong infinity axioms.[24]

6. Conclusion

As already stated, Weyl sought to provide in *DK* an arithmetical foundation of the mathematical continuum, taken as essentially associated with the continuum of space-time, on the assumption of the field of natural numbers as the most basic ontological-categorial domain. In doing so, he was convinced that arithmetical assertions, that is, those whose variables are exclusively restricted to the domain of natural numbers (or to any isomorphic structures) may have a definite truth-value attached to them in virtue of the fact that the arithmetical theory pertaining to the naturals is most fitting to a straightforward sense perception.

On the other hand, transfinite assertions whose variables range over domains of higher-level and not "self-standing" objectivities, i.e., those which do not correspond to direct sensory intuition, may not have a definite truth-value attached to them. In view of this, he espoused the restriction principle to allow for expressions whose variables range solely over the basic domain for which he offered the following justification:

> Clearly we must take the other path—that is, we must restrict the existence concept to the basic categories (here, the natural and rational numbers) and must not apply it in connection with the system of properties and relations (or the sets, real numbers, and so on, corresponding to them). In other words, the only natural strategy is to abide by the narrower iteration procedure. (Weyl 1994, 32)

In his own elaboration of a Husserlian perpective to Weyl's *DK*, J. Da Silva refers to ultimate syntactical substrates of any analytical sentence which

[24] There is a broad discussion in foundational mathematics concerning the implicit role of actual infinity in determining infinite mathematical objects as completed wholes which cannot be dealt with further here; see, for instance, (Feferman 2009) and (Livadas 2017).

are admittedly syntactically irreducible on pain of an infinite regression and thought of as corresponding to evidences of experience, as naturally represented in Weyl's analysis by the natural numbers (Da Silva 1997, 288). On this account, ultimate individuals-substrates of formal mathematical expressions given as evidences of pre-predicative experience (meant as prior to the predicative act of making judgments) may be taken, in a revisiting of Weyl's analysis, as underlying the concept of the points-individuals of continuum by virtue of being non-causally generated individuals of intentional experience independently of any spatiotemporal concerns.

On this ground and on the evidence of the unity of constituted temporality a new light may be shed to the lack of support in the intuition of a durationless point in contrast to the immediately experienced continuity of phenomenal time.

In this perspective I undertook a re-evaluation of the notion of the points of a space-time meta-theory as reducible to vacuous (with no-"thingness" content) individuals of intentional experience whose original givenness can be, in an alternative phenomenologically motivated approach, dispensed with causal and spatio-temporal constraints. At the same time I have tried to bring about the possibility of an immanent "infinity," also independent of any causal constraints pertaining to real-world experience, which is founded instead on the continuous unity of inner-time consciousness to account for the concept of completed wholes in the postulation of infinity assumptions and also of transfinite objects in the mathematics of continuum.

In view of the incompatibility between space-time points, in the sense of lowest-level individuals of a formal theory and time-intervals, founded on the notion of an immanent "infinity" ultimately reducible to a subjective temporal origin, Weyl's attempt to a description of the real continuum by purely arithmetical means was eventually doomed to fail on the phenomenological grounds discussed in this paper.

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