VARIATION OF SURFACE GRAVITY UPON TWO CEPHEIDS — δ CEPHEI AND η AQUILAE

BY

A. KIPPER

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§ 1. Introduction and Summary.

The importance of the solution of the problem of Cepheids is not only restricted to the fact that we possess a certain idea about the variable processes which occur upon their surface, but the solution of this question is important too from the standpoint of general astrophysics and of the investigation of the behaviour of substances, because the Cepheids form a sort of a natural laboratory, where substance must in a short time undergo various changes of temperature and pressure. The astrophysical interest of the problem has found its best expression in Whipple's words: "Studies based on non-variable stars of various temperatures and spectral types are handicapped by the fact that no two stars can be exactly alike, especially with regard to their mass and composition. In a Cepheid the mass must remain constant, and the composition of the layers contributing to the observed spectrum can vary only slightly."

Many hypotheses have been put forward and referred to as being able to explain the changes which we may observe in a Cepheid in the variation of luminosity, of spectral type, of radial velocity, *etc.*; but nowadays only two theories may be regarded as containing a considerable amount of correctness: 1) the double-stars theory by J. H. Jeans[1]), and 2) the pulsation theory by A. S. Eddington[2]). In spite of the fact that both of these theories have their weak points, it seems that the pulsation theory, though in a somewhat altered form, better explains the appearances than the double-star theory.

Assuming the radial-symmetric pulsation of the star, the temperature and the pressure of the atmosphere must change, because by the contraction of the gas-sphere its temperature and its radiation become increased, which at once influences the

1) M. N. 85, page 797.

2) The Internal Constitution of the Stars, page 180.

atmosphere. In accordance with that the pressure upon the surface of the Cepheid must change too, for in case of the alteration of the radius the acceleration of gravity changes upon its surface and, as a consequence, the weight of the atmosphere also. Change in temperature displays itself in the variation of colour, while pressure together with temperature calls forth the variation in the spectrum. It must be especially pointed out that the influence of temperature upon the intensity of spectral lines is much greater than the effect of pressure, and, therefore, great dispersion is wanted, in order to catch the details in the change of spectral lines as distinctly as possible. The aim of this paper is, by making use for this purpose of the results published by Fred. L. Whipple[1]), to try to find out by calculation the change of the acceleration of gravity upon the surface of δ Cephei and η Aquilae and thus to give a certain support to the pulsation theory.

The first investigator who took an interest in the change of the acceleration of gravity upon the surface of a Cepheid was Pannekoek[2]), and, later on, using the same method, Reesnick[3]). Making use of spectrograms of sufficient dispersion, the intensity of every single spectral line was estimated and from this the variation of surface gravity and of temperature was calculated. It appeared that the change of temperature was very easily noticeable, but with regard to the change in surface gravity only qualitative results could be obtained, maintaining the possibility of alterations.

The same problem is discussed in C. Payne's monograph "Stars of High Luminosity"[4]). Miss Payne compared the intensity of the spectral lines $H+H_\varepsilon$; K; Sr^+; G band; Ca^+/Ca; Fe^+/Fe at half of the ascending branch and at that of the descending and became sure of the change. Taking the temperature as equal in both of these phases, the change of the intensity of the spectral lines might be explained by the effect of pressure. At any rate, qualitatively, the results stood in agreement with the suggested effect.

1) L. O. B. 442.
2) B. A. N. 3, page 47.
3) B. A. N. 4, page 42.
4) Page 210—217.

F. Henroteau, investigating the change of ionization in the atmosphere of the Cepheids[1]), was able to show the influence of temperature, but a change of surface gravity was not found. The later papers by the same author do not add anything to the problem in this respect, for the measurement of spectral lines was made by the estimation of intensities with the bare eye, which was not exact enough to discover such a slight effect as the change of the acceleration of gravity in a Cepheid. P. ten Bruggencate[2]) in making use of the spectrograms of *Y* Sagittarii, *U* Sagittarii, *Y* Ophiuchi, which were obtained by means of the objective-prism, obtained better results as to the exactness of the observations. By comparing the H_γ, H_δ, and G lines in phases of equal temperature it appeared that a difference was to be noticed in the intensity, but in the opposite direction to that drawn from the pulsation theory. Ten Bruggencate does not ascribe it to the pulsation theory but suggests that the non-conformity is produced by the fact that the relation between temperature and colour is not identical on the ascending and descending branch of the light-curve.

Determinations of line contours by means of precise photometry have been made by Fred. L. Whipple[3]). The exactness of the results of Whipple's observations must be sufficient to discover the change of the surface gravity of a Cepheid, while — as we shall see later on — this change is not so small at all. Using the theoretical formulae as they are given by Eddington, Milne, Pannekoek, and Mentzel, Whipple computed the value of g at the time of the maximum and minimum of light. It appeared that the change of the acceleration of gravity is greater than that which follows from the pulsation theory. An increase in the hydrogen abundance of about 6000 times Russell's estimate for the sun gives values of the surface gravity in agreement with those computed from the period and mass-luminosity law.

In this paper, taking as a basis the theoretical relations between temperature, ionization, and surface gravity, the ionization of calcium has been computed in different phases and the

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1) Publ. D. Obs. Ottawa, vol. 9, No. 1.
2) Ann. Boscha Sternw. 5 A 1. 1931.
3) L. O. B. 442.

constants appearing in the formulae are derived from observation-materials. In using the results of Whipple's observations the following conclusions were drawn as summary:

1) In calculating the variation of surface gravity based upon the pulsation theory the following facts must be taken into account: I) the change of radius or the so-called "level effect" and II) the change of radial velocity or "velocity effect". The last component is especially of great importance, for the greater part of the changes both in surface gravity and atmospheric pressure must be ascribed to it, while "level effect" forms only a fractional part. All the investigators till now have, perhaps, left this very important factor outside the scope of their calculations.

2) The surface gravity of δ Cephei and η Aquilae is constant throughout the greater part of the time. Only about 0,1 of the period before the maximum light does it grow four times as much as its general value and then falls again rapidly (see fig. 2).

3) The variation of the surface temperature of δ Cephei and η Aquilae has been calculated on the basis of the obtainable material, and the results have been given in table V.

4) The change of ionization has been calculated from Whipple's measurements of the Ca^+ (K) and Ca (λ 4227) lines, and the results have been given in table VI.

5) By means of the theoretical relations found in this paper there has been computed, on the basis of the observed temperature and the surface gravity obtained through the assumption of the pulsation theory, the ionization of calcium, and this has been compared to the ionization found through observation. The agreement is sufficient if we assume that the amplitude of temperature does not surpass 600° C.

§ 2. The Method of Investigation and Theoretical Formulae.

As Miss C. Payne has already pointed out, the effect of surface gravity may be found if we compare the ionization at median decreasing and at median increasing light. In comparing the two median magnitudes we have effects related only to the differences in the surface gravity if we assume the temperatures to be the same at these phases. However, these conclusions are not confirmed, because, as was shown by Reesnick[1]), ten Bruggencate[2]) and Whipple[3]), the temperatures on the descending branch of the light-curve are lower than those on the corresponding light phases of the ascending branch. After all, Miss C. Payne's method is purely qualitative. In order to obtain quantitative results there must be found a theoretical relation among ionization, temperature, and surface gravity in the atmosphere of a Cepheid, whereby the constants which appear in the formulae ought to be derivable from the results of the observations. The last fact is especially important, for the constants found experimentally are very hypothetical, because laboratory conditions never totally correspond to the conditions existing upon stars. In the following we derive the needed formulae.

Let $n_r^{(s)}$ be the number of atoms of an element r times ionized and in the s-th state. Then we obtain:

$$n_r = n_r^{(s)} \frac{b_r(T)}{G_r^{(s)}\, e^{-\frac{E_s}{kT}}} = \frac{n_r^{(s)}}{B_r(T)} \quad \ldots (1)$$

B_r being the Boltzmann factor, $b_r(T)$ the partition function, $G_r^{(s)}$ the weight of the (s)-th level, E_s the excitation potential of the quantum level, and n_r the number of atoms r-times ionized.

According to Fowler and Milne there is in the stellar atmosphere in thermodynamical equilibrium:

$$p_\varepsilon \,.\, \frac{n_{r+1}}{n_r} = 2\,\frac{(2\ \pi m)^{\frac{3}{2}}\ (kT)^{\frac{5}{2}}}{h^3}\,\frac{b_{r+1}(T)}{b_r(T)}\, e^{-\frac{I_r}{kT}} \quad \ldots (2)$$

where p_ε is the partial electronic pressure, I_r the r-th ionization potential, and the other symbols have their customary significance.

The algebraic solution of (1) and (2) gives us, in the case of an element in two states $(r+1,\ s)$ and $(r,\ t)$, the following new equation:

$$p_\varepsilon\ \frac{n_{r+1}^{(s)}}{n_r^{(t)}} = A\,.\,T^{\frac{5}{2}} e^{-\frac{1}{KT}(I_r + E_s - E_t)} \quad \ldots (3)$$

where
$$A = 2\frac{(2\ \pi m)^{\frac{3}{2}} k^{\frac{5}{2}}}{h^3}\ \frac{G_{r+1}^{(s)}}{G_r^{(t)}}$$

is a constant, independent of temperature and pressure.

1) B. A. N. 4, page 42.

2) Ann. Boscha Obs. 2 C 23 and 2 B 58.

3) L. O. B. 442.

Setting

$$K(T) = T^{\frac{5}{2}} e^{-\frac{1}{kT}(I_r + E_s - E_t)}$$

we obtain

$$p_\varepsilon \frac{n_{r+1}^{(s)}}{n_r^{(t)}} = AK(T) \quad . \; . \; . \; (4)$$

We consider now a single line which is absorbed by atoms in the r, s state. Only a certain fraction x of all such atoms will absorb this particular line. This fraction, probably, depends upon quantum relations, though the details are not yet worked out except for the simplest case. As a working hypothesis we consider that the number of atoms which are able to absorb a certain line α will be:

$$n_r^{(s,\alpha)} = x n_r^{(s)} \quad . \; . \; . \; (5).$$

where x is a constant $0 < x < 1$ which is independent of temperature and pressure.

Thus formula (4) may be written

$$p_\varepsilon \frac{n_{r+1}^{(s,\alpha)}}{n_r^{(t,\beta)}} = A_1 K(T)$$

$$A_1 = A \frac{x_\beta}{x_\alpha}$$

or in the logarithmic form:

$$\log j = \log K(T) - \log p_\varepsilon + \text{const} \quad . \; . \; . \; (6)$$

$$\text{where } j = \frac{n_{r+1}^{(s,\alpha)}}{n_r^{(t,\beta)}}.$$

In her monograph "Stars of High Luminosity" Miss C. Payne gives a measurement of the K and 4227 lines of ionized and neutral calcium. In this particular case formula (6) takes the form:

$$\log j = 2{,}5 \log T - \frac{30700}{T} - \log p_\varepsilon + \text{const} \quad . \; . \; . \; (7)$$

Taking p_ε constant and computing by this formula $\log j + \text{const}$ we may compare it with C. Payne's measurements of calcium ionization.

Table I.

Temperature	$\log j$ + const according to formula	$\log j$ according to C. Payne
3000	—1,50	0,70
3500	+0,10	1,00
4000	1,35	1,20
4500	2,32	1,35
5000	3,13	1,45
5500	3,79	1,58
6000	4,34	1,65
6500	4,82	1,70
7000	5,25	1,75

If we take $\log j$ + const computed after formula (7) as abscissa and $\log j$ after Payne as ordinate, we must get a straight line, the equation of which is

$$(\log j)_{\text{theoretical}} = (\log j)_{\text{after Payne}}$$

But as it appears from fig. 1 this is not the case. The straight line

$$0{,}160\ (\log j)_{\text{theoretical}} = (\log j)_{\text{after Payne}}$$

better represents the condition.

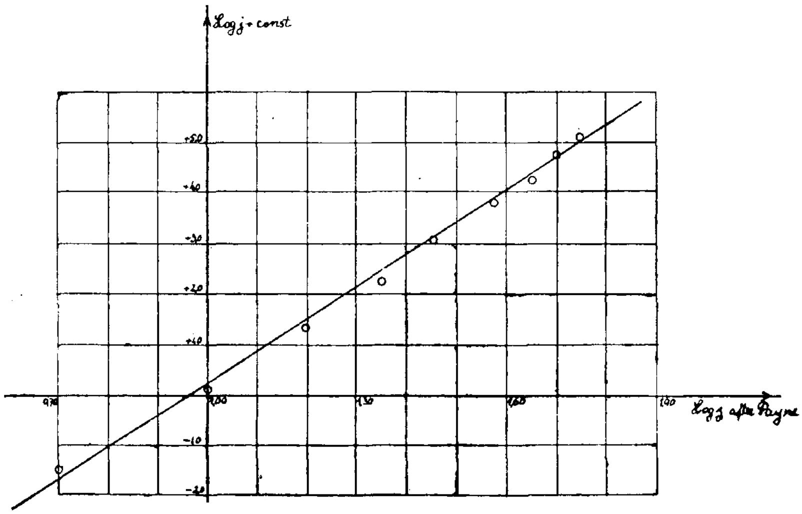

Fig. 1.

As an empirical formula, therefore, we take instead of (7) the following relation:

$$æ \log j = 2{,}5 \log T - \frac{30700}{T} - \log p_\varepsilon + \text{const} \quad . \ . \ . \ (8)$$

$$æ = 0{,}160$$

In our Cepheid problem the range of temperature variation is comparatively small. In this particular case we may simplify formula (8) without making any great mistake while writing it in the form:

$$æ \log j = \frac{T}{1000} - \log p_\varepsilon + \text{const} \quad . \; . \; . \; (9)$$
$$(5000^0 < T < 7000^0)$$

From the radial velocity curve we obtain only the variation of the surface gravity, but in formula (9) we need to know the variation of partial electronic pressure. It is a well-known fact that the electronic pressure is correlated to the surface gravity, but the exact formulae are not yet worked out. According to Pannekoek[1]) the pressure should be proportional to the surface gravity, and Milne suggests[2]) it varies just as the square root of gravity at the surface. We take as a working hypothesis the following relation:

$$p_\varepsilon = \text{const} \, . \, g^k \quad . \; . \; . \; (10)$$

where k is an exponent which may vary in a great range of g, but in our special problem of Cepheids we assume it to be a constant. According to Pannekoek it will be 1, according to Milne $^1/_2$[3]). According to (10) the formula (9) will be:

$$æ \log j = \frac{T}{1000} - k \log g + \text{const} \; . \; . \; . \; (11)$$

or if we measure the temperature not in grades but in other units, as for example in amplitudes of temperature variation, the formula may be written:

$$\log j = æ_1 \, T - k_1 \, \log g + \text{const} \quad . \; . \; . \; (12)$$

where $æ_1 = \frac{a}{æ}$ and a is an amplitude of the variation of temperature in thousand grades. This important formula we have to use in the following discussions.

§ 3. The Material of Observations used for the Investigation of the Change of Surface Gravity on δ Cephei and η Aquilae.

A. *The change of surface gravity on δ Cephei and η Aquilae, as calculated from the pulsation theory.*

Adopting the pulsation theory to explain the chief qualities of Cepheid variables, the surface gravity should vary, because:

1) B. A. N. 19. 1922.

2) M. N. 85, page 783.

3) Compare also C. Payne and F. Hogg: On the Pressures etc. H. C. 334.

1) the radius changes, 2) the velocity of the surface changes too producing forces in the atmosphere that increase or diminish the effect of gravitation. The first term is usually called "level effect", but the second term we propose to call "velocity effect". As mentioned already in the first chapter, the "velocity effect" is much more important than the "level effect" and, therefore, it must not be neglected as has been done hitherto throughout all discussions about the variation of surface gravity in Cepheids.

Let *v(t)* be the analytical form of the radial velocity curve, and let r be the variable radius of the star, then the surface gravity g will be:

$$g = g_0 + 2\, g_0 \frac{\Delta\, r}{r} + 1{,}5\, \frac{dv(t)}{dt}$$

g_0 being the surface gravity at $r = r_0$ without the "velocity effect". $2\, g_0 \frac{\Delta\, r}{r}$ is thus the "level effect", $\Delta\, r$ being the increase of radius. The term $1{,}5 \frac{dv}{dt}$ is the "velocity effect". Constant 1,5 is a correction factor for the integrated value of the whole disc in case the darkening across the disc might be neglected. When the darkening is the same as for the sun, the correction factor will be $^{27}/_{17}$.

Using the curve of the variation of radial velocity we are able to calculate $\Delta\, r$ in kilometres for each phase by integrating the curve. Forming a hypothesis concerning the $\frac{\Delta\, r_0}{r_0}$ — taking for example as a basis the relation inferred from the theory of A. S. Eddington — we can find out the value of the "level effect" if we know g_0. It must be pointed out that the "level effect" forms always a fixed $^0/_0$ of g_0 at any phase. The "velocity effect" may be found out by differentiating the curve of radial velocity. "Velocity effect" with regard to its absolute value remains independent of the magnitude of g_0. In summarizing both the effects — while one of them remains the same at a given phase, the other again is related to the value of g_0 — we obtain various curves of the change of g depending on the fact which g_0 we take as a basis. In the following table II, "level effect" and "velocity effect" are given. According to this, g is calculated at different assumptions of g_0 (table III).

Table II.

δ Cephei			η Aquilae		
Phase in days	$1{,}5\frac{dv}{dt}\left[\frac{cm}{sec^2}\right]$	$\frac{\Delta r}{r}$[1]	Phase in days	$1{,}5\frac{dv}{dt}\left[\frac{cm}{sec^2}\right]$	$\frac{\Delta r}{r}$
0,0	∓ 0,0	0,073	0,0	∓ 0,0	0,069
0,5	+14,8	0,044	1,0	+14,6	0,042
1,0	+16,6	0,021	2,0	+14,6	0,019
1,5	+15,4	0,007	3,0	+14,6	0,002
2,0	+15,4	0,001	4,0	+14,6	0,004
2,5	+17,3	0,004	5,0	+14,6	0,045
3,0	+17,2	0,016	6,0	+14,6	0,100
3,5	+19,5	0,036	6,5	—164,0	0,100
4,0	—	0,064	7,0	—29,0	0,075
4,3	± 0,0	—	7,2	∓ 0,0	0,069
4,5	—63,5	0,098			
4,75	—98,8	—			
5,0	—90,7	0,100			

For the calculation of table II the results of Jacobsen[2]) have been used. It may be mentioned here that the curve of the radial velocity of a Cepheid is variable especially in the case of η Aquilae and causes essential errors in $1{,}5\frac{dv}{dt}$. We adopt as the mean error of $1{,}5\frac{dv}{dt}$ the $\pm 5\frac{cm}{sec^2}$.

Table III.

δ Cephei				η Aquilae			
Phase in days	log g			Phase in days	log g		
	if $g_0=50$	$g_0=100$	$g_0=200$		if $g_0=50$	$g_0=100$	$g_0=200$
0,0	1,76	2,06	2,36	0,0	1,76	2,06	2,36
0,5	1,60	1,97	2,31	1,0	1,59	1,97	2,31
1,0	1,55	1,94	2,28	2,0	1,57	1,95	2,29
1,5	1,55	1,93	2,27	3,0	1,54	1,94	2,27
2,0	1,54	1,93	2,26	4.0	1,54	1,93	2,27
2,5	1,54	1,92	2,26	5,0	1,60	1,98	2,31
3,0	1,54	1,93	2,27	6,0	1,65	2,02	2,35
3,5	1,55	1,94	2,29	6,5	2,35	2,45	2,61
4,0	1,72	2,05	2,35	7,0	1,94	2,16	2,41
4,5	2,09	2,26	2,46				
4,75	2,20	2.34	2,53				
5,0	2,18	2,3[illegible]	2,52				

We represent the results of table III graphically in fig. 2. As we can see from it, the g remains more or less constant within the interval of the 0 and 0,8 phase. At the 0,8 phase there is the straight maximum. The greater g_0 is, the smaller is the

1) $\left[\frac{\Delta r}{r}\right]_{max}=0{,}1$. See Eddington: Internal Const. of Stars.

2) L. O. B. 12.

amplitude of g, for the influence of the "velocity effect" is absolute, and, relatively taken, its influence is the smaller the greater g_0 is. In the table there are given those values of g_0 which are probably in existence upon δ Cephei and η Aquilae.

B. *The variation of temperature in the atmosphere of δ Cephei and η Aquilae.*

In 1899 K. Schwarzschild pointed out the fact that the visual amplitude of η Aquilae is about twice as large as the photographic one, which might be explained by means of the change of colour, and, as a direct result, also of the temperature of Cepheid named. After Schwarzschild's discovery the temperature of Cepheids was investigated in various ways, but the results have not always been in perfect harmony, especially with regard to the amplitude of temperature. In the following table the most important results of the calculation of the surface temperature upon η Aquilae and δ Cephei have been summarized:

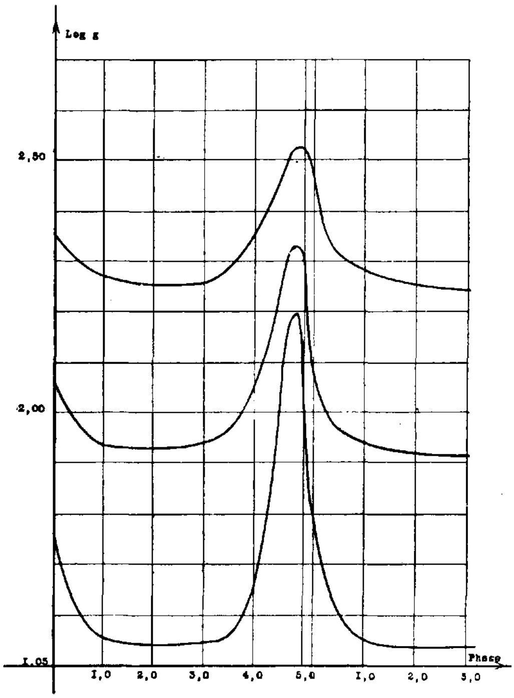

Fig. 2.
Variation of log g upon the surface of δ Cephei by g_0 being $50\,\frac{cm}{sec^2}$, $100\,\frac{cm}{sec^2}$ and $200\,\frac{cm}{sec^2}$.

Table IV.

δ Cephei.

Range of temp.	Method	References
6700—4780	colorimetric	Hopmann. A. N. 226, 1. 1925
6200—4580	radiometric	Pettit and Nicholson. Mt. W. Contr. 369. 1928
6400—5400	spectrometric	Pannekoek and Reesnick. BAN 3, 46. 1925
6650—4500	spectrophotometric	Whipple. LOB 442
5500—4700	radiometric	Pettit and Nicholson. Ap. J. 78, 5
400	spectrometric	K. Ogorodnikoff. Astron. Journ. of Soviet Union 10, 3

η Aquilae.

5240—3960	colorimetric	Hopmann. AN 226, 1. 1925
4950—3900	radiometric	Pettit and Nicholson. Mt. W. Contr. 369. 1928
5800—4300	spectrophotometric	Whipple. LOB 442
5100—4400	radiometric	Pettit and Nicholson. Ap. J. 78, 5

It appeared through comparison of the curves of temperature, calculated by the authors given in table IV, that the maximum of temperature corresponds to the maximum of luminosity, while the minimum of temperature appears earlier than the minimum of luminosity. This is the so-called asymmetry of temperature which had been pointed out already by Reesnick and others. It also appeared that the curves of temperature are identical as to their shape, but the zero point of temperature and its amplitudes given by different authors display an enormous difference. In order to find out the curve of a certain median temperature, the various temperatures, as given by several authors, were counted in the units of amplitude and the minimum of temperature was taken provisorily as zero. In such a scale all the curves of temperature, though given by different authors, are identical; and they may be used to find out the median curve. Taking at each phase the arithmetical mean of all the given temperature measurements we obtain a number of dots; and drawing through them a slight curve we get the change of the surface temperature of δ Cephei and η Aquilae. In table V there has been given after each phase the temperature found in the above mentioned way:

Table V.

δ Cephei				η Aquilae			
Phase in days	Temp.	Phase	Temp.	Phase	Temp.	Phase	Temp.
0,0	1,00	3,0	0,02	0	1,00	5	0,02
0,5	0,82	3,5	0,00	1	0,70	6	0,30
1,0	0,54	4,0	0,05	2	0,35	7	1,00
1,5	0,33	4,5	0,25	3	0,12		
2,0	0,20	5,0	0,82	4	0,02		
2,5	0,09						

C. *The ionization of calcium in the atmosphere of δ Cephei and η Aquilae.*

In order to obtain a degree of ionization in stellar atmosphere we need have measurements of the ionized and neutral lines of an element in question. In the case of δ Cephei and

η Aquilae we have the measurements of the line breadth of the calcium K (Ca^+) and λ 4227 (Ca) lines made by Whipple[1]). Using these observations, we can compute the value of $\log j$ given in formula (12). It is known that for line a which is absorbed by atoms $n_r^{(s\alpha)}$, the following relation can be written:

$$\log n_r^{(s\alpha)} = 2 \log (\lambda - \lambda_0) + C \quad . \; . \; . \; (13)$$

where $(\lambda - \lambda_0)$ is the line breadth and C is an atomic constant. According to formula (6) $j = \frac{n_{r+1}^{(s\alpha)}}{n_r^{(t\beta)}}$, and using (13) we can write:

$$\log \frac{n_{K\,Ca+}}{n_{\lambda\,4227\,(Ca)}} \equiv \log j = 2 \log \frac{[\lambda - \lambda_0]\,Ca+}{[\lambda - \lambda_0]\,Ca} \quad . \; . \; . \; (14)$$

In Whipple's publication $(\lambda - \lambda_0)$ is given at various r. We compute from formula (14) $\log j$ for each r. The following table gives the mean curve of $\log j$ with regard to r:

Table VI.

δ Cephei		η Aquilae	
Phase in days	$\log j$	Phase in days	$\log j$
0,0	0,23	0,0	0,30
0,5	0,29	1,0	0,35
1,0	0,33	2,0	0,32
1,5	0,33	3,0	0,24
2,0	0,31	4,0	0,17
2,5	0,28	5,0	0,15
3,0	0,26	6,0	0,18
3,5	0,23	6,5	0,13
4,0	0,20	7,0	0,15
4,75	0,10		
5,0	0,15		

After scattering of $\log j$ computed at various r and from the deviation of the mean curve, we adopt as the probable error of $\log j$ $\pm 0{,}04$.

Comparing table VI with Miss C. Payne's measurements of calcium ionization, it appears that she has obtained a somewhat larger amplitude in variation. The behaviour of the curve, however, is quite similar to that obtained from Whipple's data.

1) L. O. B. 442.

§ 4. Relation between Ionization, Temperature, and Surface Gravity in the Atmosphere of δ Cephei and η Aquilae.

The foregoing formula:

$$\log j = æ_1 T - k_1 \log g + C \quad . \; . \; . \; (12^{a})$$

gives the relation between ionization, temperature, and surface gravity in the atmosphere of a Cepheid. Presuming the radial-symmetric pulsation of the star and computing log g (table III) on basis of this hypothesis, the ionization (table IV) and the temperature (table IV) must satisfy the given formula. In table IV the temperatures have been given in units of amplitude and as zero has been taken the minimum temperature of the Cepheid; therefore:

$$t = \frac{T}{T_{max} - T_{min}} - \frac{T_{min}}{T_{max} - T_{min}}$$

Setting it into formula (12^{a}) we obtain:

$$\log j = æ' t - k_1 \log g + C_1 \quad . \; . \; . \; (12^{c})$$

where

$$æ' = æ_1 (T_{max} - T_{min}) \text{ and } æ_1 \text{ by page } 9 = 0{,}160$$

$$k_1 \text{ by page } 9 = 0{,}160\, k$$

If we set in formula (12^{c}) in place of log j, t and log g at each phase their values obtained correspondingly from tables III, IV and V, then, in case of certain values of $æ'$, k_1 and C_1 — if the pulsation theory be really true — the relation obtained must be satisfied. We obtain the values of the unknown constants by means of the method of least squares. As the values of log g depend upon which g_0 we take as a basis for the computation, there ought to be found a normal equation for each assumed g_0; and these group equations must be solved separately. The following represents the normal equations and their solutions together with the errors.

Table VII.

δ Cephei.

$g_0 = 50 \frac{cm}{sec^2}$

$$3{,}09\, æ' - 8{,}35\, k_1 + 4{,}60\, C_1 - 1{,}06 = 0$$
$$-8{,}35\, æ' + 36{,}90\, k_1 - 20{,}82\, C_1 + 0{,}98 = 0$$
$$4{,}60\, æ' - 20{,}82\, k_1 + 12{,}00\, C_1 - 2{,}83 = 0$$

Solution:

$æ' = 0{,}07 \pm 0{,}03$ $\quad k_1 = 0{,}30 \pm 0{,}07$ $\quad C_1 = 0{,}72 \pm 0{,}10$

$\mu = \pm 0{,}028$

$g_0 = 100 \frac{cm}{sec^2}$

$$3{,}09\, æ' - 9{,}65\, k_1 + 4{,}60\, C_1 - 1{,}06 = 0$$
$$-9{,}65\, æ' + 50{,}68\, k_1 - 24{,}59\, C_1 + 5{,}67 = 0$$
$$4{,}60\, æ' - 24{,}59\, k_1 + 12{,}00\, C_1 - 2{,}83 = 0$$

Solution:

$æ' = 0{,}08 \pm 0{,}03$ $\quad k_1 = 0{,}49 \pm 0{,}07$ $\quad C_1 = 1{,}22 \pm 0{,}10$

$\mu = \pm 0{,}029$

$$g_0 = 200 \frac{cm}{see^2}$$

$$3{,}09\ æ' - 10{,}95\ k_1 + 4{,}60\ C_1 - 1{,}06 = 0$$
$$-10{,}0\ æ' + 66{,}15\ k_1 - 28{,}15\ C_1 + 6{,}56 = 0$$
$$4{,}60\ æ_1 - 28{,}15\ k_1 + 12{,}00\ C_1 - 2{,}83 = 0$$

Solution:

$æ' = 0{,}08 \quad \pm 0{,}03 \quad k_1 = 0{,}77 \quad \pm 0{,}07 \quad C_1 = 2{,}02 \quad \pm 0{,}13$

$\mu = \pm 0{,}030$

η Aquilae.

$$g_0 = 50 \frac{cm}{sec^2}$$

$$3{,}14\ æ' - 7{,}63\ k_1 + 4{,}16\ C_1 - 0{,}98 = 0$$
$$-7{,}63\ æ' + 27{,}40\ k_1 - 15{,}54\ C_1 + 3{,}35 = 0$$
$$4{,}16\ æ' - 15{,}54\ k_1 + 9{,}00\ C_1 - 1{,}99 = 0$$

Solution:

$æ' = 0{,}15 \quad k_1 = 0{,}27 \quad C = 0{,}62$

$\pm 0{,}05 \quad \pm 0{,}09 \quad \pm 0{,}13$

$\mu = \pm 0{,}035$

$$g_0 = 100 \frac{cm}{sec^2}$$

$$3{,}14\ æ' - 8{,}79\ k_1 + 4{,}16\ C_1 - 0{,}98 = 0$$
$$-8{,}79\ æ' + 38{,}08\ k_1 - 18{,}46\ C_1 + 4{,}03 = 0$$
$$4{,}16\ æ' - 18{,}46\ k_1 + 9{,}00\ C_1 - 1{,}99 = 0$$

Solution:

$æ' = 0{,}13 \quad k_1 = 0{,}39 \quad C_1 = 0{,}96$

$\pm 0{,}05 \quad \pm 0{,}09 \quad \pm 0{,}15$

$\mu = \pm 0{,}037$

$$g_0 = 200 \frac{cm}{sec^2}$$

$$3{,}14\ æ' - 9{,}90\ k_1 + 4{,}16\ C_1 - 0{,}98 = 0$$
$$-9{,}90\ æ' + 49{,}70\ k_1 - 21{,}13\ C_1 + 4{,}64 = 0$$
$$4{,}16\ æ' - 21{,}13\ k_1 + 9{,}00\ C_1 - 1{,}99 = 0$$

Solution:

$æ' = 0{,}10 \quad k_1 = 0{,}47 \quad C = 1{,}28$

We assume:

$$(æ')_{\delta \text{Cephei}} = (æ')_{\eta \text{Aquilae}}$$
$$(k^1)_{\delta \text{Cephei}} = (k_1)_{\eta \text{Aquilae}}$$
$$(C^1)_{\delta \text{Cephei}} = (C_1)_{\eta \text{Aquilae}}$$

Under these conditions we have as best solutions for δ Cephei $g_0 = 100 \frac{\text{cm}}{\text{sec}^2}$, and for η Aquilae $g_0 = 200 \frac{\text{cm}}{\text{sec}^2}$.

As temperature amplitude we found by (12.)

$$T_{max} - T_{min} = \frac{æ'}{0{,}160} \approx 600^0$$

This value is less than that obtained from the observations of colour which give in the mean an amplitude about 1000° C. However, we shall pay attention to the results of K. Ogorodnikoff[1])

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1) Astron. Journ. of Soviet Union 10, 3.

who found from *Fe* lines the temperature amplitude of 400^0 C, and Pettit and Nicholson, who found 800^0 and 700^0 as amplitudes for δ Cephei and η Aquilae respectively. Further, the amplitude of temperature found by us depends upon the value of the æ's that is derived from very limited observation material, and thus is correct only with regard to the order.

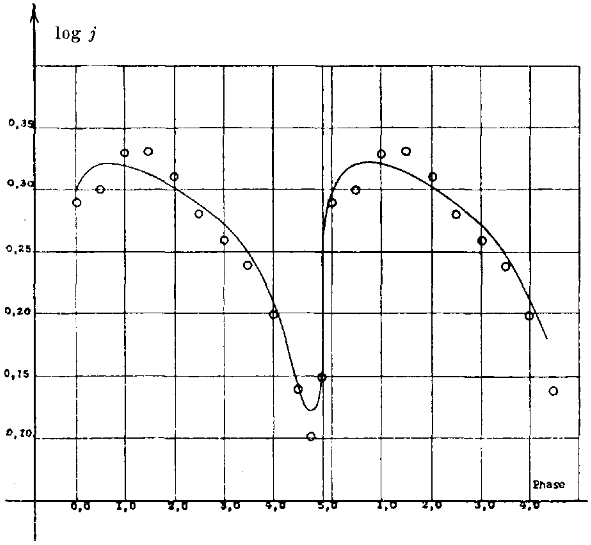

Fig. 3.

In order to illustrate how far the ionization found from the pulsation theory stands in accordance with the observed ionization, it must be computed the log j by the formula

$$\log j = 0{,}08\,t - 0{,}49 \log g + 1{,}22 \quad . \; . \; . \quad (14)$$

by means of table III, and the result must be compared with those of table V. This is illustrated in fig. 3 with regard to δ Cephei at $g_0 = 100 \frac{cm}{sec^2}$.

The observed ionization is represented by dots, while the curve represents ionization found from the pulsation theory. As it seems, the curve follows the observed dots quite well. From about 0,1 until 0,8 of the period the surface gravity is, according to fig. 2, more or less constant. In that time the ionization-curve is parallel with that of the temperature. From 0,8 of the period onwards the surface gravity grows very rapidly, becomien four times as great as its original value and causing an extensivg diminution of the ionization, though the temperature at this time grows rather intensely. While the surface gravity drops again quickly after its maximum, the ionization-curve has at about 0,9 of the period a sharp minimum where the curve rises almost vertically, reaching its maximum at about 0,1 of the period after the highest luminosity. Theoretically, the ionization-curve must be identical with the curve of temperature; only at about 0,9 of the period there is a certain deviation of the temperature-curve forming a sharp and deep minimum almost before the maximum of luminosity. The ionization-curve, therefore, in contrast with the curve of temperature, which is quite symmetrical, is much more asymmetrical than the curve of luminosity. The ionization of calcium discussed in this paper shows that effect. But also the ionization-curves obtained by estimating the intensity of the *Ti* lines λ 4534(*Ti*+) and λ 4534,7 (*Ti*)[1]) are much more asymmetrical than the light-curve and possess a deep minimum at about the zero phase and a rapid growth towards the maximum, thus pointing to the possible effect of surface gravity variation.

As the variation of surface gravity based upon the results obtained in this paper changes about three times its minimum value, this effect must display itself in the variation of the spectroscopic parallax of the Cepheid, for the spectroscopic absolute magnitude is really the effect of gravitation. Rich collections of radial velocity spectrograms of the Cepheids are in this respect a very grateful material, as their investigation would add very much to the understanding of the changes which occur in the atmosphere of the stars referred to.

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1) Publ. D. Obs. Ottawa, vol. 9, page 173.

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