COMPOSITE STELLAR MODELS

BY

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TARTU 1938

Printed by Ch. Mattiesen Ltd., Tartu, 1938.

Abstract.

Numerical illustrations to certain ideas of stellar structure and evolution, developed by the writer in a previous treatise[1], are given in this publication. "Dwarf" and "semi-giant" models, with a ratio of central to mean density ($\varrho_c : \varrho_m$) ranging from 6 to 265, and "giant" models, with $\varrho_c : \varrho_m$ from 5.10^6 to 4.10^{21}, are computed. In these computations, the assumed opacity is the usual convention, i. e. the Kramers opacity *plus* electron scattering with due allowance for the hydrogen content. The wide range in $\varrho_c : \varrho_m$ is attained partly as the result of the difference in the mean molecular weight assumed for different portions of the star, this difference being considered the result of evolution on an atomic synthesis basis with a gradual exhaustion of hydrogen (cf.[1]); partly the range in $\varrho_c : \varrho_m$ is influenced by a peculiar distribution of the energy sources (cf.[1]); the possibility of varying the type of hydrostatic structure, as represented by different combinations of the adiabatic and the radiative equilibrium states, adds to the range in $\varrho_c : \varrho_m$.

Formulae for numerical computations, although most of them are not new, are summarized; special corrections for the mass of radiant energy, and for the relativity red-shift effect upon the net flux of radiation are considered. The depression of luminosity in some supergiant stars, such as Trumpler's massive O-type stars, may be ascribed to the red-shift effect which may asymptotically tend to reduce the luminosity of a superdense contracting star to zero; in such a case these stars should possess a superdense core containing the major fraction of mass, in which the red-shift effect is considerable $\left(\frac{GM_r}{r} \sim c^2\right)$; and a vast inflated outer shell which we actually observe.

Introduction.

In a preceding paper[1] we have shown that composite stellar structures must originate in the course of normal stellar evolution. A central core of smaller (or no) hydrogen content is formed, around which one or several hydrogen-containing shells are placed. The hydrogen content in a given shell is larger, and consequently the mean molecular weight lower, than the corresponding mean values inside the shell. The state of equilibrium may be either radiative, or adiabatic (convective); additional variety in the resulting configurations is produced by the possibility of different states of equilibrium for each separate shell.

General and qualitative considerations referring to the conditions of origin of composite models and to their role in stellar evolution may be found in our above-mentioned paper[1]; here we propose to illustrate our former conclusions by a few actual computations of stellar models.

1. Formulae and Assumptions.

Our aim is to study the influence upon stellar configurations of a discontinuous distribution of the molecular weight, and of special distributions of the energy sources. Of the various physical conditions characterizing the state of matter inside a star, only the opacity is not known well enough. For our purposes it suffices to assume the same law of opacity in all cases; this we take as the combination of Kramers' opacity *) with electron scattering (non-relativistic, non-degenerate state of matter being presumed). The coefficient of opacity we thus assume to be given by

$$k = k_0 \varrho T^{-\frac{7}{2}} + 0.2\,(1 + X) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where ϱ is the density, T the temperature, X the hydrogen content (fraction of mass); k_0 is a function of the composition; for variable hydrogen content and Eddington's mixture as to the other elements we assume values of k_0 as quoted in Section 5. *h* of [1].

*) Using a certain mean value of the guillotine-factor.

The computations will be made by mechanical quadratures. The formulae, mostly well-known, are quoted here for the sake of completeness, although many of them may be found elsewhere, or follow in an obvious way from published formulae (cf.[2] and[3]). We notice that for our general case the ratio of radiation pressure cannot be considered as constant throughout the star, which circumstance renders unusable certain simplifications which Eddington introduced in his "standard" case.

The universal constants we assume according to[2]: $\log G = \bar{8}.8235$ (constant of gravitation); $\log \mathfrak{R} = 7.9168$ (gas constant); $\log a = \overline{15}.8832$ (constant of radiation).

The basic equations of the ideal gas in static equilibrium are:

$$\frac{dP}{dr} = -\frac{G\varrho M_r}{r^2} \qquad (2);$$

$$\frac{dP}{dx} = G\varrho M_r \qquad (2');$$

$$\frac{dM_r}{dr} = 4\pi\varrho r^2. \qquad (3);$$

$$\frac{dM_r}{dx} = -\frac{4\pi\varrho}{x^4} \qquad (3');$$

$$\Delta M_r = \frac{4}{3}\pi\bar{\varrho}(r_2^3 - r_1^3) \qquad (3'');$$

$$P = p_g + p_R \qquad (4);$$

$$p_R = \frac{1}{3} a T^4 \qquad (5);$$

$$p_g = \frac{\mathfrak{R}}{\mu}\varrho T \qquad (6);$$

$$\beta = \frac{p_g}{P} \qquad (7).$$

Here: P = total pressure; p_g = gas pressure; p_R = radiation pressure; r = radius; $x = \frac{1}{r}$; ϱ = density; M_r = mass inside a sphere of radius r; μ — mean molecular weight.

The actual use of one or another variant of a formula depends upon the character of the mechanical quadrature.

The equations of heat transfer are:

$$L_r = Q_r + C_r \quad \ldots \ldots \ldots \ldots \quad (8);$$

$$Q_r = \frac{16\pi ac T^3}{3k\varrho} \frac{dT}{dx} \quad \ldots \ldots \ldots \quad (9);$$

$$\frac{dT}{dx} = -r^2 \frac{dT}{dr} \quad \ldots \ldots \ldots \ldots \quad (10).$$

Here L_r is the net total outward flow of heat, Q_r the transfer of heat by radiation, C_r the transfer of heat by convection all through a sphere of radius r. As shown in [1], the luminosity of the star is equal to the maximum value of Q_r.

When one of the members in the opacity formula (1) prevails, it may be convenient to use the latter in a different form. When the Kramers opacity prevails,

$$k = k_0 \varrho T^{-\frac{7}{2}} F \quad \ldots \ldots \ldots \quad (1'),$$

$$\text{with } F = 1 + \frac{0.2\,(1+X)\,T^{\frac{7}{2}}}{k_0 \varrho}$$

differing little from 1.

When electron scattering prevails,

$$k = 0.2\,(1+X)\,F' \quad \ldots \ldots \ldots \quad (1''),$$

$$\text{with } F' = 1 + \frac{k_0 \varrho}{0.2\,(1+X)\,T^{\frac{7}{2}}}.$$

The above equations already permit us to treat numerically the problem of radiative equilibrium, i. e. of heat transfer by radiation only: $C_r = 0$; $L_r = Q_r$. For this purpose we must set Q_r equal to the sum of the physical energy sources inside r. A particular case is represented by the point-source model, for which $Q_r = L =$ const. However, in the point-source model convection necessarily takes place, at least in a central region; therefore the point-source radiative equilibrium model is physically (although not mathematically) impossible (cf.[1]).

A physically possible analogon of the point-source model is considered below; it is represented by an outer shell in radiative equilibrium and Q_r = const. for $r \geqslant r_1$; the physical energy sources are all placed inside r_1; r_1 cannot be chosen arbitrarily.

If no prescriptions with respect to Q_r are given, the problem can be rendered definite by assuming a certain equation of state; this may be done quite generally by the polytropic equation

$$n = \frac{d(\log \varrho)}{d(\log T)} \quad \ldots\ldots\ldots \quad (11),$$

where n is the (generally variable) polytropic index defining the density-temperature relation; the equations may be integrated by prescribing n as a function of radius. For the particular case of n = const. within the given limits r_1 and r_2 we have:

$$\varrho = \varrho_1 \left(\frac{T}{T_1}\right)^n \quad \ldots\ldots\ldots \quad (11').$$

The above equations [up to, and including (11), without using (11′)] lead to a highly convenient general formula for the temperature gradient:

$$\frac{dT}{dx} = \frac{GM_r}{\dfrac{(n+1)\,\Re}{\mu} + \dfrac{4}{3}\,a\,\dfrac{T^3}{\varrho}} \quad \ldots\ldots\ldots \quad (12).$$

For the particular case of (11′) (or n = const.), equation (12) can be approximately integrated and yields (for $n \neq 4$):

$$T_1 - T + \frac{4\,a\,\mu}{3\,\Re\,(4-n)(n+1)}\,\frac{T_1^{\,n}}{\varrho_1}\left(T_1^{\,4-n} - T^{4-n}\right) =$$

$$= \frac{G\,\mu\,\overline{M}_r}{\Re\,(n+1)}\,(x_1 - x) \quad \ldots\ldots \quad (12'),$$

where $\overline{M}_r$ is an average value of M_r which is supposed to vary little within the limits r_1 and r_2; (12′) evidently applies well to the outer regions of a star (extended atmospheres). For $n = 4$ we have instead of (12′):

$$T_1 - T + \frac{4}{15}\,\frac{a\,\mu}{\Re \log e}\,\frac{T_1^4}{\varrho_1}\log\frac{T_1}{T} = \frac{G\,\mu\,\overline{M}_r}{5\,\Re}\,(x_1 - x) \quad \ldots \quad (12'').$$

Further, we have:

$$\beta = \frac{1}{1 + \dfrac{a\,\mu}{3\,\Re}\,\dfrac{T^3}{\varrho}} \quad \ldots\ldots\ldots \quad (13);$$

$$Q_r = \frac{16\pi a c}{3k} \frac{G M_r}{\left[\frac{(n+1)\Re}{\mu}\frac{\varrho}{T^3} + \frac{4a}{3}\right]} \quad . \; . \; . \; . \; . \; . \quad (14).$$

The above formulae are general and may be used for computations of arbitrary stellar configurations. The variable polytropic index practically has an inferior limit in the adiabatic index of the material:

$$n \geqslant n_a \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (15).$$

When $n > n_a$, pure radiative equilibrium takes place; convection currents are absent (except those stimulated by rotation). When $n < n_a$, convection currents set in, and a considerable fraction of the heat is transported by convection; in [1] we have shown that the transport of heat by convection is so efficient that the departure from static adiabatic equilibrium is negligible; therefore, as soon as convection has started, $n = n_a$ represents an excellent approximation for practical purposes, and hence n_a may be considered an inferior limit for n. It is only in exceptional cases of a very low density and of a large flux of heat, that n may drop considerably below n_a.

In the computations made below we have neglected the influence of ionization heat upon the specific heats of the material, assuming thus the ratio of specific heats of the material to be $\Gamma = \frac{5}{3}$. Although at temperatures of the order of 10^7 ionization must considerably influence Γ, its neglect does not change our results in principle, and the quantitative changes are but slight. With higher temperatures, when complete ionization takes place, our assumption is probably correct.

With the value of $\Gamma = \frac{5}{3}$, we have (cf. also [2], p. 191):

$$n_a = 3 - \frac{1.5\beta}{4 - 3\beta} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (16).$$

For considerations of convectional stability it may seem more natural to use the density-pressure relation for the definition of the polytropic index, as has been done by Eddington on several occasions; however, our criterion of convectional stability (15) remains valid whatever definition of the effective polytropic index we may choose, provided the definition is the same for both n and n_a; the definition through the temperature-density relation which we have adopted has the advantage of making our formulae less complicated than would have been the case with the $P - \varrho$ relation.

When the polytropic index differs from 3, the ratio of radiation pressure to total pressure is not constant (cf. formula (13)); therefore, even when n and μ are constant, the well-known formula defining the central temperature as a function of mass, radius, molecular weight, and polytropic index (Eddington's formula (58.4), [2], p. 85), cannot be applied strictly; nevertheless, for practical use we may write the formula as follows:

$$T_c = \frac{R'}{(n+1)M'} \frac{G\mu}{\Re} \frac{\bar{\beta} M}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

Here $\bar{\beta}$ is a certain average value of β, M — the total mass, R — the radius of the boundary of the star; R' and M' are certain constants of Emden's tables (functions of n alone; cf.[2], p. 89).

In calculating $\bar{\beta}$, it appears to be natural to apply weighting by the increment of temperature itself; thus,

$$\bar{\beta} = \frac{\int_0^{T_c} \beta\, dT}{\int_0^{T_c} dT} = \int_0^1 \beta\, du, \text{ where } u = \frac{T}{T_c}.$$

With the aid of (13) and (11′) this leads finally to the following series ($n =$ const.):

$$1 - \bar{\beta} = \frac{a}{4-n} - \frac{a^2}{7-2n} + \frac{a^3}{10-3n} - \frac{a^4}{13-4n} + \ldots \quad (18),$$

where

$$a = \frac{1-\beta_0}{\beta_0}, \text{ and } \beta_0 = \frac{1}{1 + \frac{a\mu}{3\Re} \frac{T_c^3}{\varrho_c}}.$$

The series is convergent for $\beta_0 > 0.5$; however, the practical usefulness of (17) and (18) begins only with the larger values of β_0, especially so because a large radiation pressure (small β) must in one way or another lead to a variable polytropic index. For a very small $1-\beta$ the average radiation pressure $(1-\bar{\beta})$ does not differ very much from the value $1-\beta_3$ computed from Eddington's quartic equation with $n=3$, as shown by the following table (for β_0, cf. [2], p. 130):

$n =$	3	2.5	2.33	2.0	1.5
$\dfrac{1-\bar{\beta}}{1-\beta_3} =$	1.00	0.82	0.82	0.83	0.90.

According to (13), for $n<3$, β increases, $1-\beta$ decreases, with the decreasing temperature, i. e. with the increasing distance from the centre; for $n>3$ the reverse occurs. An interesting peculiar case is that of adiabatic equilibrium, when $n=n_a$ is itself variable, being defined by equation (16). From equations (11), (13), and (16) the following differential equation can be derived:

$$\frac{d\beta}{dT}=-\frac{(1-\beta)\beta^2}{T(4-3\beta)}.$$

Upon integration this yields:

$$\log\frac{T_1}{T_2}=1.1582\left(\frac{1}{\beta_1}-\frac{1}{\beta_2}\right)+\frac{2}{3}\log\frac{\beta_2(1-\beta_1)}{\beta_1(1-\beta_2)} \quad \ldots (19).$$

This equation represents the dependence of β upon T; by (16) and (13) it defines also the variable polytropic index, and the equation of state $\varrho=f(T)$ which corresponds to the assumed adiabatic variation of the polytropic index. When β is close to 1, or $1-\beta$ is small, $n=$ const. ~ 1.5 can nevertheless be assumed as a useful approximation.

It is difficult to derive an accurate formula for the heat transfer by convection; however, the order of magnitude of this quantity may be fairly well estimated. For our present purposes, it is not important to obtain here great precision: the formulae given below served us later chiefly to ascertain that, in cases when convection starts, adiabatic equilibrium sets in almost precisely, because convection turns out to be a much more powerful means of heat transport in stellar interiors than radiation is; how much more powerful, is not a very important question under these circumstances (cf. [1], Section 5. *a*).

Let $H=r_2-r_1$ be the absolute depth of the circulating current, ΔT its effective excess of temperature ($+\Delta T$ in the ascending branch, $-\Delta T$ in the descending one). In the absence of perceptible friction (as in the case of a large-scale current) the velocity of the current may be set equal to

$$v_c \simeq \sqrt{\frac{2g\,H\,\Delta T}{T}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (20),$$

where $g = \frac{GM_r}{r^2}$ is the acceleration of gravity. Further, let $\bar{\xi} = \frac{dT}{dr}$ be the average temperature gradient, and ξ_a the adiabatic temperature gradient; in the absence of a lateral exchange of heat, the change of state along the current is purely adiabatic; ΔT in this case should be equal to the difference of temperature between the ascending and the descending currents, the corresponding change ΔT taking place at the top and at the bottom of the circulating current; therefore

$$\Delta T \simeq 2\,(\xi_a - \bar{\xi})\;H \quad . \; . \; . \; . \; . \; . \; . \quad (21).$$

Hence the heat transfer by convection becomes

$$C_r \simeq 4\,\pi\,c_p \sqrt{G}\;r\,\varrho\,H^2 \sqrt{\frac{M_r\,(\xi_a - \bar{\xi})^3}{T}} \quad . \; . \; . \quad (22),$$

where c_p is the specific heat of the material; in formula (22) it has been assumed that only one quarter of $4\,\pi r^2$ is covered by the cross-section of the ascending current. A minimum value of c_p we obtain by neglecting the latent heat of ionization and atomic transmutations; in such a case we have

$$c_p = c_g \left[1 + \frac{32}{5}\,\frac{(1-\beta)}{\beta}\right] \quad . \; . \; . \; . \; . \; . \; . \quad (23),$$

where

$$c_g = \frac{\Re\,\Gamma}{\mu\,(\Gamma - 1)} = \frac{5}{2}\,\frac{\Re}{\mu}$$

is the specific heat of the monatomic gas (cf. [2]).

Most difficult it is to estimate the value of H. In certain experiments by Bénard [4], considered by Deslandres [5] as the clue for understanding the circulation of flocculi in the solar atmosphere, a shallow layer of liquid heated uniformly at the bottom produces a more or less steady system of circulation; the system consists of distinct cellules, with an ascending current at the centre and a descending one at the periphery; the average diameter of a cellule varies directly as the depth of the fluid; the latter must be small as compared with the width of the container to obtain a tolerably stable network of convective cellules. Inside a star a number of radially superposed systems of convective circulation may be supposed to exist; a limitation to the depth of each system is set partly by rotational deflection (cf.[1], Section 4. *f*), partly by the condition that the material cross-section of the current should not vary too suddenly, or too much. The material cross-section may be defined as the product $\varrho\sigma$, where σ is the geometric cross-section of the current. For a system of circulation placed inside a spherical layer the material cross-section is evidently proportional to ϱr^2. The assumption that the depth of the circulation is determined by the condition $\frac{\varrho_2 r_2^2}{\varrho_1 r_1^2} = e^{\pm 1}$, falls probably not far beside the point. For the terrestrial atmosphere, at least, with $\frac{r_2}{r_1} = 1$ practically, the condition yields a depth of about 7—8 kilometers, which is close to the observed order of magnitude. In the place of the above condition which implies discontinuity between the superposed systems of circulation, we introduce the fictitious conception of H varying continually with the radius, according to the formula

$$\frac{1}{H} \cong \frac{1}{\log e} \frac{d(\log \varrho)}{dr} + \frac{2}{r} \Bigg| \quad . \quad . \quad . \quad . \quad . \qquad (24),$$

which is the continuous differential equivalent of the first mentioned condition. Formula (24) may under certain circumstances yield $H = \infty$, a value which certainly cannot possess a real meaning with respect to (22); the following limitation of H to an upper limit appears, therefore, necessary as well as plausible:

$$H \leqslant \frac{1}{2} r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24').$$

In the above formulae the mass-density of radiant energy is neglected; although below, as in most other cases of stellar structure, the ratio of the mass-density of radiation to the "material" density, $\frac{\varrho'}{\varrho}$, is small and may be neglected, for the sake of completeness we quote the procedure by which the mass-density of radiation can be taken into account:

$$\frac{\varrho'}{\varrho} = \frac{a}{c^2} \frac{T^4}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),$$

with $\log \frac{a}{c^2} = 36.9290$. The right-hand sides of formulae (12), (14), (2), (2′), (3), (3′), and (3″) must be multiplied by $(1 + \frac{\varrho'}{\varrho})$. The procedure still implies the condition of the potential of gravitation to be small as compared with the square of the velocity of light. If the potential of gravitation is not small, an additional correction of formula (9) representing the net flux of radiation is required. The relativity red-shift correction, $\frac{d\nu}{\nu} = -\frac{G\,M\,dr}{c^2 r^2}$, is valid for a quantum of frequency ν travelling from a distance r to the distance $r + dr$; c is the velocity of light; this correction, if applied to the derivation of Q_r in the same manner as formula (9) is usually derived (variability of k, ϱ, or $k\varrho$ does not influence the formula), leads to the following "relativistic" formula for the flux of radiation:

$$Q_r = \frac{16\,\pi\,a\,c}{3\;k\varrho}\,T^3 r^2 \left(-\frac{dT}{dr}\right) \cdot \left[1 - \frac{G M_r T}{4\,c^2 r^2 \left(-\frac{dT}{dr}\right)}\right]. \quad . \quad . \quad (9').$$

The "relativistic" correction factor, bracketed in (9′), may reduce the radiation to arbitrarily small values; the factor, however, never can become negative, because the increase of $\frac{M}{r}$, necessary for this purpose, can take place only when accompanied by a positive flux Q_r: the extra energy of gravitational contraction ought to be radiated into space, otherwise contraction is not possible. Therefore, a star in the stage of gravitational collapse (as a whole, or in a massive core only) may show a luminosity decreasing according to (9′) as $\frac{M}{R}$ increases; the luminosity should approach zero asymptotically, and the rate of evolution should become slower as the luminosity decreases; it is, therefore, probable that very massive stars, of a fast initial rate of evolution (cf. T. P. **30.**$_1$, VIII; also[1]), will at present often be found in a stage of advanced contraction, with a considerably reduced luminosity; perhaps Trumpler's O-stars (cf.[1], loc. cit.) are specimens of such a class of objects, in which case, of course, the main mass of the star must be concentrated in an overdense core, whereas the observed surface would belong to an extensive atmosphere (an analogon to the envelopes of planetary nebulae).

2. Radiative Equilibrium with Constant Net Flux of Radiation.

This problem occurs when all physical sources of energy are concentrated inside a shell of $r = r_1$; outside r_1 radiative equilibrium with $Q_r = Q_{r1}$ takes place when convection is absent. A particular case is that of the so-called point-source model; difficulties arising in the numerical solution of this model have been described by Eddington (cf.[2], p. 124 f.); however, Eddington's mathematical solution, with the density partly decreasing inwards, cannot satisfy us: actually, a convective core is formed instead of the region of the computed negative density gradient. In the convective core, any non-uniformity of the distribution of the physical energy sources is smoothed out by the convectional transport of heat. Therefore, the true distribution of the physical energy sources in the core is of no importance for the type of the resulting structure; the same stellar structure results in the case of a physical point-source, and in the case of a more uniform distribution of the energy sources inside the core. For the sake of simplicity we may assume, in a particular case, that all physical energy sources are in the convective core, no energy being generated outside of it; in such a case $Q_r =$ const. is the equation of radiative equilibrium outside the core. A fixed value of Q_r, for fixed dimensions and structure of the core, generally does not lead to a solution of radiative equilibrium, as has been pointed out by Eddington for the point-source case (cf.[2], p. 125): "we thus overcondition the problem and generally fail to reach a solution"; the solution is to be found by a method of trial-and-error, involving laborious calculations. The situation is, however, somewhat different from the situation considered by Eddington, because we have to allow for the possibility of convective equilibrium, a circumstance disregarded in Eddington's purely mathematical solution.

In our present problem the effective polytropic index as defined by (11) is a variable quantity; the character of its variation, depending upon the assumed initial conditions, determines the character of the resulting solution; the polytropic index serves as the most useful guiding criterion in our trial-and-error computations.

It is proposed to consider first the case when the radiation pressure is small, or when β is close to 1; formula (13) tells us in such a case that the member $\frac{4a}{3}$ can be neglected as compared with $\frac{(n+1)\Re}{\mu}\frac{\varrho}{T^3}$ in (14), and we obtain, with a sufficient degree of approximation:

$$n+1=\frac{M_r}{Q_r}\frac{T^3}{k\varrho}\times \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

For $Q_r = \text{const.}$ and $k = \text{const.}$ (case of prevailing electron scattering), logarithmic differentiation of (26), together with (11), yields:

$$\frac{dn}{dr}=\frac{(n+1)}{M_r}\frac{dM_r}{dr}+\frac{(3-n)(n+1)}{T}\frac{dT}{dr} \quad . \quad . \quad . \quad (27).$$

We notice that $\frac{dM_r}{dr}$ is always positive, $\frac{dT}{dr}$ negative; therefore, the first member on the right-hand side of (27) is always positive, whereas the second member is positive when $n>3$, and negative when $n<3$. Thus, when at a certain distance r from the centre n happens to exceed 3, by (27) $\frac{dn}{dr}>0$, and n will steadily increase at an accelerated rate towards $n=\infty$; as a result, the solution in the outer portions of the model approaches an isothermal one, of a temperature T_∞ which is not zero (mostly a large fraction of T_c), although ϱ may approach zero rapidly; this solution cannot be considered a real one for individual stars belonging to our universe (where

$T_\infty \sim 3^0$ K in galactic space is practically zero), although it may apply to stellar models in an imaginary universe where the temperature of interstellar space is as high as T_∞.

When n is less than 3, the first member in (27) is positive, the second negative; according to the relative size of the two members, n may either increase or decrease. Characteristic of the final solution is the behaviour of $\frac{dn}{dr}$ in the outer portions of the star: $\frac{dM_r}{dr}$ approaches zero there, and the first member in (27) may be neglected; when $n < 3$ at the same time, $\frac{dn}{dr}$ is negative, and n decreases at an accelerated rate: as soon as n falls below n_a, convection starts; thus a second, outer convectional region originates, separated from the inner one by an intermediate shell in radiative equilibrium. And only in the limiting case, when in the outer portions n approaches 3, $\frac{dn}{dr} \rightarrow 0$, does the outer region of radiative equilibrium become complete. Summarizing we may say that $n = 3$ is a critical value for models in radiative equilibrium with $Q_r =$ const. and $k =$ const.: when n happens somewhere to exceed 3, there does not exist a real solution; when n never attains the value 3, a second convective region originates in the outer regions, of a width depending upon the initial conditions; and when n approaches asymptotically 3 as the surface of the star is approached, complete radiative equilibrium takes place outside the convective core.

For the Kramers law of opacity, and β near 1, the critical value of n is 3.25 (the proof is trivial, and the theorem not new; we omit the proof). For combined Kramers and electronic scattering opacity, the critical value of n lies somewhere between 3.25 and 3.

Not very different results are obtined when $1 - \beta$, although not negligible, is a small fraction of 1 (less than 0.2).

For fixed size and structure of the convective core the character of the ensuing solution is actually determined by the initial value of n according to (26), i. e. the value of the polytropic index just outside the boundary of the core; evidently $n+1$ is inversely proportional to Q_{r1} in such a case; a certain value of $Q_{r1} = Q_0$ leads to complete radiative equilibrium outside the core; when $Q_{r1} > Q_0$, an outer convective region is formed (composite model, cf.[1], Section 5. *f*, *h*), increasing in extent as Q_{r1} increases, joining the inner convective region at a certain value of $Q_{r1} = Q_a$ (complete adiabatic model, cf.[1], Section 5. *e*, *g*). When $Q_{r1} < Q_0$, no reasonable solution does exist.

Let Q_i be the net flux of radiation just inside the boundary of the convective core; when the initial conditions are arbitrarily chosen, Q_i generally need not be equal to Q_0. When $Q_i \geqslant Q_a$, in no portion of the star can radiative equilibrium exist: the model is of the complete adiabatic type (actually, in the normal course of stellar evolution, Q_i never can exceed Q_a, cf.[1], Section 5. *g*). When $Q_a > Q_i > Q_0$, the model is of the composite type, with two convective regions separated by a shell in radiative equilibrium. When $Q_i = Q_0$, the composite model consists of a complete radiative shell surrounding a convective core. When $Q_i < Q_0$, no solution of $Q_r = Q$ exists; Q_0 is a minimum value of the total flux of heat at the boundary of the convective core, and, in the absence of adequate physical sources of energy at, or outside r_1, an additional supply of heat C_i by convection from the core must be stimulated inevitably, so that $Q_0 = Q_i + C_i$, and the case becomes again one of complete radiative equilibrium in the outer shell. It is also conceivable that convection provides even more than this necessary minimum, $Q_a \geqslant Q_i + C_i \geqslant Q_0$, or $Q_{r1} = Q_i + C_i$, so that the other types of stellar structure described above become possible for the case $Q_i < Q_0$, too.

When there is a core of smaller hydrogen content, or of greater mean μ as compared with the shell, the convectional currents of the core cannot continue by inertia far into the shell: a considerable transport of heat by convection at the boundary of the core can exist, and, at the same time, the convectional motion may cease abruptly outside the core (cf.[1], Section 5. *h*). Therefore, for such a model with a heavy core, the above types of stellar structure can be obtained, for a fixed core, by a suitable choice of $Q_{r1} = Q_i + C_i$, which depends upon the more or less arbitrary value of C_i.

When the star is of a uniform composition throughout, the search for a solution is somewhat more complicated. There is no reason for a system of convection currents to stop at the boundary of the core unless it gradually dies out inside the core already, as the boundary of the latter is approached; in such a case, however, the convectional transport of heat must also gradually die out and reach the value $C_i = 0$ at the boundary of the core. Hence $Q_{r1} = Q_i$; for a core of fixed structure (including central conditions) and for a given radius r_1 there exists only one type of solution; Q_{r1} cannot be varied at will, and the different types of solutions in the case of homogeneous composition can be obtained only by varying r_1, the radius of the convective core. Two critical values of the radius exist, $r_1 = r_{min}$, corresponding to $Q_{r1} = Q_0$, and $r_1 = r_0 > r_{min}$, corresponding to $Q_{r1} = Q_a$ (at the same time this is Q_{max} in the polytropic structure of the core, cf.[1], Section 5. *b*); for $r_1 < r_{min}$ no solution exists; for $r_1 = r_{min}$ the outer shell is in complete radiative equilibrium; when $r_0 > r_1 > r_{min}$, the type of structure with two convective regions results; when $r_1 \geqslant r_0$, the structure is adiabatic throughout.

Near the critical initial value $r_1 = r_{min}$, the solution is extremely sensitive with respect to the initial conditions chosen; small changes in r_1 lead to large changes in the resulting mass M and the radius R of the model. Thus, in Table 1 below, Models No. 6 and 7, the relative difference in r_1 is only $\frac{\Delta r_1}{r_1} = -0.00001$, whereas $\frac{\Delta M}{M} = +0.009 = -900 \frac{\Delta r_1}{r_1}$, and $\frac{\Delta R}{R} = +0.33 = -33000 \frac{\Delta r}{r_1}$. In the present case this, however, is a purely computational instability which cannot be reflected in actual circumstances of nature. For a given star, the mass M is a constant, and our models of variable mass cannot be considered as referring to one and the same star. For $M = \text{const.}$, the character of the instability depends upon an additional condition. For the rather artificial condition $R = \text{const.}$, $\frac{dR}{dr_1} = 0$ and no instability exists. With $T_c = \text{const.}$ (cf. Table 3), the difference between Models No. 6 and 7 is: $\frac{\Delta r_1}{r_1} = -0.009$, $\frac{\Delta R}{R} = +0.30 = -33 \frac{\Delta r_1}{r_1}$, or 1000 times smaller than in our original data of Table 1; here the changes in r_1 and R are now comparable with respect to their order of magnitude.

As to the evolutional course, if the different types of structure can change one into the other, the completely radiative outer shell should be the first type of equilibrium at which the contracting star settles down because this type corresponds to minimum luminosity ($= Q_0$) and maximum radius (cf. below), thus to a minimum central temperature and density sufficient to stimulate the subatomic energy sources and to keep a balance with the radiation into space, removing thus the need of a further supply of gravitational energy by contraction; even for constant T_c, the configuration of the complete radiative shell possesses the greatest radius, and the greatest potential energy of gravitation among the types of structure considered here. Therefore, the natural order of evolution may be the following: the composite model of complete radiative structure outside the convective core; the composite model of two convective regions separated by a layer in radiative equilibrium, the width of the layer exhibiting a decreasing tendency; the complete adiabatic model. This course of evolution may, or may not take place; if in the core the exhaustion of hydrogen is accomplished before the complete adiabatic model is reached, the collapse of the core devoid of subatomic energy sources must change the direction of evolution, leading to giant structure (cf.[1], Sections 5 and 6). We are not going to discuss here in more detail the question of the evolution of the composite model, although this would be of considerable interest; a special discussion of this subject, based on more computational and observational data, may follow.

3. Composite Models of Uniform Molecular Weight.

A set of models of uniform composition based upon the same structure (not extent) of the convective core is computed below (Table 1). As shown in the preceding section, the choice of the model is determined by the boundary radius of the core. The latter is assumed to represent the central portion of a certain polytrope, built according to a constant adiabatic polytropic index; for the sake of simplicity we assumed the latter to be $n_a = 1.5$, although in the dwarf model here considered, n_a may vary between 1.7 and 1.6; the variation of n_a along the radius is also neglected, because $1 - \beta$ is small [cf. formula (16)]. No serious difference in the conclusions can result from such a small difference in n_a, where the advantage of the round constant value adopted is that it makes possible the use of Emden's tables. The molecular weight is assumed as equal to $\bar{\mu} = 1.063$, and $\log k_0 = 25.0660$; these figures are supposed to correspond to $33\frac{1}{3}$ per cent hydrogen.

The absorption coefficient is most conveniently calculated from (1′) in the present case: the role of electron scattering is small here, and $F = 1$ might have been assumed; nevertheless, this simplification was not introduced, the correction factor F having been always taken into account. In the external portions of the complete radiative shell (Model No. 7, Table 1), as soon as n approached 3, $k = \text{const.} \sim \varrho T^{-3}$ was assumed for the sake of simplification; the last stage of the calculation could be accomplished in this case according to the polytropic scheme ($n = 3$), which does not involve so much labour as the radiative equilibrium scheme does. Such a simplification for the outer layers of a star may correspond to actual circumstances: our simplification corresponds to the assumption of a progressively smaller opacity, as compared with the Kramers value; now, the decreasing relative number of free electrons, caused by the decreasing degree of ionization, should tend to reduce the Kramers component of opacity as the boundary of the star is approached.

Table 1.

Models of Uniform Composition ($33\frac{1}{3}$ per cent hydrogen; C. G. S. units and degrees K).

a) Model No. 1.

Complete adiabatic, $n_a = 1.5$. $T_c = 13{,}2.10^6$ deg; $\log \varrho_c = 0.9277$; $1 - \beta_c = 8{,}84.10^{-3}$; $1 - \bar{\beta} = 3{,}54.10^{-3}$; $R = 6{,}971.10^{10}$ cm; $M = 20{,}016.10^{32}$ g; $\log Q_{max} = 33.6707 \frac{\text{erg}}{\text{sec}}$; $m_{bol} = 4.41$;

$$\varrho_c : \varrho_m = 6.00.$$

Table 1. Continued.

β) Model No. 2.

Composite, two convectional regions separated by a shell in radiative equilibrium.

Central conditions as before; $r_1 = 2{,}376.10^{10}$ cm; $R = 6{,}986.10^{10}$ cm; $M = 19{,}964.10^{32}$ g; $\log Q_{r1} = 33.6578$; $m_{bol} = 4.45$; $\varrho_c : \varrho_m = 6.06$.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{32} g	p_g 10^{15} $\frac{\text{dyne}}{\text{cm}^2}$	p_R 10^{13} $\frac{\text{dyne}}{\text{cm}^2}$	n
			Central convective region			
0	13.200	0.9277	0	8.682	7.736	1.50
2.376	10.136	0.7554	3.773	4.482	2.687	
		Shell in radiative equil., $Q_r = Q_{r1} =$ const.				
2.4	10.080	0.7518	3.871	4.422	2.631	1.50
2.5	9.849	0.7366	4.289	4.172	2.397	1.52
2.6	9.617	0.7208	4.726	3.927	2.178	1.57
2.8	9.148	0.6868	5.651	3.455	1.783	1.58
3.0	8.667	0.6497	6.634	3.005	1.436	1.49
3.2	8.158	0.6107	7.663	2.586	1.128	
		External convective region, $n = n_a = 1.50$				
3.4	7.644	0.5683	8.725			
3.6	7.127	0.5227	9.805			
3.8	6.611	0.4737	10.888			
4.0	6.098	0.4211	11.958			
4.2	5.591	0.3645	13.001			
4.4	5.092	0.3037	14.003			
4.6	4.604	0.2381	14.952			
4.8	4.128	0.1670	15.837			
5.0	3.665	0.0895	16.647			
5.2	3.218	0.0047	17.375			
5.4	2.788	$\bar{1}.9113$	18.016			
5.6	2.375	$\bar{1}.8067$	18.565			
6.0	1.604	$\bar{1}.5511$	19.371			
6.4	0.907	$\bar{1}.1797$	19.819			
6.8	0.282	$\bar{2}.4186$	19.957			
6.986	0	$-\infty$	19.964			

Table 1. Continued.

γ) Model No. 3.

Type of structure and central conditions as before;

$r_1 = 2{,}281.10^{10}$ cm; $R = 7{,}168.10^{10}$ cm; $M = 20{,}358.10^{32}$ g;
$\log Q_{r1} = 33.6433$; $m_{bol} = 4.49$; $\varrho_c : \varrho_m = 6.42$.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{32} g	p_g 10^{15} dyne/cm²	p_R 10^{13} dyne/cm²	n
			Central convective region			
0	13.200	0.9277	0	8.682	7.736	1.50
2.281	10.348	0.7690	3.393	4.722	2.919	
		Shell in radiative equil., $Q_r = Q_{r1} = \text{const.}$				
2.3	10.305	0.7664	3.466	4.674	2.872	1.50
2.4	10.079	0.7520	3.864	4.423	2.63	1.63
2.6	9.633	0.7200	4.718	3.927	2.19	1.76
2.8	9.193	0.6843	5.640	3.452	1.82	1.83
3.0	8.756	0.6456	6.616	3.007	1.50	1.87
3.2	8.316	0.6038	7.632	2.594	1.22	1.84
3.4	7.866	0.5595	8.675	2.216	0.976	1.69
3.6	7.394	0.5138	9.733	1.875	0.761	
		External convective region, $n = n_a = 1.50$				
3.8	6.881	0.4672	10.797			
4.0	6.372	0.4169	11.854			
4.2	5.869	0.3634	12.891			
4.4	5.374	0.3060	13.895			
4.6	4.889	0.2444	14.854			
4.8	4.415	0.1780	15.756			
5.0	3.955	0.1064	16.593			
5.2	3.509	0.0284	17.356			
5.4	3.079	$\bar{1}.9432$	18.039			
5.6	2.665	$\bar{1}.8492$	18.637			
6.0	1.888	$\bar{1}.6246$	19.559			
6.4	1.182	$\bar{1}.3195$	20.131			
7.0	0.254	$\bar{2}.3178$	20.353			
7.168	0	$-\infty$	20.358			

Table 1. Continued.

δ) Model No. 4.

Type of structure and central conditions as before;
$r_1 = 2{,}234.10^{10}$ cm; $R = 7.743.10^{10}$ cm; $M = 21.368.10^{32}$ g;
$\log Q_{r1} = 33.6349$; $m_{bol} = 4.51$; $\varrho_c : \varrho_m = 7.70$.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{32} g	p_g 10^{15} $\frac{\text{dyne}}{\text{cm}^2}$	p_R 10^{13} $\frac{\text{dyne}}{\text{cm}^2}$	n
Central convective region						
0	13.200	0.9277	0	8.682	7.736	1.50
2.234	10.452	0.7756	3.212	4.843	3.04	
Shell in radiative equil., $Q_r = Q_{r1} =$ const.						
2.3	10.304	0.7665	3.464	4.675	2.87	1.55
2.4	10.083	0.7519	3.862	4.424	2.63	1.70
2.6	9.649	0.7195	4.715	3.929	2.21	1.85
2.8	9.225	0.6835	5.635	3.458	1.85	1.98
3.0	8.810	0.6439	6.608	3.014	1.54	2.11
3.2	8.404	0.6007	7.619	2.603	1.27	2.18
3.4	8.007	0.5546	8.653	2.230	1.05	2.29
3.6	7.617	0.5051	9.695	1.8932	0.858	2.31
3.8	7.234	0.4532	10.731	1.5952	0.697	2.33
4.0	6.855	0.3989	11.750	1.3340	0.562	2.27
4.2	6.474	0.3427	12.742	1.1072	0.448	2.08
4.4	6.079	0.2858	13.699	0.9118	0.348	1.81
4.6	5.657	0.2290	14.619	0.7445	0.261	
External convective region, $n = n_a = 1.50$						
4.8	5.191	0.1731	15.500			
5.0	4.737	0.1133	16.339			
5.2	4.297	0.0499	17.127			
5.4	3.871	$\bar{1}.9818$	17.859			
5.6	3.460	$\bar{1}.9087$	18.529			
6.0	2.682	$\bar{1}.7427$	19.660			
6.4	1.967	$\bar{1}.5408$	20.506			
7.0	1.012	$\bar{1}.1079$	21.219			
7.6	0.191	$\bar{2}.0114$	21.365			
7.743	0	$-\infty$	21.368			

Table 1. Continued.

ε) Model No. 5.

Type of structure and central conditions as before; $r_1 = 2{,}224.10^{10}$ cm; $R = 8{,}401.10^{10}$ cm; $M = 22{,}116.10^{32}$ g; $\log Q_{r1} = 33.6332$; $m_{bol} = 4.52$; $\varrho_c : \varrho_m = 9.51$. The computations start at $r = 4{,}4.10^{10}$ cm, by interpolation of trial models.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{32} g	p_g 10^{15} $\frac{\text{dyne}}{\text{cm}^2}$	p_R 10^{13} $\frac{\text{dyne}}{\text{cm}^2}$	n
Central convective region						
0	13.200	0.9277	0	8.682	7.736	1.50
2.224	10.473	0.7770	3.177	4.868	3.07	
Shell in radiative equil., $Q_r = Q_{r1} =$ const.						
4.4	6.242	0.2789	13.663	. . .	. . .	
						2.63
4.6	5.914	0.2173	14.564	0.7576	0.312	
						2.61
4.8	5.594	0.1541	15.415	0.6196	0.249	
						2.55
5.0	5.277	0.0895	16.213	0.5038	0.198	
						2.38
5.2	4.957	0.0247	16.958	0.4075	0.154	
						2.11
5.4	4.619	$\bar{1}.9600$	17.651	0.3272	0.116	
						1.67
5.6	4.241	$\bar{1}.8982$	18.297	0.2606	0.0824	
External convective region, $n = n_a = 1.50$						
6.0	3.472	$\bar{1}.7678$	19.447			
6.4	2.762	$\bar{1}.6188$	20.400			
7.0	1.802	$\bar{1}.3404$	21.419			
7.6	0.964	$\bar{2}.9333$	21.969			
8.2	0.238	$\bar{2}.0207$	22.110			
8.401	0	$-\infty$	22.116			

Table 1. Continued.

ζ) Model No. 6.

Type of structure and central conditions as before;

$r_1 = 2{,}22242.10^{10}$ cm; $R = 9{,}648.10^{10}$ cm; $M = 22{,}802.10^{32}$ g; $\log Q_{r1} = 33.6330$; $m_{bol} = 4.52$; $\varrho_c : \varrho_m = 13.96$. The computations start at $r = 5{,}6.10^{10}$ cm, by interpolation of trial models.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{32} g	p_g 10^{14} dyne/cm²	p_R 10^{12} dyne/cm²	n
Central convective region						
0	13.200	0.9277	0	. . .	. . .	1.50
2.222	10.477	0.7772	3.172	. . .	. . .	
Shell in radiative equil., $Q_r = Q_{r1} =$ const.						
5.6	4.563	$\bar{1}.8801$	18.237	. . .	. . .	2.98
5.8	4.318	$\bar{1}.8090$	18.808	2.161	0.887	2.96
6.0	4.083	$\bar{1}.7367$	19.326	1.730	0.708	2.95
6.2	3.857	$\bar{1}.6636$	19.795	1.3807	0.564	2.91
6.4	3.636	$\bar{1}.5894$	20.217	1.0972	0.445	2.79
6.6	3.418	$\bar{1}.5147$	20.595	0.8687	0.348	2.59
6.8	3.200	$\bar{1}.4402$	20.934	0.6846	0.267	2.29
7.0	2.970	$\bar{1}.3663$	21.237	0.5361	0.198	1.77
7.2	2.710	$\bar{1}.2960$	21.509	0.4161	0.137	
External convective region, $n = n_a = 1.50$						
7.6	2.166	$\bar{1}.1500$	21.969			
8.2	1.434	$\bar{2}.8814$	22.456			
8.8	0.791	$\bar{2}.4942$	22.721			
9.4	0.226	$\bar{3}.6780$	22.797			
9.648	0	$-\infty$	22.802			

Table 1. Continued.

η) Model No. 7.

Complete radiative outer shell, central conditions as before; $r_1 = 2{,}22240.10^{10}$ cm; $1-\bar{\beta} = 5{,}21.10^{-3}$; $R = 12{,}658.10^{10}$ cm; $M = 22{,}989.10^{32}$ g; $\log Q_{r1} = 33.6329$; $m_{bol} = 4.52$; $\varrho_c : \varrho_m = 31.27$. The computations start at $r = 5{,}6.10^{10}$ cm, by interpolation of trial models.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm^3	M_r 10^{32} g	p_g 10^{14} $\frac{\text{dyne}}{\text{cm}^2}$	p_R 10^{12} $\frac{\text{dyne}}{\text{cm}^2}$	n
			Central convective region			
0	13.200	0.9277	0	. . .	. . .	1.50
2.222	10.477	0.7772	3.172	. . .	. . .	
			Shell in radiative equil., $Q_r = Q_{r1} =$ const.			
5.6	4.568	$\bar{1}.8799$	18.236	. . .	. . .	3.01
5.8	4.325	$\bar{1}.8086$	18.806	2.162	0.891	3.04
6.0	4.093	$\bar{1}.7360$	19.324	1.7315	0.715	3.03
6.2	3.872	$\bar{1}.6626$	19.792	1.3829	0.572	3.05
6.4	3.659	$\bar{1}.5879$	20.213	1.1004	0.457	
			Same, simplified calculation: $n =$ const. $= 3$; $k =$ const. $= k$ at $r = 6{,}4.10^{10}$.			
7.0	3.069	$\bar{1}.3584$	21.217			
7.6	2.549	$\bar{1}.1169$	21.910			
8.2	2.094	$\bar{2}.8604$	22.368			
8.8	1.694	$\bar{2}.5847$	22.655			
9.4	1.342	$\bar{2}.2811$	22.824			
10.0	1.030	$\bar{3}.9367$	22.915			
10.6	0.753	$\bar{3}.5281$	22.958			
12.658	0	$-\infty$	22.989			

The seven models presented in Table 1 cannot be transformed one into another by homologous changes of mass, radius, temperature, etc. Thus each of these seven models is a representative of a particular type of stellar structure, non-polytropic, except Model No. 1. Each of these models, however, allows of a homologous transmutation of the variables. To compare the properties of the models under similar conditions, we reduced the data to $M = M_{\odot}$, $R = R_{\odot}$, thus to the observed mass and radius of the sun; the results are contained in Table 2. In this table, L is the luminosity $(\frac{\text{erg}}{\text{sec}})$, m_{bol} —

Table 2.

Characteristic Data for the Models of Table 1, reduced to Solar Mass and Solar Radius.

($n = 1.50$; $33\frac{1}{3}$ per cent hydrogen; uniform composition)

Model No.	1	2	3	4	5	6	7
$M_{r1} : M$	0.2496	0.1890	0.1667	0.1503	0.1434	0.1392	0.1381
$(1-\bar{\beta}) \times 10^3$	3.4	3.9	3.7	3.6	3.5	3.6	3.9
$10^{-6} \cdot T_c$	13.1	13.2	13.3	13.6	14.2	15.8	20.6
$\varrho_c : \varrho_m$	6.00	6.06	6.42	7.70	9.51	13.96	31.27
$\log L - 33$	0.652	0.650	0.594	0.487	0.421	0.379	0.417
m_{bol}	4.46	4.47	4.61	4.88	5.04	5.15	5.05
α	2.018	2.025	2.302	2.951	3.439	3.788	3.457

the bolometric magnitude, α — Eddington's divisor of the luminosity [cf. [1], form. (35)]; $\alpha = 2.5$ has been considered by Eddington as a fair guess, allowing for the uncertainty in the distribution of the energy sources inside the star (cf. [2]); for the average of the models of Table 2, $\alpha = 2.5$ appears to be quite a satisfactory approximation. $M_{r1} : M$ is the fraction of mass contained in the convective core. The observed bolometric magnitude of the sun, 4.65, corresponds most closely to Model No. 3; however, this result depends entirely upon the assumed hydrogen content, $33\frac{1}{3}$ per cent. With 35 per cent hydrogen, Model No. 1, and with 29 per cent, Model No. 7 lead to an agreement between the observed and the computed luminosity. As shown in [1], there are several reasons to believe that the sun is a complete adiabatic structure, like Model No. 1. If this were so, $T_c \sim 13.10^6$ may be considered the more or less fixed central temperature for stars of solar mass and of approximately the solar luminosity: the high sensitivity of the yield of subatomic energy to changes in T_c warrants the practical constancy of the central temperature. In such a case the different models of Table 2 cannot represent stages of evolution of the same star; in the eventual course of evolution, T_c should remain nearly constant, and the radius should vary. Disregarding evolution, we may state that a star of given mass and composition may assume configurations corresponding to one of our models, in such a manner, that the central temperature remains nearly constant. Table 3 represents data for such configurations.

Table 3.

Radii and Luminosities of the Models of Table 1, for $T_c = 13{,}1.10^6$ and $M = M_\odot$.

($n_a = 1.5$; $33\frac{1}{3}$ per cent hydrogen; uniform composition)

Model No.	1	2	3	4	5	6	7
$R : R_\odot$	1.000	1.007	1.013	1.042	1.092	1.217	1.584
$\bar{\varrho} : \varrho_\odot$	1.00	0.98	0.96	0.88	0.77	0.55	0.25
$\Delta m_b = 1.25\ R/R_\odot$	0.00	0.00	+0.01	+0.02	+0.05	+0.11	+0.25
m_{bol}	4.46	4.47	4.62	4.90	5.09	5.26	5.30
ϱ_c	6.0	6.0	6.2	6.8	7.3	7.7	7.8

In the third line of the table, Δm_b is the correction to be added to m_b of Table 2; the correction is equivalent to Eddington's effective temperature term, $-2 \log T_e$ (cf.[2], p. 137), and follows directly from [1], form. (35), if electron scattering is disregarded; for our models the latter is small indeed, being of the order of 0.01 or less of the Kramers opacity.

In Table 3 we notice that the luminosity steadily decreases as the radiative equilibrium zone widens: the lowest luminosity occurs for No. 7, the complete radiative shell model. In Table 2, the trend of the luminosity was not so regular as that: at constant radius, the minimum luminosity occurs for Model No. 6. As already mentioned, the conditions for actual stars should be those of Table 3.

If the models can turn one into the other, their order should be from No. 7 to No. 1, because this would correspond to a contractional trend of evolution: a decreasing radius, leading to a considerable contraction in the outer portions of the star, where the released gravitational energy exceeds the amount spent upon expansion of the central core (cf.

Table 3, ϱ_c); and an increasing luminosity, requiring a slight increase of the central temperature, thus leading to a slight contraction in the central regions, too. Judging from the ratios $M_{r1} : M$ as given in Table 2, this trend of evolution must be accompanied by an increase of the mass and of the extent of the central convective region.

Of course, a model of uniform composition may remain such only when throughly mixed, otherwise the progress of the atomic synthesis soon introduces a differentation of composition between core and periphery (cf. [1]). Thus, only Model No. 1 can be considered of a more or less permanent character, whereas Models No. 2—7 may appear only as transient configurations at an early stage of the life of a star; these models are soon transformed into those of non-uniform composition, examples of which are given in the following section.

All the above computations have been made with the conventionally constant value of $n_a = 1.5$; if for stars of about the solar mass this assumption is perhaps not far from the truth, larger stars, of smaller β, must have a larger value of n_a [form. (16)]; there cannot, however, exist a difference of principle on this account, although the quantitative picture may differ considerably; thus the fraction of mass comprised in the core must be larger for a larger n_a (cf.[1], Table 2).

4. Composite Models of Non-Uniform Molecular Weight.

Below are given computations of models consisting of a core of constant dimensions and structure, devoid of hydrogen, surrounded by a shell of $33\frac{1}{3}$ per cent hydrogen. These models are supposed to represent thus an advanced stage of the composite adiabatic-radiative configuration (cf.[1]). $n_a = 1.5$ is assumed as before. For the core $\mu = 2.24$ and $\log k_0 = 25.4660$ were taken; for the envelope, the constants are the same as in the preceding section. For an invariable core, the different models can be obtained only by a variation of the luminosity $L = Q_e$, where Q_e denotes the net flux of radiation just outside the boundary (r_1) of the core; $Q_e >> Q_i$, where Q_i is the net flux of radiation just inside r_1; the difference $Q_e - Q_i = C_{r1}$ evidently must be supplied by convection. The results of the computation are collected in Table 4.

Table 4.

Models of Non-Uniform Composition.

(Core with 0 per cent, shell with $33\frac{1}{3}$ per cent hydrogen.)

α) Structure of the adiabatic core, the same for Models No. 8—12 ; $n = n_a = 1.5$.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{32} g	$\log Q_r$	$1-\beta$
0	15.710	1.0345	0	$-\infty$	0.0240
2.222	9.268	0.6907	3.172	33.4521	0.0111
(r_1)				(Q_i)	

β) Model No. 8

[core, cf. α)].

Complete adiabatic, $n_a = 1.5$; two non-mixing principal convective regions. Log $Q_e = 34.0915$; $1-\bar{\beta} = 0.0105$ (average for the whole model, including the core); $R = 10{,}230.10^{10}$ cm ; $M = 17{,}353.10^{32}$ g ; $\varrho_c : \varrho_m = 27.98$.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{32} g	r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{32} g
2.222	9.268	0.3670	3.172	5.2	4.568	$\bar{1}.9020$	10.073
2.3	9.100	0.3550	3.287	5.6	4.070	$\bar{1}.8246$	11.154
2.4	8.892	0.3400	3.441	6.0	3.596	$\bar{1}.7420$	12.197
2.6	8.498	0.3105	3.773	6.4	3.145	$\bar{1}.6520$	13.179
2.8	8.127	0.2815	4.135	6.8	2.717	$\bar{1}.5537$	14.081
3.0	7.775	0.2525	4.525	7.2	2.313	$\bar{1}.4449$	14.886
3.2	7.438	0.2237	4.942	7.6	1.932	$\bar{1}.3226$	15.582
3.4	7.114	0.1947	5.385	8.0	1.576	$\bar{1}.1837$	16.162
3.6	6.801	0.1654	5.851	8.4	1.243	$\bar{1}.0204$	16.622
3.8	6.498	0.1365	6.337	8.8	0.932	$\bar{2}.8167$	16.962
4.0	6.202	0.1052	6.841	9.2	0.645	$\bar{2}.5588$	17.189
4.2	5.914	0.0737	7.361	9.6	0.379	$\bar{2}.1638$	17.314
4.4	5.633	0.0414	7.892	10.230	0	$-\infty$	17.353
4.8	5.089	$\bar{1}.9738$	8.977				

Table 4. Continued.

γ) Model No. 9 [core, cf. α)].

Composite model, with an outer convectional region separated from the core by a region in radiative equilibrium.

Log. $Q_e = 34.0417$; $1-\bar{\beta} = 0.0104$; $R = 10{,}327.10^{10}$ cm;
$M = 17{,}624.10^{32}$ g; $\varrho_c : \varrho_m = 28.34$.

r 10^{10} cm	T 10^6 deg	log ϱ g/cm³	M_r 10^{32} g	p_g 10^{14} $\frac{\text{dyne}}{\text{cm}^2}$	p_R 10^{12} $\frac{\text{dyne}}{\text{cm}^2}$	n
		Shell in radiative equil., $Q_r = Q_e =$ const.				
2.222	9.268	0.3670	3.172	16.760	18.8	1.82
2.3	9.116	0.3539	3.287	15.995	17.6	1.74
2.4	8.929	0.3382	3.441	15.111	16.2	1.69
2.6	8.565	0.3078	3.771	13.518	13.7	1.54
2.8	8.209	0.2791	4.131	12.122	11.6	
		External convective region, $n = n_a = 1.50$				
3.0	7.857	0.2506	4.519			
3.2	7.521	0.2222	4.935			
3.4	7.197	0.1934	5.376			
3.6	6.884	0.1645	5.840			
3.8	6.581	0.1352	6.325			
4.0	6.286	0.1054	6.829			
4.2	5.998	0.0748	7.349			
4.4	5.717	0.0436	7.881			
4.8	5.173	$\bar{1}.9784$	8.970			
5.2	4.652	$\bar{1}.9094$	10.072			
5.6	4.154	$\bar{1}.8354$	11.163			
6.0	3.679	$\bar{1}.7564$	12.219			
6.4	3.227	$\bar{1}.6710$	13.218			
6.8	2.798	$\bar{1}.5782$	14.140			
7.2	2.392	1.4760	14.968			
7.6	2.009	$\bar{1}.3623$	15.690			
8.0	1.650	$\bar{1}.2340$	16.297			
8.4	1.314	$\bar{1}.0857$	16.785			
8.8	1.000	$\bar{2}.9078$	17.154			
9.2	0.709	$\bar{2}.6837$	17.408			
9.6	0.439	$\bar{2}.3716$	17.558			
10.0	0.189	$\bar{3}.8226$	17.618			
10.327	0	$-\infty$	17.624			

Table 4. Continued.

δ) Model No. 10 [core, cf. α)].

Type of structure as before. Log $Q_e = 34.0031$; $1 - \bar{\beta} = 0.0107$; $R = 12{,}772.10^{10}$ cm ; $M = 21{,}116.10^{32}$ g ; $\varrho_c : \varrho_m = 44.75$. The computations start at $r = 2{,}8.10^{10}$ cm, by interpolation of trial models.

r 10^{10} cm	T 10^6 deg	log ϱ g/cm³	M_r 10^{32} g	p_g 10^{14} $\frac{\text{dyne}}{\text{cm}^2}$	p_R 10^{12} $\frac{\text{dyne}}{\text{cm}^2}$	n
		Shell in radiative equil., $Q_r = Q_e =$ const.				
2.222	9.268	0.3670	3.172	16.760	18.8	
2.8	8.329	0.2734	4.125	. . .	. . .	2.07
3.0	8.045	0.2423	4.507	10.921	10.7	2.09
3.2	7.779	0.2116	4.914	9.834	9.33	2.15
3.4	7.525	0.1806	5.343	8.859	8.18	2.18
3.6	7.283	0.1496	5.793	7.985	7.17	2.23
3.8	7.051	0.1184	6.261	7.195	6.30	2.24
4.0	6.829	0.0871	6.745	6.481	5.54	2.31
4.2	6.614	0.0552	7.242	5.834	4.88	2.28
4.4	6.407	0.0233	7.750	5.249	4.29	2.37
4.8	6.011	$\bar{1}.9580$	8.789	4.239	3.33	2.41
5.2	5.640	$\bar{1}.8912$	9.844	3.410	2.58	2.41
5.6	5.286	$\bar{1}.8233$	10.899	2.733	1.99	2.37
6.0	4.943	$\bar{1}.7544$	11.938	2.181	1.52	2.24
6.4	4.603	$\bar{1}.6853$	12.951	1.7328	1.14	1.96
6.8	4.250	$\bar{1}.6174$	13.913	1.3681	0.831	1.46
7.2	3.847	$\bar{1}.5542$	14.879	1.0704	0.558	
		External convective region, $n = n_a = 1.50$				
7.6	3.463	$\bar{1}.4858$	15.790			
8.0	3.098	$\bar{1}.4132$	16.650			
8.4	2.751	$\bar{1}.3358$	17.450			
8.8	2.421	$\bar{1}.2526$	18.182			
9.2	2.108	$\bar{1}.1624$	18.838			
9.6	1.812	$\bar{1}.0639$	19.414			
10.0	1.532	$\bar{2}.9546$	19.907			
11.0	0.900	$\bar{2}.6079$	20.742			
12.0	0.358	$\bar{2}.0074$	21.079			
12.772	0	$-\infty$	21.116			

Table 4. Continued.

ε) Model No. 11

[core, cf. (α)].

Type of structure as before.

$\text{Log } Q_e = 34.0027$; $1 - \bar{\beta} = 0.0110$; $R = 15{,}158 . 10^{10}$ cm; $M =$
$= 22{,}560 . 10^{32}$ g; $\varrho_c : \varrho_m = 69.98$.

The computations start at $r = 6{,}8 . 10^{10}$ cm, by interpolation of trial models.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{32} g	p_g 10^{13} dyne/cm²	p_R 10^{11} dyne/cm²	n
Shell in radiative equil., $Q_r = Q_e =$ const.						
2.222	9.268	0.3670	3.172	167.60	188	
6.8	4.438	$\bar{1}.6072$	13.885	. . .	. . .	
7.2	4.174	$\bar{1}.5332$	14.800	11.067	7.73	2.77
7.6	3.925	$\bar{1}.4586$	15.662	8.764	6.04	2.79
8.0	3.688	$\bar{1}.3831$	16.467	6.922	4.71	2.80
8.4	3.461	$\bar{1}.3074$	17.215	5.456	3.65	2.75
8.8	3.239	$\bar{1}.2316$	17.906	4.288	2.80	2.63
9.2	3.016	$\bar{1}.1565$	18.542	3.359	2.11	2.42
9.6	2.781	$\bar{1}.0835$	19.127	2.618	1.52	2.07
10.0	2.506	$\bar{1}.0165$	19.669	2.022	1.00	1.48
External convective region, $n = n_a = 1.50$						
11.0	1.876	$\bar{2}.8278$	20.825			
12.0	1.325	$\bar{2}.6013$	21.685			
13.0	0.843	$\bar{2}.3067$	22.243			
14.0	0.422	$\bar{3}.8560$	22.519			
15.0	0.054	$\bar{4}.5166$	22.560			
15.158	0	$-\infty$	22.560			

Table 4. Continued.

ζ) Model No. 12

[core, cf. (α)].

Complete radiative outer shell.

$\log Q_e = 34.0027$; $1-\bar{\beta} = 0.0121$; $R = 23{,}886.10^{10}$ cm;
$M = 23{,}346.10^{32}$ g;

$\varrho_c : \varrho_m = 264.6$.

The computations start at $r = 8{,}8.10^{10}$ cm, by interpolation of trial models.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{32} g	p_g 10^{13} $\frac{\text{dyne}}{\text{cm}^2}$	p_R 10^{11} $\frac{\text{dyne}}{\text{cm}^2}$	n
Shell in radiative equil., $Q_r = Q_e$ const.						
2.222	9.268	0.3670	3.172	167.60	188	
8.8	3.290	$\bar{1}.2282$	17.894	. . .	. . .	
9.2	3.097	$\bar{1}.1503$	18.523	3.401	2.34	2.98
9.6	2.915	$\bar{1}.0721$	19.097	2.673	1.84	2.96
Same, simplified calculation: n = const. = 3; k = const. = k at $r = 9{,}6.10^{10}$						
10.0	2.743	$\bar{2}.9929$	19.617			
11.0	2.353	$\bar{2}.7931$	20.697			
12.0	2.013	$\bar{2}.5900$	21.512			
13.0	1.715	$\bar{2}.3812$	22.112			
14.0	1.454	$\bar{2}.1661$	22.541			
15.0	1.224	$\bar{3}.9417$	22.840			
16.0	1.021	$\bar{3}.7053$	23.041			
17.0	0.840	$\bar{3}.4512$	23.170			
18.0	0.679	$\bar{3}.1740$	23.249			
23.886	0	$-\infty$	23.346			

The data for the models of Table 4, reduced to $M = M_\odot$ and $R = R_\odot$ by homologous transformation, are collected in Table 5.

Table 5.

Characteristic Data for the Models of Table 4, reduced to Solar Mass and Radius.

(Central core devoid of hydrogen; outside the core $33\frac{1}{3}$ per cent hydrogen.)

Model No.	8	9	10	11	12
$M_{r_1}:M$	0.1828	0.1800	0.1503	0.1406	0.1360
$(1-\bar{\beta}).10^3$	13.1	12.6	9.0	8.2	8.5
$10^{-6}.T_c$	26.5	26.3	27.2	30.3	46.1
$\varrho_c:\varrho_m$	27.98	28.34	44.75	69.98	264.6
$\log L - 33$	1.4883	1.4054	0.9938	0.8757	0.8947
m_{bol}	2.37	2.58	3.61	3.90	3.86
Hydrogen per cent, true	27.2	27.3	28.3	28.6	28.8
Hydrogen per cent, apparent	13	15	22	25	24

The "true" hydrogen content is the assumed percentage of hydrogen in the whole model; the "apparent" hydrogen content (in the last line of the table) is the percentage computed from the luminosity, on the assumption of a polytropic structure $n = 3$ (determing the central conditions), and a divisor of luminosity $a = 2.5$. We see that, in view of the possibility of the existence of different types of composite structure, the computed "apparent" hydrogen content may differ considerably from the true one, the deviation being negative in all cases; in other words, this means that, with the exhaustion of hydrogen in the central core, the luminosity of the star increases more rapidly than it would in the case of a uniform distribution of hydrogen throughout the star.

In a preceding paper[1], we raised a number of arguments in favour of the sun being built according to the complete adiabatic model (Model No. 1, preceding section); also, we compared the theoretical rate of the increase of the luminosity of the sun with certain geologic facts; we found that this theory, with the adiabatic model for a basis, is not contradicted by the scanty observational facts. If a composite structure for the sun were postulated, the figures of Table 5 would require a from two to four times more rapid increase of the solar luminosity during the geological ages; in such a case permanent glaciation in moderate latitudes should be found in the late Archaean, which does not seem to be the case. Thus, from this standpoint, too, a complete adiabatic model of uniform composition seems to be the more probable one.

For a constant central temperature the data are represented in Table 6, which is an analogon of Table 3.

Table 6.

Radii and Luminosities of the Models of Table 4, for $T_c = 13{,}1.10^6$ and $M = M_{\odot}$ (core devoid of hydrogen; envelope with 33 ⅓ per cent hydrogen).

Model No.	8	9	10	11	12
$R : R_{\odot}$	2.025	2.013	2.078	2.309	3.519
$\bar{\varrho} : \varrho_{\odot}$	0.12	0.12	0.11	0.081	0.023
$\Delta m_b = 1.25 \log {}^{R}/_{R_{\odot}}$	+0.38	+0.38	+0.40	+0.45	+0.65
m_{bol}	2.75	2.96	4.01	4.35	4.54
$\varrho_c : \varrho_{\odot}$	3.4	3.4	4.9	5.7	6.1

The condition $T_c =$ const. must be regarded as a good approximation to actual conditions from the standpoint of stellar energy generation; therefore the radii and luminosities of Tables 3 and 6 may be considered more or less representative of the differences due to internal structure for stars of the solar mass. In Table 3, Model No. 1 is the only permanent one; Models Nos. 2 to 7, in the course of evolution determined by the progressive exhaustion of hydrogen in the core, must gradually become more and more similar to the models of Table 6. The line of evolution is probably determined by the initial fraction of mass in the core, $M_{r1} : M$; e. g., if this fraction is assumed to remain constant, Model No. 4 may be supposed to turn ultimately into No. 10, Model No. 7 into an intermediate one between Nos. 11 and 12. Certain considerations, however, render it probable that the effective ratio $M_{r1} : M$ in the course of evolution decreases (cf. [1], the core may become stratified); in such a case all the composite models should approach something similar to Model No. 12, which may be considered an advanced stage of the semi-giant class (cf.[1], Section 6. *h* and 7. *f*). In the course of evolution with constant central temperature, the radius of a star, according to our computations, may increase to from two to three times its original value, and more, depending upon the original content of hydrogen and the degree of exhaustion attained.

5. "Giant" Composite Models.

It is possible to construct "inflated" models, of low mean density and of large concentration $\varrho_c : \varrho_m$, along the lines of the preceding section: with a suitable choice of the ratio of molecular weights in the core and in the envelope, a "giant" star may be obtained, with all the energy sources still concentrated in the core. In [1], however, we considered as the specifically "giant" structure a model consisting of an exhausted, eventually overdense core, devoid of subatomic energy sources, surrounded by a hydrogen-containing envelope, able to generate subatomic energy.

Below are given sample computations for an extreme case of the "giant" model — for extremely large masses, partly corresponding to those of the largest supergiants, partly exceeding any stellar mass observed, or imaginable. All the models are built up upon the same core. The chief purpose was to show that arbitrary degrees of "inflation" can be obtained, without making any *ad hoc* assumptions with respect to the coefficient of opacity or other physical laws. The coefficient of opacity as defined by (1) in all the computations cited below is practically determined by electron scattering, independent of temperature and density, the Kramers term of opacity being quite small. Further, for the large masses considered here, β is small, often very small, and n_α is always close to 3 [cf. (16)]; therefore, the effective polytropic index remains here close to 3, too, fluctuating eventually on both sides of this value.

In the models below three distinct regions are considered schematically: (1) a central core, devoid of hydrogen, and of subatomic energy sources; electronic opacity $= 0.2$; $\log k_0 = 25.4660$; $\bar{\mu} = 2.24$; $\bar{n} = 3$; (2) an intermediate shell, devoid of hydrogen, but containing 75 per cent helium, considered conventionally as the product of an already terminated process of the synthesis from hydrogen; the helium is supposed to change gradually into heavier elements, releasing thus some subatomic energy (at the bottom of the shell); further, the hydrogen supplied by the outer envelope may be considered a powerful source of subatomic energy at the top of the intermediate shell; in this shell we assume the electronic opacity [not by (1)] $= 0.206$, $\log k_0 = 25.4100$, $\bar{\mu} = 1.48$; (3) an outer shell, of 75 per cent of hydrogen, electronic opacity $=$ $= 0.350$, $\log k_0 = 24.4200$, $\bar{\mu} = 0.631$. Thus, in addition to the gravitational energy of the superdense core, and of the intermediate shell, all the intermediate shell, and eventually the bottom of the outer shell, may supply subatomic energy. The introduction of the intermediate shell is not necessary from the standpoint of our computations; we introduce it only to avoid the grotesque impression of a hydrogen-containing material which, at temperatures of the order of 10^9, still gradually releases subatomic energy, instead of having used up all its store of hydrogen long before such high temperatures had been reached (cf. [1]). In other respects, we did not care to represent the rate of subatomic energy generation by definite formulae; we are content to study what happens when energy sources are present in an intermediate shell. The computational results of our qualitative picture cannot be changed in principle if a definite law of energy generation is assumed; only the computations would become much more complicated in such a case.

In a real giant model we have to assume a gradual, more or less balanced, flow of the material into the central core (cf. [1]); the material outside the core sinks concentrically inwards, the temperature increasing and the atomic synthesis steadily advancing at the same time; the gravitational energy of the sinking material adds to the subatomic energy released; a gradual stratification of the material in the intermediate shell must be the result, the molecular weight steadily increasing inwards; therefore, convection currents in the intermediate shell are not very likely to occur; such currents may be assumed to start only in the outer shell, at a level where the temperature is low enough for atomic synthesis to proceed not too violently. These considerations should be kept in mind when dealing with the following computations; they justify the introduction of the intermediate helium shell as a schematic substitute for the more complicated stratified structure of a steadily varying composition. Table 7 contains the detailed results, and Table 8 gives a synopsis of the most important characteristic data. Some comments on the features of the latter table may be made. Thus, the heat output per unit of mass (fraction of heat: fraction of mass) is largest for the intermediate shell, and smallest for the outer shell in all three cases; this circumstance reflects the peculiar distribution of the energy sources in the giant model, discussed qualitatively in[1]. The effective temperature of the surface of Model No. 13 corresponds to an early O-type star; Model No. 14 corresponds to spectrum F (supergiant), whereas Model No. 15 requires 496° abs. or + 223° C: the star, in spite of its large output of heat, would be invisible. Although of very low density, the dimensions of No. 15 (about 0.2 parsec diameter) are still too small for a nebula. The conditions of radiation pressure being highly variable inside the same model, a homologous transmutation of temperature, radius,

Table 7.

"Giant" Models.

(Core devoid of hydrogen or helium; intermediate shell with 75 per cent helium, no hydrogen; outer shell with 75 per cent hydrogen.)

a) Model No. 13.

Adiabatic outside the core. Log $Q_{max} = 40.2001$; $R = 5{,}102.10^{11}$ cm; $M = 2{,}4406.10^{35}$ g; $\varrho_c : \varrho_m = 4{,}74.10^6$.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{35} g	$\log \frac{1-\beta}{\beta}$	$\log Q_r$	n
Core, $n = 3 =$ const. assumed						
0	4962.0	6.3184	0	0.6080	0	3
0.4347	1782.3	4.9845	1.3177	0.6080	40.0901	3

Table 7. Continued.

(Model No. 13.)

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm^3	M_r 10^{35} g	$\log \frac{1-\beta}{\beta}$	$\log Q_r$	n
Intermediate shell, adiabatic, $n = n_a$ defined by equation (16)						
0.4347	1782.3	4.8046	1.3177	0.6080	40.0977	2.9128
0.4400	1761.1	4.7894	1.3255	0.6076	. . .	2.9127
0.4500	1721.1	4.7603	1.3404	0.6067	40.1045	2.9124
0.5000	1539.1	4.6188	1.4096	0.6024	40.1246	2.9118
0.5556	1365.6	4.4677	1.4773	0.5978	40.1430	2.9108
0.6250	1181.8	4.2850	1.5494	0.5921	40.1611	2.9097
0.7143	986.7	4.0572	1.6237	0.5850	40.1781	2.9083
0.8333	779.5	3.7594	1.6958	0.5756	40.1922	2.9064
1.0000	559.1	3.3402	1.7577	0.5619	40.2001	2.9036
Outer shell, adiabatic, $n = n_a$ defined by equation (16)						
1.0000	559.1	2.9699	1.7577	0.5619	39.9969	2.9036
1.1111	510.4	2.8550	1.7704	0.5580	. . .	2.9028
1.2500	460.9	2.7264	1.7854	0.5537	. . .	2.9020
1.4286	410.5	2.5804	1.8034	0.5488	. . .	2.9010
1.6667	359.1	2.4119	1.8257	0.5430	. . .	2.8998
2.0000	306.3	2.2116	1.8543	0.5360	. . .	2.8983
2.273	273.9	2.0710	1.8759	0.5311	. . .	2.8972
2.500	251.9	1.9655	1.8928	0.5274	40.0213	2.8964
2.778	229.5	1.8485	1.9124	0.5232	. . .	2.8953
3.125	206.7	1.7171	1.9354	0.5184	. . .	2.8943
3.571	183.38	1.5666	1.9628	0.5129	. . .	2.8930
4.167	159.44	1.3908	1.9963	0.5063	. . .	2.8914
5.000	134.68	1.1789	2.0383	0.4983	. . .	2.8896
5.556	121.97	1.0547	2.0637	0.4935	. . .	2.8885
6.250	108.96	0.9129	2.0929	0.4880	. . .	2.8872
7.143	95.63	0.7496	2.1269	0.4815	40.0605	2.8856
8.333	81.88	0.5552	2.1669	0.4737	. . .	2.8838
10.000	67.65	0.3159	2.2145	0.4640	. . .	2.8814
11.111	60.31	0.1727	2.2417	0.4581	. . .	2.8798
12.500	52.80	0.0060	2.2715	0.4511	. . .	2.8780
14.286	45.07	$\bar{1}.8083$	2.3039	0.4427	. . .	2.8758

Table 7. Continued.
(Model No. 13.)

r 10^{10} cm	T 10^{6} deg	$\log \varrho$ g/cm³	M_r 10^{35} g	$\log \frac{1-\beta}{\beta}$	$\log Q_r$	n
Outer shell, adiabatic, $n = n_a$ defined by equation (16), continued						
16.667	37.11	$\bar{1}.5657$	2.3385	0.4321	. . .	2.8731
20.000	28.86	$\bar{1}.2521$	2.3743	0.4181	. . .	2.8693
22.73	23.74	$\bar{1}.0089$	2.3953	0.4069	40.0905	2.8662
25.00	20.24	$\bar{2}.8104$	2.4085	0.3975	40.0898	2.8634
27.78	16.66	$\bar{2}.5686$	2.4204	0.3858	. . .	2.8601
31.25	12.98	$\bar{2}.2588$	2.4302	0.3704	. . .	2.8555
35.71	9.18	$\bar{3}.8294$	2.4371	0.3483	. . .	2.8486
41.67	5.18	$\bar{3}.1227$	2.4404	0.3095	. . .	2.8360
50.00	0.52	$\bar{6}.7177$	2.4406	0.1615	. . .	2.779
51.02	0	$-\infty$	2.4406	$-\infty$	. . .	1.500

β) Model No. 14.

$$\text{Log } Q_{max} = 42.4736;\ R = 1{,}915.10^{15} \text{ cm};\ M = 435{,}5.10^{35} \text{ g};$$
$$\varrho_c : \varrho_m = 1{,}41.10^{15}.$$

r 10^{10} cm	T 10^{6} deg	$\log \varrho$ g/cm³	M_r 10^{35} g	p_g $10^{21}\frac{\text{dyne}}{\text{cm}^2}$	p_R $10^{21}\frac{\text{dyne}}{\text{cm}^2}$	n
Core: same as No. 13.						
Intermediate shell, radiative equilibrium with $\log Q_r = 40.0901 =$ const.						
0.4347	1782.3	4.8046	1.3177	6.340	25.706	3.00
0.4400	1760.5	4.7902	1.3257	6.057	24.460	(2.67)
0.4500	1720.6	4.7582	1.3405	5.500	22.334	3.23

r 10^{10} cm	T 10^{6} deg	$\log \varrho$ g/cm³	M_r 10^{35} g	$\log \frac{1-\beta}{\beta}$	$\log Q_r$
Intermediate shell, continued: radiative equilibrium with $n = 3.25 =$ const. and variable Q_r					
0.5000	1544.6	4.6058	1.4085	0.6202	40.1211
0.5556	1380.1	4.4469	1.4737	0.6324	40.1426
0.6250	1212.5	4.2642	1.5424	0.6465	40.1647

Table 7. Continued.
(Model No. 14.)

r 10^{10} cm	T 10^{6} deg	$\log \rho$ g/cm^3	M_r 10^{35} g	$\log \frac{1-\beta}{\beta}$	$\log Q_r$
Intermediate shell, continued: radiative equilibrium with $n = 3.25 =$ const. and variable Q_r					
0.7143	1042.3	4.0507	1.6144	0.6629	40.1873
0.8333	870.4	3.7962	1.6890	0.6825	40.2103
1.0000	698.9	3.4865	1.7654	0.7063	40.2321
1.1111	613.7	3.3032	1.8039	0.7204	40.2430
1.2500	529.3	3.0942	1.8422	0.7365	40.2544
1.4286	446.2	2.8531	1.8799	0.7550	40.2652
1.6667	365.0	2.5697	1.9165	0.7768	40.2760
2.0000	286.4	2.2274	1.9514	0.8032	40.2870
Intermediate shell, continued: radiative equilibrium with $n = 3.30 =$ const. and variable Q_r					
2.0833	270.79	2.1449	1.9581	0.8128	40.2883
2.1739	255.52	2.0614	1.9644	0.8204	. . .
2.2727	240.34	1.9736	1.9708	0.8284	. . .
2.500	210.70	1.7852	1.9831	0.8455	. . .
2.778	181.88	1.5743	1.9947	0.8647	. . .
3.125	154.06	1.3364	2.0055	0.8863	. . .
3.571	127.35	1.0635	2.0154	0.9111	. . .
4.167	101.95	0.7447	2.0243	0.9401	. . .
5.000	78.05	0.3619	2.0321	0.9749	. . .
5.556	66.74	0.1375	2.0355	0.9953	. . .
6.250	55.90	$\bar{1}.8834$	2.0386	1.0184	. . .
7.143	45.60	$\bar{1}.5917$	2.0413	1.0449	. . .
8.333	35.90	$\bar{1}.2488$	2.0436	1.0761	. . .
10.000	26.86	$\bar{2}.8330$	2.0455	1.1139	. . .
11.111	22.620	$\bar{2}.5869$	2.0463	1.1362	. . .
12.500	18.593	$\bar{2}.3060$	2.0470	1.1628	40.3279
14.286	14.798	$\bar{3}.9787$	2.0476	1.1915	40.3286
16.667	11.252	$\bar{3}.5863$	2.0480	1.2272	40.3289
20.000	8.006	$\bar{3}.0982$	2.0483	1.2719	40.3285
25.000	5.116	$\bar{4}.4564$	2.0485	1.3299	40.3271

Table 7. Continued.

(Model No. 14.)

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm^3	M_r 10^{35} g

Outer shell, $n = 3 =$ const.; adiabatic equilibrium (n_a for such a high value of the radiation pressure, $\beta = 0.04469 =$ const., differs from 3 but negligibly);

$$\log \frac{Q_r}{M_r} = 4.8346 \text{ when } k \text{ is const.}$$

25.00	5.116	$\bar{4}.0861$	2.0485
50.00	2.786	$\bar{5}.2944$	2.0487
100.0	1.6208	$\bar{6}.5885$	2.0489
200.0	1.0382	$\bar{6}.0083$	2.0494
500.0	0.6885	$\bar{7}.4731$	2.0516
1000.	0.5717	$\bar{7}.2310$	2.0591
2000.	0.5126	$\bar{7}.0888$	2.0992
3000.	0.4923	$\bar{7}.0360$	2.1911
4000.	0.4815	$\bar{7}.0072$	2.3540
5000.	0.4745	$\bar{8}.9880$	2.6044
6000.	0.4692	$\bar{8}.9736$	2.9691
8000.	0.4609	$\bar{8}.9502$	4.1049
10000	0.4538	$\bar{8}.9301$	5.886
12000	0.4470	$\bar{8}.9103$	8.423
15000	0.4368	$\bar{8}.8803$	13.844
20000	0.4183	$\bar{8}.8239$	27.626
25000	0.3977	$\bar{8}.7582$	47.366
30000	0.3755	$\bar{8}.6832$	72.41
40000	0.3288	$\bar{8}.5101$	133.63
50000	0.2825	$\bar{8}.3124$	198.52
60000	0.2395	$\bar{8}.0973$	259.60
80000	0.1680	$\bar{9}.6353$	350.74
100000	0.1153	$\bar{9}.1448$	400.92
120000	0.0762	$\overline{10}.6054$	423.8
150000	0.0354	$\overline{11}.6034$	432.6
191500	0	$-\infty$	435.5

Table 7. Continued.

γ) Model No. 15.

Log $Q_{max} = 42.5641$; $R = 2{,}899.10^{17}$ cm; $M = 534.10^{35}$;

$\varrho_c : \varrho_m = 3{,}98.10^{21}$.

Core and intermediate shell up to $r \leqslant 25{,}000.10^{10}$ cm same as in Model No. 14.

r 10^{10} cm	T 10^6 deg	$\log \varrho$ g/cm³	M_r 10^{35} gr
Intermediate shell, radiative equilibrium, $n = 3.30$			
25.000	5.116	$\bar{4}.4564$	2.0485
29.412	3.577	$\bar{5}.9435$	2.0486
Outer shell, $n = 3 =$ const. (cf. Model No. 14); $\beta = 0.04030 =$ const.; $\log \frac{Q_r}{M_r} = 4.8366$ when $k =$ const.			
29.412	3.577	$\bar{5}.5732$	2.0486
50.00	2.106	$\bar{6}.8832$	2.0487
200.0	0.5290	$\bar{7}.0832$	2.0489
500.0	0.2136	$\bar{9}.9015$	2.0490
1000.0	0.10847	$\bar{9}.0186$	2.0491
2000.0	0.05591	$\overline{10}.1552$	2.0492
5000.	0.02437	$\overline{11}.0734$	2.0493
10000	0.01386	$\overline{12}.3375$	2.0494
20000	0.008597	$\overline{13}.7159$	2.0497
50000	0.005442	$\overline{13}.1201$	2.0507
100000	0.004389	$\overline{14}.8399$	2.0538

r 10^{15} cm	T deg	$\log \varrho$ g/cm³	M_r 10^{35} g
Same, continued			
2.0	3860	$\overline{14}.6725$	2.0695
5.0	3528	$\overline{14}.5552$	2.2621
10.0	3384	$\overline{14}.5009$	3.473
15.0	3302	$\overline{14}.4691$	6.512
20.0	3226	$\overline{14}.4388$	12.023
25.0	3146	$\overline{14}.4058$	20.468

Table 7. Continued.
(Model No. 15.)

r 10^{15} cm	T deg	$\log \rho$ g/cm³	M_r 10^{35} g
Same, continued			
30.0	3058	$\overline{14}.3689$	32.096
40.0	2863	$\overline{14}.2831$	64.89
50.0	2649	$\overline{14}.1817$	107.88
60.0	2425	$\overline{14}.0668$	158.62
80.0	1985	$\overline{15}.8061$	265.7
100.0	1591	$\overline{15}.5178$	359.6
120.0	1255	$\overline{15}.2085$	430.0
150.0	862	$\overline{16}.7192$	493.5
200.0	423	$\overline{17}.7916$	528.4
289.9	0	$-\infty$	534.0

and mass does not seem possible, even as an approximation; therefore, we had to leave all our results unchanged, thus deviating from the procedure employed with respect to the types of structure discussed in the preceding sections. Although a quantitative application of our result to smaller masses is not possible, qualitatively similar types of structure may be expected to result for stars of different masses; Models No. 14 and No. 15 are of a size which does not occur in our observational data, but we may expect similar structures to exist for masses of the observed order of magnitude.

With respect to the distribution of mass between the different sections of the star, our examples refer to two extreme cases: Model No. 13, with the major fraction of mass in the core, but at the same time with the mass of the outer shell comparable to the mass of the core; Models No. 14 and 15, with almost the whole mass in the outer shell, and an almost negligible fraction of mass in the core. Without doubt, intermediate cases could be constructed, as well as such where the major fraction of the mass is in the core, the outer shell figuring as an extended atmosphere of a relatively small mass.

Table 8.

Characteristic Data for the Models of Table 7.

($T_c = 4{,}96.10^9$ deg; $\varrho_c = 2{,}08.10^6$ g/cm³.)

Model No.			13	14	15
$R : R_\odot$			7.36	27500	$4{,}18.10^6$
$M : M_\odot$			123.0	22000	26900
m_{bol}			− 11.91	− 17.59	− 17.82
$\varrho_c : \varrho_m$			$4{,}74.10^6$	$1{,}41.10^{15}$	$3{,}98.10^{21}$
Fraction of mass	core		0.54	0.0030	0.0025
	intermediate shell		0.11	0.0017	0.0014
	outer shell		0.35	0.9953	0.9961
Fraction of heat output *)	core		0.78	0.0041	0.0034
	intermediate shell		0.22	0.0031	0.0024
	outer shell		0	0.9928	0.9942
Bottom of outer shell	T deg		559.10^6	$5{,}12.10^6$	$3{,}58.10^6$
	ϱ g/cm³		933	$1{,}22.10^{-4}$	$3{,}74.10^{-5}$
β	centre		0.198	0.198	0.198
	interm. shell	bottom	0.198	0.198	0.198
		top	0.215	0.045	0.040
	outer shell	bottom	0.215	0.045	0.040
		top	1.000	0.045	0.040
Effective temperature of the surface, deg abs.			95900	5790	496

Our computations were made without consideration being given to any definite properties of the physical source of stellar energy. Of course, gravitational energy, mainly as a source in the core, is self-regulating under all circumstances. However, from the standpoint of the transmutation of the elements, the conditions at the bottom of the hydrogen-containing outer shell determine the secular stability of the model. In Model No. 13, the temperature and density at the bottom of the outer shell (cf. Table 8) are rather high, requiring a very intense development of subatomic energy, whereas the amount which the outer shell is able to get rid

*) Net output radiated into space.

of is zero; therefore, Model No. 13 would not be secularly stable, and should go on expanding instead. In Model No. 14, and still more so in No. 15, the temperature and density at the bottom of the outer shell are too low for an appreciable amount of subatomic energy to be produced, whereas the loss of energy by this shell is very large; the models will contract. From such considerations it appears that a secularly stable type of structure may be an intermediate one between No. 13 and No. 14. Given a definite law of subatomic energy generation it may be possible to calculate secularly stable giant models of a great variety of structure depending upon the initial conditions. This problem we hope to discuss in future in a cooperative investigation.

Tartu, March 5, 1938.

Note added in proof.

The importance of convection as a means of heat transport, and the fact that the adiabatic value of the temperature gradient cannot be much exceeded, has been pointed out by Jeffreys (Monthly Notices **91**, 121, 1930), and has been more thoroughly studied by L. Biermann (Zeitsch. für Astroph. **5**, 117, 1932; Astron. Nachr. **257**, 269, 1935 and **264**, 361, 1938), who also computed some models with convective zones and with a small degree of concentration of the energy sources ($\varepsilon \sim T$, cf. ibidem, and Astr. Nachr. **258**, 257, 1936). The only case of a convective model with a concentrated source of energy ("point-source") hitherto computed is Cowling's (Monthly Notices **96**, 58, 1935).

References.

[1] E. Öpik, Stellar Structure, Source of Energy, and Evolution, Acta et Comm. Univ. Tartu **A 33.**₉, 1938; Tartu Obs-y Pub. **30.**₃, 1938.

[2] A. S. Eddington, The Internal Constitution of the Stars, Cambridge 1926.

[3] R. Emden, Gaskugeln, Leipzig und Berlin 1907.

[4] H. Benard, Les tourbillons cellulaires etc., Revue générale des Sciences, 1900, pp. 1261, 1309; Ann. de Physique et de Chimie, 7-e série, **23**, 62, 1901.

[5] H. Deslandres, Recherches sur l'atmosphère solaire, Annales de l'Observatoire de Meudon **4**, 116 f., 1910.