FOUNDATIONS OF ARITHMETIC

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In the previous paper[1] arithmetic was used as the foundation of metric geometry. The paper aimed at some contribution towards bridging the gap between primitive concepts and immediate percepts in geometry. What follows aims at the same end in arithmetic.

1. The primitive ideas.

B. Russel and *A. N. Whitehead*[2] have tried to derive the concepts of arithmetic from those of logic. Percepts are the concrete bottom of concepts. The idea of perceiving involves the following primitive ideas: I. *existence*, II. *difference*, III. *parts*, IV. *becoming* and V. *tendency*. The idea of difference in its purest form is the logical concept of the negation "not", as the idea of parts is the concept of the conjunction "and".

There are constituents [III][3] of the process [IV] of perceiving: *sensation* [I II], *memory* [IV II IV], *thought* [IV III] and *imagination* [V].

There are similar (partly not different) percepts. The imagination tends [V] to associate [III IV] similar percepts with similar names (denotative words) or [III] other [II] similar symbols.

2. Symbols and rules of mathematical deduction.

The identical (not different) part of similar percepts is called (named) the content of a concept, and these percepts together [III] the extension of this concept or the class whose elements or members are these percepts separately (not together). A class is symbolized by a Latin letter, e. g. a, and its members by the same letter with an index, e. g. a_i.

Thought analyses [IV III] perception into concepts. The expression (naming) of a perception in the words denoting the concepts involved is called a proposition. The proposition "a is called (or will be or will mean or will denote the same as) b"

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1 Acta et Comm. Univ. Tartuensis A XIX. 4 (1931).

2 Amer. Journ. of Math. XXIV (1902), p. 378. Mind **171** (1934), p. 297.

3 The square brackets contain references, here to the primitive ideas involved.

is called a definition. It will be written (symbolized) by *Heyting*'s[1] symbol "$\equiv$":

$$a \equiv b$$

The proposition "a_i is a member of the class a" will be written by *Peano*'s[2] symbol "ε":

$$a_i \, \varepsilon \, a$$

So

$$\varepsilon \equiv \text{"is a (member of the class)"}$$

The negation "not" will be written by *Whitehead*'s and *Russel*'s[3] (or *Gabelsberger*'s[4]) symbol "$\sim$":

$$\sim \equiv \text{"not"}$$

Brackets will often be replaced by *Leibniz*'s[5] dots, greater bracket by more dots, a group of brackets within a proposition by the dots replacing the greatest in the group, omitted at the beginning and end, e. g.

$$(\sim (a \, \varepsilon \, b)) \equiv : \sim . \, a \, \varepsilon \, b$$

$$\sim . \, a \, \varepsilon \, b : \equiv : a \sim \varepsilon \, b$$

$$\sim \varepsilon \, . \equiv \text{"is not a"}$$

A dot or dots will replace also the conjunction "and".

There are different steps of perception: by sensation, in memory, in thought, and in imagination. There are perceptions p in thought connected with another perception q and not thought without this, if fully [II III] thought of. This connectedness is called implication of q by p and is written by *Gergonne*'s[6] symbol "$\supset$":

$$p \supset q \quad \text{or} \quad q \subset p$$

i. e.

$p \supset q \, . \equiv .$ "if p, then q" or "from p follows q" or "p implies q"

For the sake of brevity instead of

$p \supset q \, . \, q \supset p$ (from p follows q and from q follows p)

there will be written

$$p \underset{\subset}{\supset} q$$

1 Sitzungsb. d. preuss. Akad. 1930, p. 57.
2 G. Peano, I Principii di Geometria (1889).
3 A. N. Whitehead and B. Russel, Principia Mathematica I (1910).
4 A. Padoa, La Logique Déductive (1912), p. 55.
5 A. Padoa, l. c., p. 46.
6 Ibidem.

The mental (in memory, thought, and imagination, or reproductive, analysing, and constructive) activity [V IV] produces [IV] new (non existent in memory) facts (perceptions) implied by the old (existent in memory) ones. There are immutable [II IV] elements [III] in this mental activity — so-called primitive mental facts. The expressions of these primitive facts are called primitive propositions — rules, laws, axioms, postulates. The deriving (producing) of new propositions (q, r) from the given (p) and old ones (P) is called deduction.

The rules of mathematical deduction are:

A $\quad p \supset : p \,.\, P$ (if there are any given propositions, then there are these and the old ones)

B $\quad p \,.\, q : \supset q$ (if there are p and q, then there is q)

C $\quad p \,.\, p \supset q : \supset : p \,.\, q$ (if there is p and from p follows q, then there are p and q)

D $\quad p \supset q \supset r \,.\, \supset .\, p \supset r$

$\quad :.\, p \supset q \supset r \,.\, \equiv : p \supset q \,.\, q \supset r$ (if from p follows q and from q follows r, then from p follows r)

E $\quad p \supset .\, p \equiv p$

F $\quad p \equiv q \,.\, \supset :: p \equiv r \,.\, \supset .\, q \equiv r$

$\quad :.\, p \,\varepsilon\, r \,.\, \supset .\, q \,\varepsilon\, r$

$\quad :.\, r \,\varepsilon\, p \,.\, \supset .\, r \,\varepsilon\, q$

$\quad :.\, \sim p \,.\, \supset .\, \sim q$

$\quad :.\, p \,.\, r : \supset : q \,.\, r$

$\quad :.\, r \,.\, p : \supset : r \,.\, q$

$\quad :.\, p \mathrel{\substack{\supset \\ \subset}} q$ (if p denotes the same as q, then in combinations with the symbols $\equiv$, ε, $\sim$, $.$, $\supset$ p may be substituted by q)

G $\quad \sim p \,.\, \sim \supset p$ (from not p does not follow p)

The first rule (A) explicitly justifies the real first step of mathematical deduction, i. e. the recourse to the old propositions. Similarly the second rule (B) justifies the choosing from the existent propositions of the suitable. These two rules together with the "rules of inference" (CD) are involved almost in every deduction, and therefore will not be specially referred to.

3. *Primitive concepts and primitive propositions in arithmetic.*

The primitive concepts in arithmetic are: "***one***" [I II III], "***equal to***" [II II] and "***the sum***" [III], symbolized by 1, $=$ and $+$ (plus). These concepts are indefinable in arithmetic (though logically derivable from primitive ideas indicated in square brackets). Hence we must have rules for the use of these symbols. These rules are the primitive propositions of arithmetic. The first of them is the definition of natural number.

Natural numbers are 1, $1+1 \equiv 2$, $2+1 \equiv 3$, $3+1 \equiv 4$, $4+1 \equiv 5$, $5+1 \equiv 6$, $6+1 \equiv 7$, $7+1 \equiv 8$, $8+1 \equiv 9$, $9+1 \equiv 10$, $10+1 \equiv 11$, $11+1 \equiv 12$, and so on, $19+1 \equiv 20$, $20+1 \equiv 21$, and so on, $99+1 \equiv 100$, $100+1 \equiv 101$, and so on, i. e. *one is a natural number, and if according to this definition i is so, $i+1$ will be also, and every natural number is so defined*, or in symbols:

1 $1 \varepsilon n :.$ this definition $. \supset . i \varepsilon n : \supset : i+1 . \varepsilon n$
$:. a \varepsilon n . \supset :$ this definition $. \supset . a \varepsilon n$
$:: \equiv . n \equiv$ the class of natural numbers

From this moment onwards a, b, c etc. will denote natural numbers and n the class of natural numbers, if not stated otherwise.

The remaining primitive propositions are:

2 $a \equiv b . \overset{\supset}{\subset} . a = b$

3 $a = b . c = d : \supset . a + c = b + d$

4 $a + (b + 1) = (a + b) + 1$

5 $a + b = c . \supset . a \neq c$
$: \neq \equiv . \sim =$

From these primitive propositions and suitable definitions according to the rules of deduction in the following pages, the fundamental facts of arithmetic will be deduced. The propositions and their deduction will be written in the introduced symbols. The deduction will be adjoined to the proposition by the symbol of implication in its second form "$\subset$".

Every proposition will have its specifying number (as the primitive propositions have their numbers $1 \ldots 5$, and the rules of deduction the letters $A \ldots G$).

In the course of deduction the specifying numbers or letters of propositions needed as arguments will be written under the symbols of implication and equality or inequality.

4. The values of the natural numbers.

6 $a \,.\supset.\, a = a$

$$:\subset:.\, a \,.\underset{E}{\supset}.\, a \equiv a : a \equiv a \,.\underset{2}{\supset}.\, a = a$$

7 $a = b \,.\supset.\, b = a$

$$:\subset::\, a = b \,.\underset{2}{\supset}.\, a \equiv b :.\, a \,.\underset{E}{\supset}.\, a \equiv a :.\, a \equiv b \,.\, a \equiv a :\underset{F}{\supset}: b \equiv a$$

$$:.\, b \equiv a \,.\underset{2}{\supset}.\, b = a$$

8 $a = b = c \,.\supset.\, a = c :.\, a = b = c \,.\equiv:\, a = b \,.\, b = c$

$$::\subset:.\, a = b = c \,.\underset{7.2}{\supset}: b \equiv a \,.\, b \equiv c :\underset{F}{\supset}.\, a \equiv c \,.\underset{2}{\supset}.\, a = c$$

9 $a = b \,.\equiv.\, a$ and b have the same value

10 $a + b \,.\, \varepsilon\, n$ (the sum of two natural numbers is a natural number)

$$\underset{1}{:\subset}::.\, b = 1 \,.\underset{1.3.2.F}{\supset}: a + b \,.\, \varepsilon\, n$$

$$::\, b = i \,.\supset: a + b \,.\, \varepsilon\, n :.\supset:.\, b = i + 1 \,.\supset.\, a + b$$

$$\underset{3}{=} a + (i + 1) \underset{4}{=} (a + i) + 1 \,.\underset{1.8}{\supset}: a + b \,.\, \varepsilon\, n$$

11 $a + (b + c) = (a + b) + c$

$$.\underset{1}{\subset}::\, c = 1 \,.\underset{4}{\supset}.\, a + (b + c) = (a + b) + c$$

$$:.\, c = i \,.\supset.\, a + (b + c) = (a + b) + c :\supset: c = i + 1 \,.\supset$$

$$.\, a + (b + c) \underset{3}{=} a + \big(b + (i + 1)\big) \underset{4}{=} a + \big((b + i) + 1\big)$$

$$\underset{4}{=} \big(a + (b + i)\big) + 1 \underset{3}{=} \big((a + b) + i\big) + 1$$

$$\underset{4}{=} (a + b) + (i + 1) \underset{3}{=} (a + b) + c$$

12 $a+1=1+a$

$.\underset{1}{\subset} :: a=1 . \underset{3}{\supset} . a+1=1+a$

$:. a=i . \supset . a+1=1+a : \supset : a=i+1 . \supset . a+1$

$\underset{3}{=} (i+1)+1=(1+i)+1 \underset{4}{=} 1+(i+1) \underset{3}{=} 1+a$

13 $a+b=b+a$

$.\underset{1}{\subset} :: b=1 . \underset{12}{\supset} . a+b=b+a$

$:. b=i . \supset . a+b=b+a : \supset : b=i+1 . \supset . a+b$

$\underset{3}{=} a+(i+1) \underset{4}{=} (a+i)+1 \underset{3}{=} (i+a)+1 \underset{4}{=} i+(a+1)$

$\underset{12}{=} i+(1+a) \underset{11}{=} (i+1)+a \underset{3}{=} b+a$

If x, y, z denote any objects of which equality and sum are defined and the associative law 11 and the commutative law 13 hold, then

14 $x+(y+z) \underset{13}{=} x+(z+y) \underset{13}{=} (z+y)+x \underset{13}{=} (y+z)+x \underset{11}{=} y+(z+x)$

$\underset{13}{=} y+(x+z) \underset{13}{=} (x+z)+y \underset{13}{=} (z+x)+y \underset{11}{=} z+(x+y)$

$\underset{13}{=} z+(y+x) \underset{13}{=} (y+x)+z \underset{13}{=} (x+y)+z . = x+y+z$

$= x+z+y = y+x+z = y+z+x = z+x+y$

$= z+y+x =$ the sum of x, y, and z

15 $a>b . = . b<a . = : c . a=b+c$

$:. > . = .$ (has) greater (value) than $: < . = .$ smaller than

$:. \not> . = . \sim >$ $: \not< . = . \sim <$

16 $a=b . b>c : \underset{15.3}{\supset} . a>c :. a>b . b=c : \underset{15.3}{\supset} . a>c$

17 $a>b>c . \supset . a>c :. a>b>c . = : a>b . b>c$

$:: \underset{15}{\subset} :. a \underset{15}{=} b+d . b \underset{15}{=} c+e : \supset . a \underset{3}{=} (c+e)+d \underset{14}{=} c+(e+d)$

18 $a>b . \supset . a+c>b+c$

$: \underset{15}{\subset} : a \underset{15}{=} b+d . \supset . a+c \underset{3}{=} (b+d)+c \underset{14}{=} (b+c)+d$

19 $a \neq 1 . \underset{1}{\supset} . a = b + 1$

20 $a > 1 : b < a . \supset . b \varepsilon m : a \varepsilon m : c > a . \supset . c \varepsilon m :. d \varepsilon m \supset$

$: (d \not< a . d \neq a) \supset (d > a) . (d \neq a . d \not> a) \supset (d < a)$

$. (d \not< a . d \not> a) \supset (d = a) :: \supset : d \varepsilon n . \supset . d \varepsilon m$

$:: . \underset{1}{\subset} :: . b = 1 . \supset . 1 \varepsilon m$

$:: b = i . a \underset{15}{=} b + e . e = 1 : \supset :. i \varepsilon m . a = i + 1 : \supset : i + 1 . \varepsilon m$

$:: b = i . a = b + e . e \underset{19}{=} f + 1 : \supset :. i \varepsilon m . a = i + f + 1$

$\underset{14}{=} (i + 1) + f : \supset :. i + 1 \underset{15}{<} a . \supset : i + 1 . \varepsilon m$

$:: a = i . \supset : i \varepsilon m . i + 1 = a + 1 : \underset{15}{\supset} : i + 1 > a . \supset : i + 1 . \varepsilon m$

$:: c \underset{15}{=} i = a + g . \supset : i \varepsilon m . i + 1 = a + g + 1 : \underset{15}{\supset}$

$: i + 1 > a : \supset : i + 1 . \varepsilon m$

21 $a . b : \underset{20.17.5.6.G}{\supset} :: a \neq b . a \not< b : \underset{\subset}{\supset} . a > b :. a \not< b . a \not> b : \underset{\subset}{\supset} . a = b$

$:. a \neq b . a \not> b : \underset{\subset}{\supset} . a < b :: \equiv .$ the values of the natural numbers [9] form an ordered set, and every natural number a cuts this set into inferior numbers $Ia < a$ and superior numbers $Sa > a$

22 $a + b = b + c . \supset . a = c$

$: \underset{21}{\subset} :. a + b = b + c . \underset{18.G}{\supset} : a \not> c . a \not< c$

5. *The difference, product, and quotient.*

If x, y, z denote any defined objects of which the equality and the sum are defined, then

23 $x + y = z . \equiv . x = z - y : z - y . \equiv . z$ minus y . $\equiv$. the difference between z and y

24 $a - b = c . d - e = f . a = d . b = e : \supset . c = f$

$:. \underset{22}{\subset} :: . a \underset{23}{=} c + b . d \underset{23}{=} f + e \underset{3}{=} f + b : \underset{8}{\supset} . c + b = f + b$

25 $(a+b)-a=b$

$$. \subset : a+b \underset{10}{=} c . \underset{23}{\supset} . c-a=b$$

26 $a>b+c . \supset . a-(b+c)=(a-b)-c$

$$: \underset{8}{\subset} :. a>b+c . \underset{15}{\supset} . a=b+c+d . \underset{14.23}{\supset}$$

$$: a-(b+c)=d . a-b=c+d . \underset{23}{\supset}$$

$$. (a-b)-c=d$$

27 $a>b . \supset . (a-b)+c=(a+c)-b$

$$: \subset :. a-b \underset{15.23}{=} d . \supset : (a-b)+c=d+c$$

$$\underset{25}{=} (b+d+c)-b \underset{24}{=} (a+c)-b$$

28 $a>b>c . \supset . a-(b-c)=(a-b)+c$

$$: \underset{8}{\supset} :. b \underset{15}{=} c+d . a \underset{15}{=} b+e=c+d+e$$

$$: \supset : a-(b-c) \underset{23}{=} a-d \underset{23}{=} c+e$$

$$. (a-b)+c \underset{23}{=} e+c$$

29 $d>a . b>c : \supset :. a>b . \overset{\supset}{\subset} . d-a<d-b : a>b . \overset{\supset}{\subset} . a-c>b-c$

$$:: \underset{15}{\subset} :. a \underset{15}{=} b+e . \underset{23}{\overset{\supset}{\subset}} . b=a-e . \underset{24}{\overset{\supset}{\subset}} . d-b$$

$$=d-(a-e) \underset{28}{=} (d-a)+e : (b+e)-c \underset{27}{=} (b-c)+e$$

30 $a=1 . a\times b : \equiv 1\times b \equiv b :. a=i+1 . a\times b : \equiv (i+1)\times b$

$\equiv i\times b+b :. a\times b \equiv ab :. ab \equiv$ the product of a and b

31 $ab . \varepsilon\, n$

$$: \underset{1}{\subset} ::. a=1 . \underset{30}{\supset} : ab . \varepsilon\, n$$

$$:: a=i . \supset : ab . \varepsilon\, n :. \underset{30.10}{\supset} :. a=i+1 . \supset : ab . \varepsilon\, n$$

32 $a = b . c = d :\supset . ac = bd$

$$:. \underset{1}{\subset} :: a = b = 1 . \underset{30}{\supset} . ac = bd$$

$$:. a = b = i . \supset . ac = bd : \underset{30.3}{\supset} : a = b = i + 1 . \supset$$

$$. ac = bd$$

33 $a(b + c) = ab + ac$

$$. \underset{1}{\subset} :: a = 1 . \supset . a(b + c) \underset{30}{=} ab + ac$$

$$:. a = i . \supset . a(b + c) = ab + ac : \supset : a = i + 1 . \supset$$

$$. a(b + c) \underset{32}{=} (i + 1)(b + c) \underset{30}{=} i(b + c) + b + c$$

$$\underset{15}{=} ib + b + ic + c \underset{30}{=} (i + 1) b + (i + 1) c \underset{32}{=} ab + ac$$

34 $a - b = c . \supset . d(a - b) = da - db$

$$: \subset . dc \underset{25}{=} (db + dc) - db \underset{33}{=} d(b + c) - db \underset{24}{=} da - db$$

35 $a1 = 1a$

$$. \underset{1}{\subset} :: a = 1 . \underset{32}{\supset} . a1 = 1a$$

$$:. a = i . \supset . a1 = 1a : \supset : a = i + 1 . \supset . a1 = (i + 1) 1$$

$$\underset{30}{=} i1 + 1 \times 1 = 1i + 1 \times 1 \underset{30}{=} 1(i + 1) = 1a$$

36 $ab = ba$

$$. \underset{1}{\subset} :: b = 1 . \underset{35}{\supset} . ab = ba$$

$$:. b = i . \supset . ab = ba : \supset : b = i + 1 . \supset . ab$$

$$= a(i + 1) \underset{33}{=} ai + a = ia + a \underset{30}{=} (i + 1) a = ba$$

37 $a > b . c > d : \supset . (a - b)(c - d) = (ac + bd) - (bc + ad)$

$$:. \subset :: a \underset{15}{=} b + e . c \underset{15}{=} d + f : \supset : (a - b)(c - d) \underset{23.32}{=} ef$$

$$: (ac + bd) - (bc + ad) \underset{36.33}{=} (bc + ed + ef + bd)$$

$$- (bc + bd + ed) \underset{25}{=} ef$$

38 $(ab)c = a(bc) \equiv abc \equiv$ the product of a, b, and c

$$\underset{1}{.\subset} :: a = 1 . \underset{30}{\supset} . (ab)c = a(bc)$$

$$:. a = i . \supset . (ab)c = a(bc) : \supset : a = i + 1 . \supset . (ab)c$$

$$= \big((i+1)b\big)c \underset{30}{=} (ib + b)c \underset{36 . 33 . 36}{=} (ib)c + bc = i(bc) + bc$$

$$\underset{30}{=} (i+1)(bc) = a(bc)$$

If x, y, z denote any objects of which the equality and the product are defined and the commutative law 36 and the associative law 38 hold, then

39 $$x(yz) \underset{36}{=} x(zy) \underset{36}{=} (zy)x \underset{36}{=} (yz)x \underset{38}{=} y(zx) \underset{36}{=} y(xz) \underset{36}{=} (xz)y$$

$$\underset{36}{=} (zx)y \underset{38}{=} z(xy) \underset{36}{=} z(yx) \underset{36}{=} (yx)z \underset{36}{=} (xy)z . \equiv xyz \equiv xzy$$

$$\equiv yxz \equiv yzx \equiv zxy \equiv zyx \equiv \text{the product of } x, y, \text{and } z$$

40 $a > b . \supset . ac > bc$

$$\underset{15}{:\subset} : a \underset{15}{=} b + d . \supset . ac \underset{36 . 33 . 36}{=} bc + dc$$

41 $ab = cb . \supset . a = c$

$$\underset{21}{:\supset} : ab = cb . \underset{40 . G}{\supset} : a \not> c . a \not< c$$

If x, y, z denote any defined objects of which the equality and the product are defined, then

42 $xy = z . \equiv . x = \frac{z}{y} : \frac{z}{y} \equiv . z/y : z/y . \equiv .$ the quotient of z by y

43 $a/b = c . d/e = f . a = d . b = e : \supset . c = f$

$$:. \subset :: a \underset{42}{=} bc . d \underset{42}{=} ef : \supset . bc = ef$$

$$:. bc = ef . b = e : \underset{41}{\supset} . c = f$$

44 $\frac{ab}{b} = a$

$$. \subset : ab \underset{31}{=} c . \underset{42}{\supset} . \frac{c}{b} = a$$

6. *Rational numbers.*

45 $\frac{ab}{c} = \frac{a}{c} b = a$ cths of b . $\frac{a}{c} =$ rational number $\frac{a}{b}$

46 $\frac{a}{b} bc = ac$

$$. \supset . ac \underset{8.7}{} = _{44} \frac{acb}{b} = _{45.36} \frac{a}{b} bc$$

47 $a =_{44} \frac{ab}{b} . \supset_{45} .$ natural numbers are rational

48 $\frac{a}{b} c = \frac{d}{e} c :=. \frac{a}{b} = \frac{d}{e} =_{7} \frac{a}{b}$

49 $\frac{a}{b} = \frac{c}{d} . \supset\subset . ad = bc . \supset\subset_{36} . \frac{d}{c} = \frac{b}{a}$

$$:\subset: \frac{a}{b} = \frac{c}{d} . \supset\subset_{46.48} . \frac{a}{b} bd = \frac{c}{d} bd . \supset\subset_{46.8} . ad = cb$$

50 $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} . \supset . \frac{a}{b} = \frac{e}{f}$

$$:\subset:_{46.48} \frac{a}{b} bdf = \frac{c}{d} bdf = \frac{e}{f} bdf . \supset_{46.8} . \frac{a}{b} bdf = \frac{e}{f} bdf$$

51 $\frac{a}{b} = \frac{c}{d} . = . \frac{a}{b}$ and $\frac{c}{d}$ have the same value

52 $\frac{a}{b} c + \frac{d}{e} c := . \left(\frac{a}{b} + \frac{d}{e}\right) c :. \frac{a}{b} + \frac{d}{e} . =$

. the sum of $\frac{a}{b}$ and $\frac{d}{e}$

53 $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

$$. \subset_{48} . \left(\frac{a}{b} + \frac{c}{b}\right) b =_{52} \frac{a}{b} b + \frac{c}{b} b =_{46} a + c =_{46} \frac{a+c}{b} b$$

54 $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} =_{13.\ 6} \frac{c}{d} + \frac{a}{b}$

$$. \underset{46.48}{\subset} . \left(\frac{a}{b} + \frac{c}{d}\right) bd \underset{52.45}{=} \frac{abd}{b} + \frac{cbd}{d} \underset{39.44}{=} ad + cb$$

$$\underset{46}{=} \frac{ad + cb}{bd} bd$$

55 $\frac{a}{b} = \frac{c}{d} . \supsetsubset . \frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}$

$$: \underset{D.48}{\subset} : \frac{a}{b} = \frac{c}{d} . \underset{49.32.41}{\supsetsubset} . adf = bcf . \underset{3.22}{\supsetsubset} . adf + ebd$$

$$= cbf + ebd . \underset{54.46}{\supsetsubset} . \left(\frac{a}{b} + \frac{e}{f}\right) bdf = \left(\frac{c}{d} + \frac{e}{f}\right) bdf$$

56 $\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f}$

$$. \underset{48.52.45.44.11}{\subset} . \left(\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)\right) bdf = \left(\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f}\right) bdf$$

57 $\frac{a}{b} > \frac{c}{d} . \quad . \frac{c}{d} < \frac{a}{b} . = : \frac{e}{f} . \frac{a}{b} = \frac{c}{d} + \frac{e}{f}$

58 $\frac{a}{b} > \frac{c}{d} . \supsetsubset . ad > bc . \underset{36}{\supsetsubset} . \frac{d}{c} > \frac{b}{a}$

$$: \underset{57}{\subset} : . \frac{a}{b} \underset{57}{=} \frac{c}{d} + \frac{e}{f} . \underset{48.46}{\supset} . adf = cbf + ebd . \underset{41.52}{\supset} . ad = cb + \frac{ebd}{f}$$

$$: ad \underset{57}{=} bc + g . \underset{48.52}{\supset} . \frac{ad}{bd} = \frac{bc}{bd} + \frac{g}{bd} . \underset{49.55}{\supset} . \frac{a}{b} = \frac{c}{d} + \frac{g}{bd}$$

59 $\frac{a}{b} > \frac{c}{d} . \frac{c}{d} = \frac{e}{f} : \supset . \frac{a}{b} > \frac{e}{f}$

$$: . \underset{57}{\subset} . \frac{a}{b} \underset{57}{=} \frac{c}{d} + \frac{g}{h} \underset{55}{=} \frac{e}{f} + \frac{g}{h}$$

60 $\frac{a}{b} = \frac{c}{d} . \frac{c}{d} > \frac{e}{f} : \supset . \frac{a}{b} > \frac{e}{f}$

$$: . \underset{57}{\subset} . \frac{c}{d} \underset{57}{=} \frac{e}{f} + \frac{g}{h} \underset{50}{=} \frac{a}{b}$$

61 $\frac{a}{b} > \frac{c}{d} > \frac{e}{f} :\supset . \frac{a}{b} > \frac{e}{f}$

$$:.\underset{57}{\subset}:. \frac{a}{b} \underset{57}{=} \frac{c}{d} + \frac{g}{h} . \frac{c}{d} \underset{57}{=} \frac{e}{f} + \frac{i}{j} :\supset . \frac{a}{b}$$

$$\underset{53}{=} \left(\frac{e}{f}+\frac{i}{j}\right)+\frac{g}{h} \underset{56}{=} \frac{e}{f} + \left(\frac{i}{j}+\frac{g}{h}\right)$$

62 $\frac{a}{b} > \frac{c}{d} . \stackrel{\supset}{\subset} . \frac{a}{b}+\frac{e}{f} > \frac{c}{d}+\frac{e}{f}$

$$:\underset{57}{\subset}: . \frac{a}{b} \underset{57}{=} \frac{c}{d} + \frac{g}{h} . \supset . \frac{a}{b} + \frac{e}{f} \underset{55}{=} \left(\frac{c}{d}+\frac{g}{h}\right)+\frac{e}{f}$$

$$\underset{14}{=} \left(\frac{c}{d} + \frac{e}{f}\right) + \frac{g}{h} : \frac{a}{b} + \frac{e}{f} \underset{57}{=} \left(\frac{c}{d} + \frac{e}{f}\right)+\frac{g}{h}$$

$$\underset{14}{=} \left(\frac{c}{d}+\frac{g}{h}\right)+\frac{e}{f} . \underset{55}{\supset} . \frac{a}{b} = \frac{c}{d} + \frac{g}{h}$$

63 $\frac{a}{b} > \frac{c}{d} . \supset . \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$

$$: \subset . \frac{a}{b} \underset{57}{=} \frac{c}{d} + \frac{e}{f} \underset{54}{=} \frac{cf+de}{df} . \underset{49}{\supset} . adf = bcf + bde$$

$$. \underset{23}{\supset} . adf - bcf = bde . \underset{34.41.43}{\supset} . ad - bc = \frac{bde}{f}$$

$$\underset{45}{=} \frac{e}{f} bd \underset{46}{=} \frac{ad-bc}{bd} bd . \underset{48}{\supset} . \frac{ad-bc}{bd} = \frac{e}{f} \underset{23}{=} \frac{a}{b} - \frac{c}{d}$$

64 $\frac{a}{b} > \frac{c}{d} . \frac{e}{f} > \frac{g}{h} :\underset{23.55}{\supset}: . \frac{c}{d} = \frac{e}{f} . \stackrel{\supset}{\subset} . \frac{a}{b} - \frac{c}{d} = \frac{a}{b} - \frac{e}{f}$

$$: \frac{c}{d} = \frac{e}{f} . \stackrel{\supset}{\subset} . \frac{c}{d} - \frac{g}{h} = \frac{e}{f} - \frac{g}{h}$$

65 $\frac{a}{b} > \frac{c}{d} + \frac{e}{f} . \supset . \frac{a}{b} - \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} - \frac{c}{d}\right) - \frac{e}{f}$

$$: \subset . \frac{a}{b} - \left(\frac{c}{d} + \frac{e}{f}\right) \underset{54.63}{=} \frac{adf-(bcf+ebd)}{bdf}$$

$$\underset{58.26}{=} \frac{(adf-bcf)-ebd}{bdf} \underset{63}{=} \left(\frac{a}{b} - \frac{c}{d}\right) - \frac{e}{f}$$

66 $\frac{a}{b} > \frac{c}{d} \mathrel{.\underset{54.63.27}{\supset}.} \left(\frac{a}{b}-\frac{c}{d}\right)+\frac{e}{f} = \left(\frac{a}{b}+\frac{e}{f}\right)-\frac{c}{d}$

67 $\frac{a}{b}>\frac{c}{d}>\frac{e}{f} \mathrel{.\underset{63.54.28}{\supset}.} \frac{a}{b} - \left(\frac{c}{d}-\frac{e}{f}\right)=\left(\frac{a}{b}-\frac{c}{d}\right)+\frac{e}{f}$

68 $\frac{a}{b}>\frac{c}{d} . \frac{c}{f} > \frac{g}{h} :\supset:. \frac{c}{d}>\frac{e}{f} . \overset{\supset}{\subset} . \frac{a}{b}-\frac{c}{d}<\frac{a}{b}-\frac{e}{f}$

$: \frac{c}{d}>\frac{e}{f} \mathrel{.\underset{23.62}{\overset{\supset}{\subset}}.} \frac{c}{d}-\frac{g}{h}>\frac{e}{f}-\frac{g}{h}$

$:: \underset{57}{\subset} :. \frac{c}{d} \underset{57}{=} \frac{e}{f}+\frac{i}{j} . \supset . \frac{a}{b}-\frac{c}{d} \underset{64}{=} \frac{a}{b} - \left(\frac{e}{f}+\frac{i}{j}\right)$

$\underset{65}{=}\left(\frac{a}{b}-\frac{e}{f}\right)-\frac{i}{j} . \underset{23}{\supset} . \frac{a}{b}-\frac{e}{f}=\left(\frac{a}{b}-\frac{c}{d}\right)+\frac{i}{j}$

$: \frac{a}{b}-\frac{e}{f} \underset{57}{=} \left(\frac{a}{b}-\frac{c}{d}\right)+\frac{i}{j} \underset{67}{=} \frac{a}{b}-\left(\frac{c}{d}-\frac{i}{j}\right)$

$. \underset{64}{\supset} . \frac{c}{d}-\frac{i}{j}=\frac{e}{f} . \underset{23}{\supset} . \frac{c}{d}=\frac{e}{f}+\frac{i}{j}$

69 $\frac{ab}{c}\,\varepsilon\, n . \frac{de}{f}\,\varepsilon\, n . \frac{ab}{c}\times\frac{de}{f} :\equiv \left(\frac{a}{c}\times\frac{d}{f}\right) be :. \frac{a}{c}\times\frac{d}{f} .\equiv. \frac{a}{c} \frac{d}{f}$

$.\equiv.$ the product of $\frac{a}{c}$ and $\frac{d}{f}$

70 $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}\underset{36}{=}\frac{c}{d}\times\frac{a}{b}$

$.\underset{48}{\subset}. \left(\frac{a}{b}\times\frac{c}{d}\right) bd \underset{69.46}{=} ac \underset{44}{=} \frac{ac\,bd}{bd} \underset{45}{=} \frac{ac}{bd}\, bd$

71 $\frac{a}{b}=\frac{c}{d} . \overset{\supset}{\subset} . \frac{ae}{bf}=\frac{ce}{df}$

$:\subset: \frac{a}{b}=\frac{c}{d} \mathrel{.\underset{49.32.41}{\overset{\supset}{\subset}}.} adef=bcef . \underset{49}{\overset{\supset}{\subset}} . \frac{ae}{bf}=\frac{ce}{df}$

72 $\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right)\underset{54.70.33.54}{=}\frac{ac}{bd}+\frac{ae}{bf}$

73 $\frac{a}{b}\left(\frac{c}{d}-\frac{e}{f}\right)\underset{63.70.34.63}{=}\frac{ac}{bd}-\frac{ae}{bf}$

74 $\frac{a}{b} > \frac{c}{d} . \frac{e}{f} > \frac{g}{h}$

$$: \supset . \left(\frac{a}{b} - \frac{c}{d}\right)\left(\frac{e}{f} - \frac{g}{h}\right) \underset{63.70.37.63.54}{=} \left(\frac{ae}{bf} + \frac{cg}{dh}\right) - \left(\frac{ce}{df} + \frac{ag}{bh}\right)$$

75 $\frac{a}{b} > \frac{c}{d} . \underset{\subset}{\supset} . \frac{ae}{bf} > \frac{ce}{df}$

$$: \subset : \frac{a}{b} > \frac{c}{d} \underset{58.40.41.21}{. \underset{\subset}{\supset} .} adef > bcef . \underset{58}{\underset{\subset}{\supset}} . \frac{ae}{bf} > \frac{ce}{df}$$

76 $\frac{a}{b} / \frac{c}{d} = \frac{ad}{bc}$

$$\underset{71}{. \subset :} \left(\frac{a}{b} / \frac{c}{d}\right)\frac{c}{d} \underset{42}{=} \frac{a}{b} . \frac{ad}{bc}\frac{c}{d} \underset{70}{=} \frac{adc}{bcd} \underset{49}{=} \frac{a}{b}$$

77 $\frac{a}{b} = \frac{c}{d} \underset{76.71}{. \underset{\subset}{\supset} .} \frac{a}{b} / \frac{e}{f} = \frac{c}{d} / \frac{e}{f} : \frac{a}{b} = \frac{c}{d} \underset{76.71}{. \underset{\subset}{\supset} .} \frac{g}{h} / \frac{a}{b} = \frac{g}{h} / \frac{c}{d}$

78 $\frac{a}{b} > \frac{c}{d} \underset{76.75}{. \underset{\subset}{\supset} .} \frac{a}{b} / \frac{e}{f} > \frac{c}{d} / \frac{e}{f} : \frac{a}{b} > \frac{c}{d} \underset{76.75}{. \underset{\subset}{\supset} .} \frac{g}{h} / \frac{a}{b} < \frac{g}{h} / \frac{c}{d}$

79 $\frac{a}{b} \neq \frac{c}{d} . \frac{a}{b} \not< \frac{c}{d} : \underset{49.58.21}{\underset{\subset}{\supset}} . \frac{a}{b} > \frac{c}{d} :. \frac{a}{b} \not> \frac{c}{d} . \frac{a}{b} \not< \frac{c}{d} : \underset{\subset}{\supset} . \frac{a}{b} = \frac{c}{d}$

$$:. \frac{a}{b} \neq \frac{c}{d} . \frac{a}{b} \not> \frac{c}{d} : \underset{\subset}{\supset} . \frac{a}{b} < \frac{c}{d}$$

$::\equiv.$ the values of rational numbers [51] form an ordered set and every rational number $\frac{a}{b}$ cuts this set into inferior numbers

$$I\frac{a}{b} < \frac{a}{b}$$

and superior ones

$$S\frac{a}{b} > \frac{a}{b}$$

$. \equiv . \frac{a}{b} \equiv$ rational cut $\frac{a}{b}$

80 $\frac{a}{b}>\frac{c}{d}\,.\supset:\frac{e}{f}\,.\,\frac{a}{b}>\frac{e}{f}>\frac{c}{d}$

$$:.\subset:.\frac{a}{b}>\frac{c}{d}\,.\underset{58}{\supset}.\,ad>bc\,.\underset{15}{\supset}:ad=bc+g$$

$$.\,(bc+g)(h+i)\underset{33.15}{>}bc(h+i)+gh\underset{15}{>}bc(h+i)$$

$$:\underset{75}{\supset}.\,\frac{a}{b}>\frac{bc(h+i)+gh}{bd(h+i)}>\frac{c}{d}$$

81 $x\varepsilon m\,.\supset.\,x\varepsilon n:\supset:a\,.\,a\varepsilon m\,.\,x\not< a$ (every class m of natural numbers has among its members x a smallest a)

$$:.\subset:::.\,1\,\varepsilon m\,.\underset{19.15}{\supset}.\,1=a$$

$$::.\,1\sim\varepsilon m\,.\supset::c\,.\,c\sim\varepsilon m:x\varepsilon m\,.\supset.\,c<x:(c+1)\,\varepsilon m:.\supset.\,c+1=a$$

$$::.\underset{G}{\subset}:::1\sim\varepsilon m\,.\underset{19.15}{\supset}:x\varepsilon m\,.\supset.\,1<x$$

$$::.\,i\sim\varepsilon m:x\varepsilon m\,.\supset.\,i<x:.\supset.\,(i+1)\sim\varepsilon m$$

$$::\supset:x\varepsilon m\,.\supset.\,i+1<x$$

$$(.\underset{G}{\subset}::i+1\not< x\,.\,i+1\neq x:\underset{21}{\supset}.\,i+1>x$$

$$:.\,x\underset{15}{=}i+j\,.\supset.\,i+1\underset{15}{=}x+k\underset{3}{=}i+j+k\underset{19.14.5}{\neq}i+1)$$

$$::.\underset{1}{\supset}:b\varepsilon n\,.\supset.\,b\sim\varepsilon m$$

82 $\frac{a}{b}\underset{81}{\supset}:.\,\frac{c}{d}\,.\,\frac{c}{d}=\frac{a}{b}:\frac{e}{f}=\frac{a}{b}\,.\supset.\,f\not< d:.=:\frac{c}{d}=.$ irreducible (rational number) $\frac{c}{d}$

7. *The cuts.*

83 $\frac{a}{b}\varepsilon$. irreducible $:\supset.\,\frac{aa}{bb}\neq 2$

$$:.\underset{G}{\subset}:.\,\frac{aa}{bb}=2\,.\underset{49}{\supset}.\,aa=2bb\,.\underset{15}{\supset}:aa>bb$$

$$.\,aa<(2b)(2b)\;\underset{40.32.21}{:\supset:}\;a>b\,.\,a<2b:\supset$$

$$: a \underset{15}{=} b + c \,.\, c \underset{62}{<} b : \supset : b \underset{15}{=} c + d \,.\, d \underset{23}{=} b - c : \supset$$

$$: aa \underset{33.14}{=} bb + cc + 2bc \,.\, dd \underset{37}{=} (bb + cc) - 2bc : \underset{27.25}{\supset}$$

$$: aa + dd = 2bb + 2cc \,.\, \underset{22}{\supset} \,.\, dd = 2cc \,.\, \underset{49}{\supset} \,.\, \frac{dd}{cc} = 2$$

84 $\frac{aa}{bb} < \frac{e}{f} \,.\, \equiv \,.\, \frac{a}{b}\, \varepsilon\, I \sqrt{\frac{e}{f}} : \frac{cc}{dd} > \frac{e}{f} \,.\, \equiv \,.\, \frac{c}{d}\, \varepsilon\, S \sqrt{\frac{e}{f}}$

$: Ix \,.\, \equiv \,.$ a class of rational numbers named inferior numbers of x

$: Sx \,.\, \equiv \,.$ a class of rational numbers named superior numbers of x

$: \sqrt{\frac{e}{f}} \equiv \,.$ square root of $\frac{e}{f}$

85 $\frac{a}{b} \sim \varepsilon\, Ix \,.\, \supset \,.\, \frac{a}{b}\, \varepsilon\, Sx : \frac{c}{d} \sim \varepsilon\, Sx \,.\, \supset \,.\, \frac{c}{d}\, \varepsilon\, Ix$

$: .\, \frac{a}{b}\, \varepsilon\, Sx \,.\, \frac{c}{d}\, \varepsilon\, Ix : \supset \,.\, \frac{a}{b} > \frac{c}{d}$

$:: \equiv \,.\, x$ is an irrational cut in the ordered set of rational numbers (as [84.83] the square root of 2)

86 $x \varepsilon$ cut $. \equiv :: x \sim \varepsilon \,.$ rational cut $: \supset : x \varepsilon \,.$ irrational cut
$: . x \sim \varepsilon \,.$ irrational cut $: \supset : x \varepsilon \,.$ rational cut

87 $x \varepsilon$ cut $. \supset : I_i x \,.\, S_i x \,.\, S_i x - I_i x < \frac{a}{b}$

$$: . \subset :: I_k x = \frac{c}{d} \,.\, S_k x = \frac{e}{f} \,.\, \left(\frac{c}{d} + \frac{2a}{3b}\right) \sim \varepsilon\, Sx$$

$$. \left(\frac{e}{f} - \frac{c}{d}\right) / \frac{a}{3b} \underset{63.76.44.30}{\not>} (ed - cf)\, 3b$$

$$: \underset{79.85}{\supset} : \frac{c}{d} + \left((ed - cf)\, 3b\right) \frac{a}{3b} \,.\, \varepsilon\, Sx$$

$$:. \supset : \left(\frac{c}{d} + m\,\frac{a}{3b}\right) \varepsilon\ Sx \,.\, \left((ed - cf)\,3b\right) \varepsilon m \,.\, 2 \sim \varepsilon m$$

$$:: \underset{81}{\supset} : i \,.\, i \sim \varepsilon m \,.\, (i+1)\,\varepsilon m$$

$$.\ S_i x = \frac{c}{d} + (i+1)\,\frac{a}{3b} \,.\, I_i x = \frac{c}{d} + (i-1)\,\frac{a}{3b}$$

88 $x\,\varepsilon\,\text{cut} \,.\, y\,\varepsilon\,\text{cut} : I_i x \,.\, S_i y \,.\, I_i x > S_i y :. \equiv . x > y . \equiv . y < x$

89 $x\,\varepsilon\,\text{cut} \,.\, y\,\varepsilon\,\text{cut} : x > y :. \supset y \not> x$

$$:: \subset : I_i x > S_i y \,.\, \supset .\, I_k y \underset{79.85}{<} S_i y < I_i x \underset{79.85}{<} S_k x$$

$$. \underset{61}{\supset} .\, I_k y < S_k x \,.\, \underset{79}{\supset} .\, I_k y \not> S_k x$$

90 $x\,\varepsilon\,\text{cut} \,.\, y\,\varepsilon\,\text{cut} \,.\, z\,\varepsilon\,\text{cut} : x > y > z :. \supset . x > z$

$$:: \underset{88}{\subset} : I_i x > S_i y \underset{79.85}{>} I_k y > S_k z \,.\, \underset{61}{\supset} .\, I_i x > S_k z$$

91 $x\,\varepsilon\,\text{cut} \,.\, y\,\varepsilon\,\text{cut} : x \not> y \,.\, x \not< y : \equiv . x = y \underset{88}{=} x$

92 $x\,\varepsilon\,\text{cut} \,.\, y\,\varepsilon\,\text{cut} : \supset :: x \not> y . x \not< y : \underset{91}{\supset} . x = y :. x \not> y . x \neq y : \supset . x < y$

$$:. x \not< y \,.\, x \neq y : \supset x > y$$

$$::. \underset{G}{\subset} ::. x \not> y \,.\, x \neq y : \supset . x \not< y :. \underset{91}{\supset} . x = y$$

$$:: x \not< y \,.\, x \neq y : \supset . x \not> y :. \underset{91}{\supset} . x = y$$

93 $x\,\varepsilon\,\text{cut} \,.\, y\,\varepsilon\,\text{cut} \,.\, z\,\varepsilon\,\text{cut} : x > y \,.\, y = z :. \supset . x > z$

$$:: \subset :. I_i x \underset{88}{>} S_i y \,.\, S_k z \underset{87}{=} I_k z + \tfrac{1}{2}(I_i x - S_i y) \,.\, S_i y \underset{91}{\not<} I_k z$$

$$: \supset .\, S_k z \underset{62.55}{\not>} S_i y + \tfrac{1}{2}(I_i x - S_i y) \underset{75.62.61.59}{. \supset .} S_k z < I_i x$$

94 $x\,\varepsilon\,\text{cut} \,.\, y\,\varepsilon\,\text{cut} \,.\, z\,\varepsilon\,\text{cut} : x = y \,.\, y > z :. \supset . x > z$

$$:: \subset :. I_i y \underset{88}{>} S_i z \,.\, S_k x \underset{87}{=} I_k x + \tfrac{1}{2}(I_i y - S_i z) \,.\, S_k x \underset{91}{\not<} I_i y$$

$$: \supset .\, I_k x \underset{23.68.64}{\not<} I_i y - \tfrac{1}{2}(I_i y - S_i z) \underset{75.62.61.60}{. \supset .} I_k x > S_i z$$

95 $x \,\varepsilon\, \mathrm{cut} \,.\, y \,\varepsilon\, \mathrm{cut} \,.\, z \,\varepsilon\, \mathrm{cut} : x = y = z :. \supset . \, x = z$

G . 93 . 94

96 $x \,\varepsilon\, \mathrm{cut} \,.\, y \,\varepsilon\, \mathrm{cut} : I_i x + I_k y \,.\equiv.\, I_l(x+y) : S_m x + S_n y$

$.\equiv.\, S_p(x+y)$

97 $x \,\varepsilon\, \mathrm{cut} \,.\, y \,\varepsilon\, \mathrm{cut} : \supset . \,(x+y) \,\varepsilon\, \mathrm{cut}$

$:. \underset{79.85.96}{\subset} ::: . \, S_i x > I_k x \,.\, S_l y > I_m y \,.\, S_n(x+y) = S_i x + S_l y$

$. \, I_p(x+y) = I_k x + I_m y : \underset{62.61}{\supset} . \, S_n(x+y) > I_p(x+y)$

$:. \, \frac{a}{b} > S_i x + S_i y \; \underset{62.79.85}{: \supset :} \; \frac{a}{b} = S_k x + S_i y$

$\underset{96}{=} S_l(x+y) \,.\, S_k x \underset{23}{=} \frac{a}{b} - S_i y$

$:. \, \frac{a}{b} < I_i x + I_i y : \underset{75}{\supset} : \frac{a}{b} = I_k x + I_k y \underset{96}{=} I_l(x+y)$

$. \, I_k x = \frac{a \, I_i x}{b(I_i x + I_i y)} \underset{78}{<} I_i x \;.\; I_k y = \frac{a \, I_i y}{b(I_i x + I_i y)} \underset{78}{<} I_i y$

$::: \frac{a}{b} :: \sim : S_i x \,.\, S_i y \,.\, \frac{a}{b} > S_i x + S_i y$

$:. \sim : I_i x \,.\, I_i y \,.\, \frac{a}{b} < I_i x + I_i y$

$::. \supset ::. \, \frac{c}{d} = \frac{a}{b} + \frac{e}{f} \,.\, S_k x \underset{87}{<} I_k x + \frac{e}{2f} \,.\, S_k y \underset{87}{<} I_k y + \frac{e}{2f}$

$: \supset . \, S_k x + S_k y < I_k x + I_k y + \frac{e}{f}$

$. \underset{61.62}{\supset} . \, S_k x + S_k y < \frac{a}{b} + \frac{e}{f} \,.\, \supset . \, S_k x + S_k y < \frac{c}{d}$

$:. \underset{96}{\supset} : \frac{c}{d} > \frac{a}{b} \,.\, \supset . \, \frac{c}{d} = S_l(x+y)$

$:: \frac{g}{h} = \frac{a}{b} - \frac{i}{j} \,.\, I_l x \underset{87}{>} S_l x - \frac{i}{2j} \,.\, I_l y \underset{87}{>} S_l y - \frac{i}{2j}$

$: \supset . \, I_l x + I_l y > S_l x + S_l y - \frac{i}{j}$

$$. \underset{61.62}{\supset} . I_l x + I_l y > \frac{a}{b} - \frac{i}{j} . \supset . I_l x + I_l y > \frac{g}{h}$$

$$: . \supset : \frac{g}{h} < \frac{a}{b} . \underset{96}{\supset} . \frac{g}{h} = I_l (x + y)$$

$$: : . \underset{79}{\supset} . \frac{a}{b} = x + y$$

98 $x \varepsilon \text{ cut} . y \varepsilon \text{ cut} . z \varepsilon \text{ cut} : \supset : x > y . \overset{\supset}{\subset} . x + z > y + z$

$$: . \underset{96.57.88}{\subset} : : I_i x \underset{88.57}{=} S_i y + \frac{a}{b} . S_i z \underset{87}{=} I_i z + \frac{a}{2b} : \supset . I_i (x + z)$$

$$\underset{96}{=} \left(S_i y + \frac{a}{b}\right) + \left(S_i z - \frac{a}{2b}\right) \underset{14.66}{=} (S_i y + S_i z) + \frac{a}{2b}$$

$$: . I_i x + I_i z = S_i y + S_i z + \frac{a}{2b} . S_i z \underset{87}{=} I_i z + \frac{a}{2b} : \underset{55}{\supset}$$

$$. I_i x = S_i y + \frac{a}{b}$$

99 $x \varepsilon \text{ cut} . y \varepsilon \text{ cut} . z \varepsilon \text{ cut} : \supset : x = y . \underset{G.98}{\overset{\supset}{\subset}} . x + z = y + z$

100 $x \varepsilon \text{ cut} . y \varepsilon \text{ cut} : \underset{96.54}{\supset} . x + y = y + x$

101 $x \varepsilon \text{ cut} . y \varepsilon \text{ cut} . z \varepsilon \text{ cut} : \underset{96.56}{\supset} . x + (y + z) = (x + y) + z$

102 $x \varepsilon \text{ cut} . y \varepsilon \text{ cut} . x > y : S_i z \equiv S_k x - I_l y : I_k x > S_l y$

$$. \supset . I_i z \equiv I_k x - S_l y : . \supset : z \varepsilon \text{ cut} . z = x - y$$

$$: : \subset : : : . S_k x > I_k x . S_l y > I_l y : \underset{68.61}{\supset} . S_k x - I_l y > I_k x - S_l y$$

$$: . \frac{a}{b} > S_i x - I_i y : \underset{68}{\supset} : \frac{a}{b} = S_k x - I_l y = S_k z . S_k x = \frac{a}{b} + I_l y$$

$$: . \frac{a}{b} < I_i x - S_i y : \underset{68}{\supset} : \frac{a}{b} = I_k x - S_l y = I_k z . I_k x = \frac{a}{b} + S_l y$$

$$: : . \frac{a}{b} : : \sim : S_i x . I_i y . \frac{a}{b} > S_i x - I_i y : . \sim : I_i x . S_i y . \frac{a}{b} < I_i x - S_i y$$

$$::. \supset :::. \frac{c}{d} = \frac{a}{b} + \frac{e}{f} . S_l x \underset{87}{<} I_l x + \frac{e}{2f} . I_l y \underset{87}{>} S_l y - \frac{e}{2f} : \supset$$

$$. S_l x - I_l y < I_l x - S_l y + \frac{e}{f} . \underset{61.62}{\supset} . S_l x - I_l y < \frac{a}{b} + \frac{e}{f} . \supset$$

$$. S_l x - I_l y < \frac{c}{d} :. \supset : \frac{c}{d} > \frac{a}{b} . \supset . \frac{c}{d} = S_l z$$

$$:: \frac{g}{h} = \frac{a}{b} - \frac{i}{j} . I_m x \underset{87}{>} S_m x - \frac{i}{2j} . S_m y \underset{87}{<} I_m y + \frac{i}{2j} : \supset$$

$$. I_m x - S_m y > S_m x - I_m y - \frac{i}{j} . \underset{61.62}{\supset} . I_m x - S_m y > \frac{a}{b} - \frac{i}{j} . \supset$$

$$. I_m x - S_m y > \frac{g}{h} :. \supset : \frac{g}{h} < \frac{a}{b} . \supset . \frac{g}{h} = I_m z ::. \underset{79}{\supset} . \frac{a}{b} = z$$

$$::: S_i z = S_k x - I_l y . \supset . S_k x = S_i z + I_l y \underset{96.G}{\nless} I_k (z + y)$$

$$: I_i z = I_k x - S_l y . \supset . I_k x = I_i z + S_l y \underset{96.G}{\ngtr} S_k (z + y) :. \underset{91}{\supset}$$

$$. x = z + y . \underset{23}{\supset} . z = x - y$$

103 $x \,\varepsilon\, \text{cut} . y \,\varepsilon\, \text{cut} . z \,\varepsilon\, \text{cut} . t \,\varepsilon\, \text{cut} : x > y . z > t$

$$:. \supset :. y > z . \gtrless . x - y < x - z : y > z . \gtrless . y - t > z - t$$

$$:: \underset{102}{\subset} :::. I_i y \underset{88.57}{=} S_i z + \frac{a}{b} . S_i x \underset{87}{=} I_i x + \frac{a}{2b} : \supset . S_i (x - y)$$

$$\underset{102}{=} \left(I_i x + \frac{a}{2b} \right) - \left(S_i z + \frac{a}{b} \right) \underset{65}{=} (I_i x - S_i z) - \frac{a}{2b}$$

$$:. S_i x - I_i y = I_i x - S_i z - \frac{a}{2b} . S_i x = I_i x + \frac{a}{2b}$$

$$: \underset{23.3}{\supset} . I_i y = S_i z + \frac{a}{b}$$

$$:: I_i y = S_i z + \frac{a}{b} . S_i t \underset{87}{=} I_i t + \frac{a}{2b} : \supset . I_i (y - t) \underset{102}{=} \left(S_i z + \frac{a}{b} \right)$$

$$-\left(I_it+\frac{a}{2b}\right)\underset{65}{=}(S_iz-I_it)+\frac{a}{2b}$$

$$:.\ I_iy-S_it=(S_iz-I_it)+\frac{a}{2b}\,.\,S_it=I_it+\frac{a}{2b}$$

$$\underset{23\,.\,3}{:\supset}.\ I_iy=S_iz+\ \frac{a}{}$$

104 $x\,\varepsilon\,\mathrm{cut}\,.\,y\,\varepsilon\,\mathrm{cut}\,.\,z\,\varepsilon\,\mathrm{cut}\,.\,t\,\varepsilon\,\mathrm{cut}:x>y\,.\,z>t$

$$\underset{G.103.92}{:.\supset:.}\ y=z\,.\gtrless.\,x-y=x-z:y=z\,.\gtrless.\,y-t=z-t$$

105 $x\,\varepsilon\,\mathrm{cut}\,.\,y\,\varepsilon\,\mathrm{cut}\,.\,z\,\varepsilon\,\mathrm{cut}:x>y+z:.\supset.\,x-(y+z)=(x-y)-z$

$$\underset{91}{:.\subset:}\ I_i\big((x-(y+z)\big)\underset{102}{=}I_ix-(S_iy+S_iz)\underset{65\,.\,102}{=}I_i(x-y)-S_iz$$

$$\underset{102}{\not>}\ S_i\big((x-y)-z\big)\,.\,I_i\big((x-y)-z\big)\underset{102\,.\,65}{=}I_ix-S_i(y+z)$$

$$\underset{102}{\not>}\ S_i\big(x-(y+z)\big)$$

106 $x\,\varepsilon\,\mathrm{cut}\,.\,y\,\varepsilon\,\mathrm{cut}\,.\,z\,\varepsilon\,\mathrm{cut}:x>y:.\underset{102\,.\,68\,.\,66}{\supset}.\,(x-y)+z=(x+z)-y$

107 $x\,\varepsilon\,\mathrm{cut}\,.\,y\,\varepsilon\,\mathrm{cut}\,.\,z\,\varepsilon\,\mathrm{cut}:x>y>z:.\underset{102\,.\,68\,.\,67}{\supset}.\ x-(y-z)=(x-y)+z$

108 $x\,\varepsilon\,\mathrm{cut}\,.\,y\,\varepsilon\,\mathrm{cut}:\supset:.\,(I_ix)\,(I_ky)\,.\equiv.\,I_l(xy):(S_mx)\,(S_ny)\,.\equiv.\,S_p(xy)$

109 $x\varepsilon\ \mathrm{cut}\ .\ y\varepsilon\ \mathrm{cut}:\supset.\,(xy)\ \varepsilon\ \mathrm{cut}$

$$\underset{79.85.108}{:.\subset::}\ S_ix>I_kx\,.\,S_ly>I_my\,.\,S_n(xy)=(S_ix)\,(S_ly)$$

$$.\,I_p(xy)=(I_kx)\,(I_my):\underset{75.108.61}{\supset}.\,S_n(xy)>I_p(xy)$$

$$:.\,\frac{a}{b}>(S_ix)\,(S_iy)\underset{79.85}{:\supset:}\frac{a}{b}=(S_kx)\,(S_iy)\underset{108}{=}S_l(xy)$$

$$.\,S_kx=\frac{a}{b}\ /\ S_iy$$

$$:.\,\frac{a}{b}<(I_ix)\,(I_iy):\supset:\frac{a}{b}=(I_kx)\,(I_iy)=I_l(xy)$$

$$.\,I_kx=\frac{a}{b}\ /\ I_iy$$

$$:\,:\,.\,\frac{a}{b}\,:\,:\,\sim\,:\,S_i x\,.\,S_i y\,.\,\frac{a}{b} > (S_i x)(S_i y)\,:\,.\,\sim\,:\,I_i x\,.\,I_i y\,.\,\frac{a}{b} < (I_i x)(I_i y)$$

$$:\,:\,\supset\,:\,:\,\frac{c}{d} = \frac{a}{b} \Big/ \left(1-\frac{e}{f}\right)\,.\,S_k x \underset{87.42}{<} (I_k x) \Big/ \left(1-\frac{e}{2f}\right)$$

$$.\,S_k y \underset{87}{<} (I_k y) \Big/ \left(1-\frac{e}{2f}\right)\,:\,\underset{74.75}{\supset}\,.\,(S_k x)\,(S_k y) < \frac{(I_k x)\,(I_k y)}{1-\frac{e}{f}+\frac{ee}{4ff}}$$

$$\underset{78}{<} \frac{(I_k x)\,(I_k y)}{1-\frac{e}{f}}\,.\,\supset\,.\,(S_k x)\,(S_k y) < \frac{a}{b} \Big/ \left(1-\frac{e}{f}\right).\,\supset$$

$$.\,(S_k x)\,(S_k y) < \frac{c}{d}\,:\,.\,\underset{108}{\supset}\,:\,\frac{c}{d} > \frac{a}{b}\,.\,\supset\,.\,\frac{c}{d} = S_l(xy)$$

$$:\,:\,\frac{g}{h} = \frac{a}{b}\left(1-\frac{i}{j}\right).\,I_l x \underset{87}{>} (S_l x)\left(1-\frac{i}{2j}\right).\,I_l y \underset{87}{>} (S_l y)\left(1-\frac{i}{2j}\right)$$

$$:\,\underset{74.75}{\supset}\,.\,(I_l x)\,(I_l y) > (S_l x)\,(S_l y)\left(1-\frac{i}{j}+\frac{ii}{4jj}\right) > (S_l x)\,(S_l y)\left(1-\frac{i}{j}\right)$$

$$.\,\supset\,.\,(I_l x)\,(I_l y) > \frac{a}{b}\left(1-\frac{i}{j}\right).\,\supset\,.\,(I_l x)\,(I_l y) > \frac{g}{h}$$

$$:\,.\,\supset\,:\,\frac{g}{h} < \frac{a}{b}\,.\,\underset{108}{\supset}\,.\,\frac{g}{h} = I_l(xy)\,:\,:\,.\,\underset{79}{\supset}\,.\,\frac{a}{b} = xy$$

110 $x\,\varepsilon\,\mathrm{cut}\,.\,y\,\varepsilon\,\mathrm{cut}\,.\,z\,\varepsilon\,\mathrm{cut}\,:\,\supset\,:\,x > y\,.\,\underset{\subset}{\supset}\,.\,xz > yz$

$$:\,.\,\underset{88.57}{\subset}\,:\,.\,I_i x = S_i y + \frac{a}{b}\,.\,S_i z \underset{87}{=} I_i z + \frac{c}{d}\,.\,\frac{c}{d} = \frac{a I_i z}{b I_i x}$$

$$:\,\supset\,.\,S_i(yz) \underset{108}{=} \left(I_i x - \frac{a}{b}\right)\left(I_i z + \frac{c}{d}\right) \underset{73.72.65.25}{=} (I_i x)\,(I_i z) - \frac{ac}{bd}$$

111 $x\,\varepsilon\,\mathrm{cut}\,.\,y\,\varepsilon\,\mathrm{cut}\,.\,z\,\varepsilon\,\mathrm{cut}\,:\,\underset{G.110}{\supset}\,:\,x = y\,.\,\underset{\subset}{\supset}\,.\,xz = yz$

112 $x\,\varepsilon\,\mathrm{cut}\,.\,y\,\varepsilon\,\mathrm{cut}\,:\,\supset\,.\,xy \underset{108.70}{=} yx$

113 $x\,\varepsilon\,\mathrm{cut}\,.\,y\,\varepsilon\,\mathrm{cut}\,.\,z\,\varepsilon\,\mathrm{cut}\,:\,\underset{108.70.39}{\supset}\,.\,x(yz) = (xy)\,z = xyz$

114 $x\,\varepsilon\,\mathrm{cut}\,.\,y\,\varepsilon\,\mathrm{cut}\,.\,z\,\varepsilon\,\mathrm{cut}\,:\,\underset{108.72}{\supset}\,.\,x(y+z) = xy + xz$

115 $x \,\varepsilon\, \mathrm{cut} \,.\, y \,\varepsilon\, \mathrm{cut} \,.\, z \,\varepsilon\, \mathrm{cut} \,.\, y > z : \supset .\, x(y-z) = xy - xz$

$$: . \underset{92.91}{\subset} : I_i\big(x(y-z)\big) \underset{108.102.73}{=} I_j x\, I_k y - I_j x\, S_l z \underset{79.85.75.68}{<} S_m(xy - xz)$$

$$.\, S_i\big(x(y-z)\big) \underset{108.102.73}{=} S_j x\, S_k y - S_j x\, I_l z \underset{79.85.75.68}{>} I_m(xy - xz)$$

116 $x \varepsilon\, \mathrm{cut} \,.\, y \,\varepsilon\, \mathrm{cut} \,.\, z \,\varepsilon\, \mathrm{cut} \,.\, t \,\varepsilon\, \mathrm{cut} \,.\, x > y \,.\, z > t : \supset .\, (x-y)(z-t) =$

$$= (xz + yt) - (xt + yz)$$

$$: . \underset{92.91}{\subset} : I_i(x-y)(z-t) \underset{108.102.74}{=} (I_j x\, I_k z + S_l y\, S_m t)$$

$$-(I_j x\, S_m t + S_l y I_k z) \underset{79.85.75.62.68}{<} S_n\big((xz + yt) - (xt + yz)\big)$$

$$.\, S_i(x-y)(z-t) \underset{108.102.74}{=} (S_j x\, S_k z + I_l y\, I_m t)$$

$$-(S_j x\, I_m t + I_l y\, S_k z) \underset{79.85.62.68.75}{>} I_n\big((xz + yt) - (xt + yz)\big)$$

117 $x \,\varepsilon\, \mathrm{cut} \,.\, y \,\varepsilon\, \mathrm{cut} \,.\, S_i z = S_k x / I_l y \,.\, I_i z = I_k x \,/\, S_l y : \supset : z \,\varepsilon\, \mathrm{cut} \,.\, z = x/y$

$$: . \subset :::\, . S_k x > I_k x \,.\, S_l y > I_l y : \underset{78.61}{\supset} .\, S_k x / I_l y > I_k x / S_l y$$

$$: . \frac{a}{b} > S_i x / I_i y : \underset{78}{\supset} : \frac{a}{b} = S_k x / I_i y = S_k z \,.\, S_k x = \frac{a}{b} I_i y$$

$$: . \frac{a}{b} < I_i x \,/\, S_i y : \underset{78}{\supset} : \frac{a}{b} = I_k x \,/\, S_i y = I_k z \,.\, I_k x = \frac{a}{b} S_i y$$

$$::: \frac{a}{b} :: \sim : S_i x \,.\, I_i y \,.\, \frac{a}{b} > S_i x \,/\, I_i y : . \sim : I_i x \,.\, S_i y \,.\, \frac{a}{b} < I_i x \,/\, S_i y$$

$$::. \supset ::. \frac{c}{d} = \frac{a}{b} \Big/ \left(1 - \frac{e}{f}\right) . S_l x \underset{87}{<} I_l x \Big/ \left(1 - \frac{e}{2f}\right) . I_l y \underset{87}{>} S_l y \left(1 - \frac{e}{2f}\right)$$

$$: \underset{78}{\supset} . S_l x / I_l y < (I_l x / S_l y) \Big/ \left(1 - \frac{e}{f} + \frac{ee}{4ff}\right) < (I_l x / S_l y) \Big/ \left(1 - \frac{e}{f}\right)$$

$$. \supset .\, S_l x \,/\, I_l y < \frac{a}{b} \Big/ \left(1 - \frac{e}{f}\right) . \supset .\, S_l x \,/\, I_l y < \frac{c}{d}$$

$$: . \supset : \frac{c}{d} > \frac{a}{b} . \supset . \frac{c}{d} = S_l z$$

$$:: \frac{g}{h} = \frac{a}{b}\left(1-\frac{i}{j}\right) . I_m x \underset{87}{>} S_m x \left(1-\frac{i}{2j}\right) . S_m y \underset{87}{<} I_m y \,/\, \left(1-\frac{i}{2j}\right)$$

$$: \underset{78}{\supset} . I_m x \,/\, S_m y > (S_m x \,/\, I_m y)\left(1-\frac{i}{j}+\frac{ii}{4jj}\right) > (S_m x \,/\, I_m y)\left(1-\frac{i}{j}\right)$$

$$. \supset . I_m x \,/\, S_m y > \frac{a}{b}\left(1-\frac{i}{j}\right) . \supset . I_m x \,/\, S_m y > \frac{g}{h}$$

$$:. \supset : \frac{g}{h} < \frac{a}{b} . \supset . \frac{g}{h} = I_m z ::. \underset{79}{\supset} . \frac{a}{b} = z$$

$$::: S_i z = S_k x \,/\, I_l y . \underset{42}{\supset} . S_k x = (S_i z)(I_l y) \underset{108.G}{\nless} I_k (zy)$$

$$: I_i z = I_k x \,/\, S_l y . \supset . I_k x = (I_i z)(S_l y) \underset{108.G}{\ngtr} S_k (zy)$$

$$:. \underset{91}{\supset} . x = zy . \underset{42}{\supset} . z = x/y$$

118 $x \,\varepsilon\, \text{cut} . y \,\varepsilon\, \text{cut} . z \,\varepsilon\, \text{cut} \underset{117.75.110}{:\supset:.} x > y . \supset\subset . x/z > y/z$

$$: x > y . \supset\subset . z/x < z/y$$

119 $x \,\varepsilon\, \text{cut} . y \,\varepsilon\, \text{cut} . z \,\varepsilon\, \text{cut} \underset{G.118}{:\supset:.} x = y . \supset\subset . x/z = y/z$

$$: x = y . \supset\subset . z/x = z/y$$

120 $x \,\varepsilon\, \text{cut} . y \,\varepsilon\, \text{cut} . z \,\varepsilon\, \text{cut} : \supset . x/y \underset{117.108.91}{=} xz\,/\,yz$

If x and y denote any defined objects of which the equality and the product are defined, then

121 $x = yy . \equiv . y = \sqrt{x} : \sqrt{x} \equiv .$ the square root of x

8. *Real numbers.*

In the following propositions x, y, z, t, u, v, w will denote cuts, if not stated otherwise.

122 $(x+y) - z \equiv (x-z) + y . x - z \equiv$ real number $x - z$

123 $(x-y) + y \underset{122}{=} (x+y) - y \underset{23}{=} x : (x-y) + (y+z) \underset{122.14}{=}$

$$= \big((x+z)+y\big) - y \underset{23}{=} x + z$$

124 $x \underset{23}{=} (x+y)-y \,.\underset{122}{\supset}.$ the cuts are real numbers

125 $(x-y)+z=(t-u)+z \,.\equiv.\, x-y=t-u \underset{91}{=} x-y$

126 $x-y=z-t \,.\overset{\supset}{\subset}:\, x+t=y+z \,.\underset{100}{\overset{\supset}{\subset}}.\, t-z=y-x$

$:.\subset: x-y=z-t \,.\underset{125}{\overset{\supset}{\subset}}.\,(x-y)+y+t$

$=(z-t)+y+t \,.\underset{123}{\overset{\supset}{\subset}}.\, x+t=z+y$

127 $x-y=z-t=u-v \,.\underset{95.125}{\supset}.\, x-y=u-v$

128 $((x-y)+z)+((t-u)+v) \,.\equiv.\, ((x-y)+(t-u))+(z+v)$

$:(x-y)+(t-u) \,.\equiv.$ the sum of $x-y$ and $t-u$

129 $(x-y)+(z-t)=(x+z)-(y+t) \underset{100}{=} (z-t)+(x-y)$

$.\underset{23}{\subset}: (x-y)+(z-t)+(y+t)$

$\underset{128}{=} ((x-y)+y)+((z-t)+t) \underset{123}{=} x+z$

130 $x-y=z-t \,.\overset{\supset}{\subset}:\,(x-y)+(u-v)=(z-t)+(u-v)$

$:\underset{126}{\subset}:(x-y)+(u-v)=(z-t)+(u-v)\,.\underset{129}{\overset{\supset}{\subset}}.\,(x+u)-(y+v)$

$=(z+u)-(t+v)\,.\underset{126}{\overset{\supset}{\subset}}$

$.\,x+u+t+v=z+u+y+v\,.\underset{99}{\overset{\supset}{\subset}}.\,x+t=z+y$

131 $(x-y)+((z-t)+(u-v)) \underset{129.101}{=} ((x-y)+(z-t))+(u-v)$

132 $(x-y)-(z-t)=(x+t)-(y+z)\underset{129}{=}(x-y)+(t-z)$

$.\underset{127}{\subset}:(x-y)-(z-t)=u-v\,.\underset{23}{\supset}.\,x-y=(u-v)+(z-t)\underset{129}{=}(z+u)-(t+v)$

$.\underset{126}{\supset}.\,x+(t+v)=y+(z+u)\,.\underset{101.126}{\supset}.\,u-v=(x+t)-(y+z)$

133 $(x-y)-\big((z-t)+(u-v)\big) \underset{129}{=} (x-y)-\big((z+u)-(t+v)\big)$

$\underset{132}{=} (x-y)+\big((t+v)-(z+u)\big)$

$\underset{129.14}{=} (x-y)+(t-z)+(v-u)$

134 $x=y \,.\equiv.\, x-y=0 : 0 \equiv \text{zero} :. \, x=y+z \,.\equiv.\, x-y>0$

$.\equiv: x-y \,.\, \varepsilon \,.\, \text{positive real number} : \equiv .\, y-x<0$

$.\equiv: y-x \,.\, \varepsilon \,.\, \text{negative real number}$

135 $0=0$

$.\underset{126}{\subset} :.\, x=y \,.\, z=t : \underset{99}{\supset} .\, x+t=y+z$

136 $(x-y)+0 \underset{134.135}{=} (x-y)+(z-z) \underset{129}{=} (x+z)-(y+z) \underset{126}{=} x-y$

137 $(x-y)-0 \underset{134.135.132.136}{=} x-y$

138 $x-y=z-t \,.\overset{\supset}{\subset}.\, (x-y)-(z-t)=0$

$:\subset: x-y=z-t \,.\underset{126}{\overset{\supset}{\subset}}.\, x+t=y+z$

$.\underset{134}{\overset{\supset}{\subset}}. =0=(x+t)-(y+z) \underset{132}{=} (x-y)-(z-t)$

139 $0-(x-y) \underset{132.136}{=} y-x : 0-(x-y) \,.\equiv.\, -(x-y)$

140 $x=y+z \,.\supset: x-y \underset{23}{=} z \,.\, y-x \underset{139}{=} -z$

141 $(x-y)(z-t) \,.\equiv.\, (x-y)z-(x-y)t \,.\equiv.\, x(z-t)-y(z-t)$

$.\equiv.\ \text{the product of } x-y \text{ and } z-t$

142 $(x-y)(z-t) \underset{141\ 132}{=} (xz+yt)-(xt+yz) \underset{112}{=} (z-t)(x-y)$

$\underset{100}{=} (y-x)(t-z)$

From this moment onwards *a*, *b*, *c*, etc. will denote real numbers, if not stated otherwise.

143 $a+(-b)=a-b\,.\,a-(-b)=a+b$

$$:\subset:.\,a=x-y\,.\,b=z-t:\supset:a+(-b)\underset{139}{=}(x-y)+(t-z)$$

$$\underset{129}{=}(x+t)-(y+z)\underset{132}{=}(x-y)-(z-t)$$

$$.\,a-(-b)\underset{139}{=}(x-y)-(t-z)\underset{132}{=}(x+z)-(y+t)$$

$$\underset{129}{=}(x-y)+(z-t)$$

144 $a(-b)\underset{139.142}{=}-(ab)\,.\,(-a)(-b)\underset{139.142}{=}ab$

145 $a\neq 0\,.\supset:ab=0\,.\underset{\subset}{\supset}.\,b=0$

$$:.\underset{134}{\subset}::.\,a=x-y\,.\,b=z-t:\supset::a>0\,.\underset{141}{\supset}:(x-y)z$$

$$=(x-y)t\,.\underset{111}{\underset{\subset}{\supset}}.\,z=t:.\,a<0\,.\underset{134}{\supset}:(y-x)z$$

$$=(y-x)t\,.\underset{111}{\underset{\subset}{\supset}}.\,z=t$$

146 $0\times 0\underset{134.142.111.99}{=}0$

147 $a\neq 0\,.\underset{140.144}{\underset{\subset}{\supset}}.\,aa>0$

148 $aa+bb=0\,.\underset{147.136.97.G}{\underset{\subset}{\supset}}:a=0\,.\,b=0$

149 $a(b+c)=ab+ac$

$$.\underset{142}{\subset}:.\,a=x-y\,.\,b=z-t\,.\,c=u-v:\supset$$

$$.\,a(b+c)\underset{142.114}{=}(xz+xu+yt+yv)$$

$$-(yz+yu+xt+xv)\underset{129}{=}\big((xz+yt)$$

$$-(yz+xt)\big)+\big((xu+yv)-(yu+xv)\big)$$

150 $a\neq 0\,.\supset:b=c\,.\underset{134.149.145}{\underset{\subset}{\supset}}.\,ab=ac$

151 $a(bc) \underset{141.144.113}{=} (ab)c$

152 $\frac{a}{-b} \underset{42.144}{=} -\frac{a}{b} \,.\, \frac{-a}{-b} \underset{42.144}{=} \frac{a}{b}$

153 $a \neq 0 \,.\, \supset : b = c \,.\, \underset{42.150}{\substack{\supset\\\subset}} \,.\, \frac{b}{a} = \frac{c}{a} :.\, a = b \neq 0 \,.\, \underset{42.150}{\supset} .\, \frac{c}{a} = \frac{c}{b}$

$:.\, \frac{c}{a} = \frac{c}{b} \neq 0 \,.\, \underset{42.150}{\supset} .\, a = b$

154 $\frac{a}{b} \,.\, a \neq 0 : \underset{42.145.146.G}{\supset} .\, b \neq 0$

155 $a \neq 0 \,.\, \underset{150.42}{\supset} .\, \frac{b}{c} = \frac{ab}{ac}$

9. *Complex numbers.*

156 $aa - bb \underset{149.144.143}{=} (a+b)(a-b)$

157 $aa + bb \,.\, \equiv .\, (a+bi)(a-bi) : 1i \equiv i \equiv$ the imaginary unit : $a + bi \,.\, \equiv .$ complex number $a + bi \,.\, \equiv$. the conjugate complex number to $a - bi$: $aa + bb \,.\, \equiv .$ the norm of $a + bi : \sqrt{aa + bb} \equiv |a + bi|$ $\equiv$. the absolute value of $a + bi$

158 $a + 0i \,.\, \equiv a : 0 + ai \equiv ai$ $\equiv$. purely imaginary number $ai : 0 + 0i \,.\, \equiv 0$

159 $a = b \,.\, c = d : \equiv .\, a + ci = b + di \underset{125}{=} a + ci$

160 $a + bi = c + di = e + fi \,.\, \underset{159.127}{\supset} .\, a + bi = e + fi$

161 $(a + bi) + (c + di) \,.\, \equiv .\, (a + c) + (b + d)\, i \underset{129}{=} (c + di) + (a + bi)$

162 $a + bi = c + di \,.\, \underset{161.159.130}{\substack{\supset\\\subset}} .\, (a + bi) + (e + fi) = (c + di) + (e + fi)$

163 $(a+bi)+\big((c+di)+(e+fi)\big) \underset{161.159.131}{=} \big((a+bi)+(c+di)\big)+(e+fi)$

164 $(a+bi)(c+di) \equiv .\, (ac-bd)+(ad+bc)i \underset{142.159}{=} (c+di)(a+bi)$

$:(a+bi)(c+di) \equiv .$ the product of $a+bi$ and $c+di$

165 $(a+bi)\big((c+di)+(e+fi)\big) \underset{161.164}{=} (a+bi)(c+di)+(a+bi)(e+fi)$

166 $(a+bi)\big((c+di)(e+fi)\big) \underset{164}{=} \big((a+bi)(c+di)\big)(e+fi)$

167 $a+bi=0 \,.\, \underset{158.164.145.146}{\supset} \,.\, (a+bi)(c+di)=0$

168 $(a+bi)(c+di)=0 \,.\, a+bi \neq 0 : \supset .\, c+di=0$

$:.\, \underset{148.158}{\subset} :: ac-bd \underset{164.158}{=} 0 \,.\, ad+bc \underset{164.581}{=} 0 : \supset$

$:.\, a \neq 0 \,.\, \supset .\, bdc \underset{138.146.135}{=} acc = -add \,.\, \underset{149}{\supset}$

$.\, a(cc+dd)=0 \,.\, \underset{145}{\supset} .\, cc+dd=0$

$: b \neq 0 \,.\, \supset .\, acd \underset{138.146.135}{=} bdd = -bcc \,.\, \underset{149}{\supset} .\, b(dd+cc)=0$

169 $a+bi \neq 0 \,.\, \supset : c+di=e+fi \,.\, \underset{23.134.158.165.167.168}{\supset\!\subset} .\, (a+bi)(c+di)=(a+bi)(e+fi)$

170 $a+bi \neq 0 \quad \underset{42.164.159}{.\supset .} \dfrac{c+di}{a+bi}=\dfrac{ac+bd}{aa+bb}+\dfrac{ad-bc}{aa+bb}i$

171 $a+bi \neq 0 \,.\, \supset : c+di=e+fi \,.\, \underset{42.169}{\supset\!\subset} .\, \dfrac{c+di}{a+bi}=\dfrac{e+fi}{a+bi}$

$:.\, a+bi=c+di \neq 0 \,.\, \underset{42.169}{\supset} .\, \dfrac{e+fi}{a+bi}=\dfrac{e+fi}{c+di}$

$: \dfrac{e+fi}{a+bi}=\dfrac{e+fi}{c+di} \neq 0 \,.\, \underset{42.169}{\supset} .\, a+bi=c+di$

172 $a+bi \neq 0 \,.\, c+di \neq 0 : \underset{169}{\supset} .\, \dfrac{e+fi}{c+di}=\dfrac{(a+bi)(e+fi)}{(a+bi)(c+di)}$

10. Concluding remarks.

The primitive concepts and primitive propositions together with the indispensable definitions make up the foundations. Writer has tried to start from perception and the primitive ideas involved therein (instead of *Veronese's* thought). The definitions are brought in closer accordance with the historical development of arithmetical concepts, and, perhaps, also with the development of individual thinking. Thus the rational, real, and complex numbers are defined not simply as ordered pairs of numbers already settled but, following *Peano*[1], as pairs in proper symbolical connexion. And the formulas for the fundamental operations are given not in their final form but in a more suggestive preliminary form, from which, by proper propositions, the final form is derived.

The symbolics used remain, of course, open to criticism. For it is a pure convention. The writer has tried to use no new symbol. Still *Heyting's* symbol $\equiv$ is used in a slightly different meaning as a symbol of definition. The quantity of different logical symbols is reduced to 5. This has been possible in consequence of writer's aim not to develop an algebra of arithmetical propositions but to convey to the reader the fundamental facts of arithmetic.

The writer is much indebted to Dr. *J. Nuut* and Dr. *A. Tudeberg* for very valuable criticisms.

Corrigenda.

In the proposition

19 must be $a \neq 1 . \underset{1}{\supset} : b . a = b + 1$

21 omit $a . b : \supset ::$

45 must be $\frac{a}{c}$ instead of $\frac{a}{b}$

48 must be . instead of :

52 and 69 must be . and : instead of : and :.

69 omit $\frac{ab}{c} \varepsilon n . \frac{de}{f} \varepsilon n$

[1] *G. Peano*, Arithmétique (1898), § 2 P 060.