NOTES ON STELLAR STATISTICS AND STELLAR EVOLUTION

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I.

Theoretical Luminosity-Curves and Stellar Evolution.

§ 1. The frequency-distribution of the luminosities of the fixed stars may be considered from two different standpoints. We may regard it as a mere chance distribution or, in other words, as the result of a combination of unknown factors; or we may attempt to build up the distribution on the basis of certain hypotheses concerning the nature of the stars and the laws of their evolution; if the result will agree with the observational facts, this may be regarded as a more or less weighty chance in favour of the initial hypothesis.

The following general considerations will form the basis of our hypotheses. According to a universally accepted opinion, all stars must undergo certain changes during sufficiently great intervals of time, the successive changes forming the steps of stellar evolution; this statement can hardly be disputed; whatever the source of the stellar energy may be, whether gravitational contraction, intraatomic processes or anything else, the question will be only of the time necessary to produce the changes, not of the changes themselves. As to the supposed direction of the evolution, there may exist different opinions; in the concrete schemes discussed below there has been made the general assumption of a decrease of the total energy[1]) of a star with the advance of time; but even this statement, apparently so obvious, cannot be accepted without reserves; we cannot be sure that the stars, notwithstanding their enormous loss of energy through radiation, are not centra of accumulation of energy, supplied from some unknown universal source[2]).

1) Total energy = gravitational potential + kinetic energy of the molecules + intraatomic energy etc.

2) Compare E. Wiechert. Anmerkungen zur Theorie der Gravitation

The time involved in the processes of stellar evolution is of no little importance in our speculations; if the time is great in comparison with the age of the Galactic system, so that the actual variety of the luminosities and other characteristics is the result of certain factors that acted ab initio, during the formation of the single stars, e. g. during their birth from a hypothetical universal nebula, then little or nothing can be deduced from the present distribution of the luminosities; but if the evolution of the single stars goes on more rapidly than the evolution of the whole system, a sort of a steady state in the distribution of stellar luminosities must be established, undergoing only a slow „secular" change; if we assume a one-sided direction of the stellar evolution, say, from a high luminosity towards complete extinction, the presence in our stellar system of a great number of luminous stars together with the supposed short duration of life of an individual in comparison with the age of the whole system leads to the hypothesis of the regeneration of an extinguished star; from the appearance of the Novae the suggestion may be made that this regeneration takes place in the form of sudden catastrophes, through which the extinguished star is transferred to the initial stage of evolution; thus the tempting hypothesis of a perpetual repetition of similar stages of stellar evolution, interrupted by catastrophes, presents itself to our mind. Such a hypothesis may be treated mathematically, if certain assumptions on the rate of change of the luminosity with time and on the probability of the catastrophes are made. It may be remarked that if the Novae are interpreted as actually giving origin to new stars, the relative shortness of a star's life follows as a natural consequence; indeed, the number of Galactic Novae is probably not less than one per year; that gives for a period of 10^9 years a number of Novae nearly equal to the probable number of all stars in the Galaxy: and 1000 million years is without doubt a small period in comparison with the time required for the formation of this immense system, where the period of revolution of a star along some hypothetical orbit must be of the order of 100 million years. Of course, the relative frequency of the Novae may be

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und über das Schicksal der Gestirne. Vierteljahrsschrift der Astron. Gesellschaft, 56, H. 3 (1921), p. 187.

regarded as a convincing proof of the part played by them in stellar evolution; during the short time of spectroscopic observations the transformation of these objects into planetary nebulae and even into Wolf-Rayet stars (Nova Persei) has been stated; if we take an interval during which there might without doubt occur no sensible changes in the general aspect of the heavens, say — ten million years — we should observe a corresponding amount of planetary nebulae or Wolf-Rayet stars among the stars of the Galaxy: but the actual number of these objects is several thousand times smaller. Where are the remaining millions? Evidently they are transformed into millions of ordinary stars. If so, the most natural conclusion follows that the major part — if not all — of the stars of our Galaxy have been once Novae and, vice versa, will be Novae again. All these considerations increase the probability of the hypothesis of a perpetual reiteration of the consecutive stages of stellar evolution; in the following discussion we will therefore adopt this hypothesis. We will admit too, that the chances of a catastrophe are equal for all stars, and that after the catastrophe the average star will undergo the same steps of evolution as during its previous „life-time“.

§ 2. Let M be the absolute bolometric[1]) magnitude, $\varphi(M)$ — the Luminosity-Curve or the frequency-function of the absolute bolometric magnitudes within a certain stellar universe, so that $\varphi(M)dM$ is equal to the number of stars having M between M and $M+dM$; the absolute magnitude is supposed to undergo changes with the time τ, so that $M=f(\tau)$; let

$$V(M)=\frac{\delta f(\tau)}{\delta\tau}=\frac{\delta M}{\delta\tau} \quad (1);$$

this function we shall call for convenience's sake the „rate of cooling“; if $V(M)>0$, then the absolute magnitude increases (or the total radiation diminishes) with the time and vice versa. For the same star $V(M)$ is supposed to be a function of the total radiation emitted. We shall at first limit our problem to the case when $V(M)\neq 0$.

Let q be for a single star the probability of a catastrophe occurring during the unit of time, so that $nqd\tau$ will represent

1) Or the magnitude defining the total amount of radiation lost to space. See A. S. Eddington, Monthly Notices 77 (1917), p. 605.

the probable number of catastrophes among n stars during the time $d\tau$. The simplifying assumption will be made that q is constant for all stars and all parts of the stellar universe. On the contrary, we shall admit of a different rate of cooling for physically different stars, e. g. for stars with unequal masses; the function V must therefore depend on a parameter μ,

$$V = V(\mu, M).$$

We shall suppose that the whole aggregate of stars may be exhaustively classified according to the parameter μ alone, and that the function V does not depend upon other parameters; if therefore $\psi(M, \mu)$ represents the Luminosity-Curve of stars with a given value of the parameter, $X(\mu)$ — the frequency-function of the parameter μ, we shall have

$$\varphi(M) = \int_{\mu_1}^{\mu_2} \psi(M, \mu) \,.\, X(\mu) \,.\, d\mu \quad (2),$$

where μ_1 and μ_2 are the extreme values of the parameter that may occur for the given value of M.

Let us consider the group of stars having a constant μ, so that V for this group will be a function of the absolute magnitude only; the number of stars whose absolute magnitudes will be comprised between $M_1 = M - \frac{1}{2}\, dM$ and $M_2 = M + \frac{1}{2}\, dM$ will be

$$\psi(M) dM\,;$$ [1]

during the time $d\tau$ the number

$$\psi(M) dM \,.\, q d\tau \quad \text{(a)}$$

will be lost through the catastrophes; we shall suppose that as the result of the catastrophe the absolute luminosity is increased instantaneosly so that a certain absolute magnitude M_0, called further the Initial Magnitude, will be attained; the schematizing assumption will be made that the Initial Magnitude depends only upon the parameter μ,

$$M_0 = \alpha(\mu) \quad (3),$$

and that for a given parameter μ always.

$$M \geqslant M_0 \quad (4).$$

1) The parameter μ is understood to be contained in ψ and V implicite.

If the variation of M goes on in the direction from M_1 to M_2, the class of stars with the magnitude M_1 will supply our group with new members, and the members of the group will in their turn pass over to the group M_2. The number coming from the group M_1 during the time $d\tau$ will be

$$\psi(M_1) \cdot \frac{\delta M}{\delta \tau}\bigg|_{M=M_1} d\tau = \psi(M_1)\, V(M_1) d\tau \quad \text{(b)},$$

and, similarly, the number lost towards M_2 is

$$\psi(M_2)\, V(M_2)\, d\tau \quad \text{(c)};$$

now we will make the assumption of a state of equilibrium, this assumption being the consequence of the supposed relative shortness of the life-time of an individual star; the total loss must therefore be counterbalanced by the gain, and thus from (a), (b) and (c) we have

$$\psi(M_2)\, V(M_2) - \psi(M_1)\, V(M_1) + q\psi(M)\, dM = O \quad \text{(d)}.$$

Further we may put

$$\psi(M_2) = \psi(M) + \tfrac{1}{2}\, dM \frac{\delta \psi}{\delta M} + \ldots .$$

$$\psi(M_1) = \psi(M) - \tfrac{1}{2}\, dM \frac{\delta \psi}{\delta M} + \ldots .$$

and

$$V(M_2) = V(M) + \tfrac{1}{2}\, dM \frac{\delta V}{\delta M} + \ldots .$$

$$V(M_1) = V(M) - \tfrac{1}{2}\, dM \frac{\delta V}{\delta M} + \ldots .$$

Substituting this into (d) and retaining only members of the first order with respect to dM, we obtain finally

$$V(M) \frac{\delta \psi(M)}{\delta M} + \psi(M) \frac{\delta V(M)}{\delta M} + q\psi(M) = O \ldots . \quad (5).$$

Since $V(M) \neq O$, we may transform this equation into

$$\frac{1}{\psi} \frac{\delta \psi}{\delta M} + \frac{1}{V} \frac{\delta V}{\delta M} + \frac{q}{V} = O,$$

whence it is easy to obtain with the aid of simple integration (the inferior limit of integration being chosen equal to M_0):

$$\log \psi(M) = \text{Const.} - \log V(M) - q \log e \int_{M_0}^{M} \frac{dM}{V(M)} \ldots . \quad (6);$$

The constant of integration is understood to be a function of the parameter μ.

The analytical correlation (6) between the Luminosity-Curve ψ, the rate of cooling V and the frequency of catastrophes q will hold for $V(M) \neq O$; let us suppose that this condition will be fulfilled for all $M > M_0$; but for the Initial Magnitude M_0 we will admit of an initial stationary period Θ, during which the luminosity does not vary; this would correspond to the constancy of the bolometric magnitude of the star during its giant stage of evolution, a constancy required by the theory of Radiative Equilibrium. Θ must evidently be a function of the parameter μ.

Let n be the total number of stars having the given value of the parameter μ, n_1 — the number of stars with $M = M_0$ and n_2 — the number having $M > M_0$; evidently

$$n = n_1 + n_2.$$

Since the catastrophes, according to our assumptions, cannot increase the luminosity above the Initial one, we may neglect the catastrophes occurring among the stars with $M = M_0$, for in this case nothing will be changed; thus the loss from the initial group is due only to the „cooling", and may be expressed in two forms:

1) as
$$\frac{n_1 \, d\tau}{\Theta} \quad \text{(e)},$$

for every star remains within the initial group during the interval Θ;

2) or as
$$\psi(M_0) V(M_0) \, d\tau \quad \text{(f)}$$

(to compare with (b) or (c)), where $\psi(M_0)$ and $V(M_0)$ denote the limit of the functions $\psi(M_0 + \triangle)$, $V(M_0 + \triangle)$ for $\triangle = O$. It may be remarked here that $V(M_0)$ will generally be not equal to zero.

Since (e) and (f) denote the same quantity, we have

$$\frac{n_1}{\Theta} = \psi(M_0) V(M_0) \quad \text{or}$$

$$n_1 = \Theta \psi(M_0) V(M_0) \; \ldots . \quad (7).$$

The gain of the initial group takes place only through the catastrophes occurring among the remaining stars; the gain will be therefore equal to

$$n_2 q d\tau \; \ldots . \quad (g);$$

for a steady state we have (e) = (g), or

$$n_1 = n_2 \Theta q \quad (8).$$

Combining (7) and (8), we obtain

$$n_2 = \frac{1}{q} \psi(M_0) V(M_0) \dots \quad (9) \quad \text{and}$$

$$n_1 + n_2 = n = \psi(M_0) V(M_0) \left[\Theta + \frac{1}{q}\right] \dots \quad (10).$$

Equation (10) furnishes the constant of integration in (6); if we put $M = M_0$ in (6), we obtain

$$\log \psi(M_0) = \text{Const.} - \log V(M_0), \text{ or}$$

$$\text{Const.} = \log[\psi(M_0) \,.\, V(M_0)] \text{ and from (10)}$$

$$\text{Const.} = \log \frac{n}{\Theta + \frac{1}{q}} = \log \frac{nq}{1 + q\Theta};$$

substituting this into (6), we obtain

$$\log \psi(M) = \log \frac{qn}{1 + q\Theta} - \log V(M) - q \log e \int\limits_{M_0}^{M} \frac{dM}{V(M)} \quad \text{or}$$

$$\psi(M) = \frac{qn}{1 + q\Theta} \cdot \frac{1}{V(M)} \, e^{-q \int\limits_{M_0}^{M} \frac{dM}{V(M)}} \dots \quad (6^1)$$

In a like manner (7) will be transformed into

$$n_1 = \frac{nq\Theta}{1 + q\Theta} \dots \quad (7^1).$$

Formulae (6^1) and (7^1) represent the distribution of the absolute luminosities within a group with a constant value of the parameter; this distribution we shall call the Elementary Luminosity-Curve. The Luminosity-Curve of the whole aggregate of stars will be found through integration of the Elementary distribution over the entire range of the parameter.

The constants in (6^1) and (7^1) are chosen so that the total number will be

$$n_1 + \int\limits_{M_0}^{\infty} \psi(M) \, dM = n;$$

we shall put further $n = 1$; in this case n_1 and $\psi(M)dM$ will represent the fraction of the total number of stars having $M = M_0$ or M between M and $M + dM$ respectively.

The most convenient choice of the parameter μ mentioned above will be to put it equal to the Initial Magnitude,

$$\mu = M_0 \quad (3^1).$$

With the aid of equation (2), modified according to the composite and discontinuous character of the Elementary frequency-distribution, we obtain

$$\varphi(M) = \int\limits_{M_0=-\infty}^{M_0=M} \psi(M, M_0) X(M_0)\, dM_0 + [n_1 X(M_0)]_{M_0=M} \;\ldots\; (11);$$

or, taking ψ and n_1 from (6^1) and (7^1) with $n = 1$, we have

$$\varphi(M) = q \left\{ \int\limits_{M_0=-\infty}^{M_0=M} \frac{X(M_0)}{[1 + q\Theta(M_0)] \cdot V(M, M_0)} \, e^{-q \int\limits_{M_0}^{M} \frac{dM}{V(M, M_0)}} dM_0 + \right.$$

$$\left. + \frac{\Theta(M) X(M)}{1 + q\Theta(M)} \right\} \;\ldots\; (11^1).$$

If we choose the frequency-function of the Initial Magnitudes so that

$$\int\limits_{-\infty}^{+\infty} X(M_0)\, dM_0 = 1 \;\ldots\; (12),$$

then we shall have

$$\int\limits_{-\infty}^{+\infty} \varphi(M)\, dM = 1 \;\ldots\; (13).$$

In this case $\varphi(M)dM$, computed from (11^1), will give the fraction of stars whose absolute magnitudes are contained between M and $M + dM$.

§ 3. Before the application of formula (11^1) to particular cases, we shall discuss some observational data from which certain constants needed in the numerical computations may be deduced.

a) The most frequent absolute bolometric magnitude occurring among stars, or the position of the maximum of the actual Luminosity-Curve will be one of the constants to determine. For this purpose we may use the data found by Kapteyn and Van Rhijn[1]); since these data refer to the visual magnitudes, a transition from the visual to the bolometric magnitudes is needed. Were the percentage of each spectral type occurring among a given absolute magnitude known, the transition could be performed easily; but for lack of the exact data the following effective data will fit our purpose equally well. We shall adopt the following correlation between the spectral type, colour-index, visual surface brightness (j) and integral (bolometric) surface brightness (I), and the absolute visual magnitudes of the dwarf series (M_{vis}):

Table 1.

Sp. Type	M_{vis}	j	I	$I-j$	Colour Index
B_0	—6.6	—1.93	—2.21	—0.28	—0.26
A_0	—4.3	—1.76	—1.97	—0.21	0.00
F_0	—2.6	—0.83	—0.83	0.00	+0.29
G_0	—0.7	+0.45	+0.40	—0.05	+0.55
K_0	+0.9	+2.16	+1.65	—0.51	+1.02
M_a	+4.8	+4.40	+2.90	—1.50	+1.50
(M)	(+10.)	—	—	(—2.0)	—

The absolute magnitudes are taken according to Fr. H. Seares[2]); the visual surface brightnesses, based on the temperatures by I. Wilsing, are taken from a paper by E. Bernewitz[3]); the I are computed from the same data (j and I for the sun assumed $= 0$); the colour-index is given according to H. N. Russel, the data being quoted from a paper by I. Wilsing[4]). The 5th column of the table contains the difference between the bolo-

1) Mt. Wilson Contr. 188, p. 8, Table IV.

2) The Masses and Densities of Stars. Astrophysical Journal 55 (1922), p. 179.

3) Über die Dichten der Doppelsterne. Astronomische Nachrichten 5089 (1921).

4) Astronomische Nachrichten 5124.

metric and visual magnitudes; for convenience's sake for $M_{vis} = +10$ has been asumed the difference $= -2.00$ *mg*, though no data are available for such faint stars; the value assumed would correspond to a colour-index of about $+1.60$, not much differing from the colour-index of the more luminous *M*-type dwarfs; thus the somewhat arbitrarily adopted figure in the last line of the table will not be seriously inconsistent with the fact found by E. Hertzsprung [1]), that absolutely very faint stars do not exhibit sensible variation of colour.

The curves on Fig. 1 represent the difference *I—j* as the function of the colour-index (Ist curve) and the absolute magnitude (IInd curve); curve II may be applied only to stars of the dwarf series, while curve I must be valid for all stars.

The data of table 1 can be used directly only for stars of low or moderate luminosity — say, for $M_{vis} > -3{,}5$. Among the brighter stars, besides the normal blue stars, occur in a considerable proportion yellow and red giants; for this reason we shall treat the brighter stars separately and try to determine the proportion of stars of different colour among them. Since the data for our Galactic system cannot be used for this purpose without a thorough discussion, we shall make use of the data for two globular clusters, found by H. Shapley [2]) and H. Shapley and Helen N. Davis [3]). The data limited by the magnitude for which, according to Shapley, they are complete, are contained in table 2.

The difference *I—j* has been read from curve I, fig. 1; the absolute magnitudes for $\pi = 1''$ were computed with the aid of the following reductions:

$$\text{for } M_3 \ \ldots\ldots \ M_{abs} = m_{app} - 20{,}70$$
$$\text{„ } M_{13} \ \ldots\ldots \ M_{abs} = m_{app} - 20{,}20.$$

The reductions are taken from Mt. Wilson Contrib. 176, p. 2, and rounded off to $0^{mg}{,}1$.

Since our only purpose is to determine the frequency of the various values of *I—j* within a given absolute magnitude, the

1) Effective Wave-Lengths of Absolutely Faint Stars. Astrophysical Journal, 42 (1915), pp. 111—119.

2) Thirteen Hundred Stars in the Hercules Cluster (Messier 13), Mt Wilson Contributions 116 (1915). Table XII, p. 51 „distance $\geqq 2'.0$".

3) Photometric Catalogue of 848 Stars in Messier 3. Ibidem, 176 (1920). Table VIII, p. 37, „distance 2′.0 to 11′.3".

Table 2.

Colour-Class		$< b_5$	b_5-a_0	a_0-a_5	a_5-f_0	f_0-f_5	f_5-g_0	g_0-g_5	g_5-k_0	k_0-k_5	$> k_5$	All colours
Colour-Index		<-0.20	−0.20 to 0.00	0.00 to 0.20	0.20 to 0.40	0.40 to 0.60	0.60 to 0.80	0.80 to 1.00	1.00 to 1.20	1.20 to 1.40	>1.40	—
$J-j$ (average)		−0.26	−0.24	−0.14	0.00	0.00	−0.20	−0.41	−0.66	−1.03	−1.50	—
Photovis. Magn. Apparent	Absolute ($\pi=1''$) Mean											
						Messier 3						
<12.00	−9.7	0	0	0	1	0	0	0	1	1	0	3
12.00—12.59	−8.4	0	0	0	0	0	0	0	0	1	2	3
12.60—13.19	−7.8	0	0	0	0	0	2	0	6	6	2	16
13.20—13.79	−7.2	0	0	0	0	0	1	4	8	3	0	16
13.80—14.39	−6.6	0	0	1	0	1	3	12	14	1	0	32
14.40—14.99	−6.0	1	1	0	3	5	30	24	6	0	0	70
15.00—15.59	−5.4	0	10	28	18	30	52	31	4	0	0	173
15.60—16.19	−4.8	2	12	14	4	21	59	19	2	0	0	133
16.20—16.79	−4.2	1	2	2	33	63	43	2	1	0	0	147
											Total	593
						Messier 13						
12.00—12.59	−7.9	0	0	0	0	0	0	0	4	6	3	13
12.60—13.19	−7.3	1	0	0	1	1	1	13	7	0	0	24
13.20—13.79	−6.7	0	0	0	0	0	4	13	4	1	1	23
13.80—14.39	−6.1	0	3	1	2	7	33	22	8	0	0	76
14.40—14.99	−5.5	0	6	10	9	6	35	36	7	1	0	110
15.00—15.59	−4.9	14	56	22	12	26	58	30	3	1	0	222
											Total	468

data for both clusters, separately for each magnitude-class (or for classes differing but little) may be joined; in a like manner, the different colour-classes, having similar values of $I-j$ (differing, say, by less than $0^{mg}.1$) will be joined, and thus we obtain table 3; the numbers of this table are for convenience converted into percentages.

Table 3. Frequency of the difference $J-j$. ($M_3 + M_{13}$).

Col.-Cl.	a_5-f_5	a_0-a_5	b_0-a_0 and f_5-g_0	g_0-g_5	g_5-k_0	k_0-k_5	$>k_5$	Total
$J-j$ average	0.00	−0.14	−0.23	−0.41	−0.66	−1.03	−1.50	
M_{vis} mean ($\pi = 1''$)		Percentage						
<−8.1	16.7	0	0	0	16.7	33.3	33.3	100.0
−7.85	0	0	6.9	0	34.4	41.3	17.3	99.9
−7.25	5.0	0.0	7.5	42.5	37.5	7.5	0.	100.0
−6.65	1.8	1.8	12.7	45.5	32.7	3.6	1.8	99.9
−6.05	11.6	0.7	46.6	31.5	9.6	0.	0.	100.0
−5.45	22.3	13.4	36.4	23.7	3.9	0.4	0.	100.1
−4.85	17.7	10.1	56.6	13.8	1.4	0.3	0.	99.9
−4.20	65.3	1.4	31.3	1.4	0.7	0.	0.	100.1

We will make the assumption — which seems us plausible enough — that the percentages found will hold for our Galactic aggregate of stars. This assumption does by no means signify the identity of the Galactic Luminosity-Curve with the Luminosity-Curve for the Globular Clusters; on the contrary, they may be very different from one another and, probably, are actually so [1]).

With the aid of the tables 1 and 3 table 4 was computed; this table gives the effective quantities for transforming a Luminosity-Curve arranged according to the visual magnitudes into one arranged according to the bolometric magnitudes; the data are given for the classes of absolute visual magnitude as

1) The term „Galactic" means here simply the part of the stellar universe that surrounds us.

they figure in the table by Kapteyn and Van Rhijn[1]) already mentioned.

Table 4.

Effective Quantities, representing the Number of Stars of different Absolute Bolometric Magnitude, by which 100 Stars of a given Absolute Visual Magnitude must be remplaced.

$M vis (\pi=1'')$	≤ — 8.64	— 7.64	— 6.64	— 5.64	— 4.64	— 3.64	— 2.64	— 1.64	— 0.64
M Bol. — — *M vis.*									
+ 1.0	0	0	0	0	0	0	0	2	0
0.0	23	33	55	74	85	89	100	98	95
— 1.0	60	62	44	26	15	11	0	0	5
— 2.0	17	5	1	0	0	0	0	0	0

$M vis (\pi=1'')$	+ 0.36	1.36	2.36	3.36	4.36	5.36	6.36	7.36	8.36	9.36
M Bol. — — *M vis.*										
+ 1.0	0	0	0	0	0	0	0	0	0	0
0.0	72	28	0	0	0	0	0	0	0	0
— 1.0	28	72	88	69	55	44	34	24	15	6
— 2.0	0	0	12	31	45	56	66	76	85	94

The table is arranged according to two arguments: the absolute visual magnitude (M^{vis}) and the difference between the bolometric and the visual magnitude ($M^{Bol} - M^{vis}$). Between $M_{vis} = -3.64$ and -4.64 there is no sensible break, notwithstanding the different methods by which the data above and below these limits have been computed. For $M^{vis} > 6.36$ the quantities are the result of a more or less legitimate extrapolation. With the aid of these reduction factors the Luminosity-Curve found by Kapteyn and Van Rhijn[2]) was transformed so as to make it correspond to the absolute bolometric magnitudes. Table 5 contains the result.

1) Loc. cit.

2) Mt Wilson Contr. 188. Table IV.

Table 5:

Luminosity-Curve. Number of Stars per 10^9 Cubic Parsecs Near the Sun.

Absolute Magnitude (Visual or Bolometric)	Observed		Reduced	
	log φ (M_{vis})	φ (M_{vis})	φ (M_{Bol})	log φ (M_{Bol})
— 13.64	—	0	0.3	— 0.5
— 12.64	—	0	1.7	+ 0.23
— 11.64	0.2	1.6	8.8	0.94
— 10.64	0.6	4.0	62.5	1.796
— 9.64	1.55	36	221	2.344
— 8.64	2.373	236	998	2.999
— 7.64	3.148	1 410	3 540	3.549
— 6.64	3.843	6 970	11 500	4.061
— 5.64	4.474	29 800	38 100	4.581
— 4.64	5.020	105 000	123 000	5.090
— 3.64	5.493	311 000	277 000	5.442
— 2.64	5.894	784 000	784 000	5.894
— 1.64	6.215	1 640 000	1 760 000	6.246
— 0.64	6.472	2 970 000	4 140 000	6.617
+ 0.36	6.662	4 590 000	8 420 000	6.925
+ 1.36	6.776	5 970 000	9 750 000	6.989
+ 2.36	6.836	6 860 000	7 010 000	6.846
+ 3.36	6.819	6 590 000	5 370 000	6.730
+ 4.36	6.737	5 460 000	3 390 000	6.530
+ 5.36	6.627	4 240 000	2 000 000	6.301
+ 6.36	6.364	2 310 000	1 230 000	6.090
+ 7.36	6.200	1 590 000	380 000	5.58
+ 8.36	6.00	1 000 000	(20 000)	—
+ 9.36	5.4	250 000	—	—

Here the 2nd column contains the data directly given by Kapteyn and Van Rhijn, the 5th column — the result of the reduction. These data are plotted on fig. 2, curve II (the ordinates of this curve representing the logarithm of the number of stars per 1000 cubic parsecs); the maximum of the reduced curve lies at $M_{Bol} = + 1.1$, whereas the maximum of the „visual" curve, according to Kapteyn, has its place at $M_{vis} = + 2.7$; the position of the maximum found we will adopt further. For comparison the Gaussian curve which, according to Kapteyn, represents with high accuracy the distribution of the visual luminosities, is given (Fig. 2, I), this curve being shifted by 1.6 magnitudes towards the left hand side to make its maximum to coincide with the maximum of curve II; at the first glance on these curves we may infer that the close resemblence of the distribution of the visual magnitudes to a Gaussian error-curve, found by Kapteyn, ceases after the curve is reduced to the bolometric magnitudes.

b) The dispersion of the masses of all stars is the other quantity which proves to be of importance in our computations. Fr. H. Seares[1]) and H. N. Russel[2]) discuss the dispersion of the masses within a given spectral type and for binaries and find the dispersion to be small. But when all stars together are considered, the dispersion will probably be more conspicuous. As a working hypothesis we will assume, as Fr. H. Seares[3]) has done, a Gaussian distribution of the logarithms of the masses; this assumption seems to agree better with the actual distribution than a similar distribution of the masses themselves not only for a single spectral type, but equally for the stars as a whole. The derivation of the mass-dispersion must satisfy two conditions: 1) the choice of the stars on which the determination will be based must be representative of the whole aggregate; 2) the determination of the masses must be free from any great source of accidental error which may entirely conceal the true dispersion of the masses; e. g., masses of binaries computed from their parallaxes cannot be used for this purpose, because they satisfy neither of these conditions.

The correlation between mass and spectral type for the dwarf series, discussed by Fr. H. Seares[4]), may furnish a good basis for the derivation of the dispersion; the latter being small within the limits of one spectral type and, at any rate, considerably smaller than the dispersion for all stars together, we may

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1) Loc. Cit. pp 184—186.

2) On the Calculation of Masses from Spectroscopic Parallaxes. Astrophysical Journal 55 (1922), pp 238—241.

3) Loc. cit. p. 183.

4) Loc. cit. p. 179 table IV.

assume the mass of a dwarf star equal to the geometrical mean mass for the corresponding spectral type; thus the problem is reduced to the determination of the frequency of the different spectral types within a given volume of space; to obtain a provisional value of the dispersion we shall choose the stars from the Mount Wilson spectroscopic list[1]) with $\pi^{sp} \geqslant 0''.096$; since the early types are little represented in the list, with the exception of few A-s, the following three stars were added from the list of measured parallaxes by Kapteyn and Weersma[2]): β Leonis, sp. A_2; α Aquilae, sp. A_5; α Piscis Australis, sp. A_3. According to the spectral type the stars were distributed as follows (companions of double stars were counted as individuals):

Table 6.

Sp. Type	Giants G–M	Dwarfs B_0–B	A_0	A_5	F_0	F_5	G_0	G_5	K_0	K_5	M	Total
Number	4	0	1	3	0	5	9	6	16	24	22	90
μ ($\odot$=1)	3	—	6.0	4.0	2.5	1.5	1.0	0.76	0.68	0.62	0.59	
Log μ	0.47	—	0.78	0.60	0.40	0.19	—0.01	—0.12	—0.17	—0.21	—0.23	

μ denotes here the probable mass of a single star adopted according to Fr. H. Seares.

The smallest masses are the most numerous; it appears therefore that the distribution in table 6 represents only one half of the frequency-curve with the maximum at about log $\mu = -0{,}22$ or $\mu = 0{,}60\,\odot$; assuming this, the mean square deviation of the log μ from their adopted mean value (—0.22), computed from the data of table 6, results as.

$$\triangle = \pm 0{,}265.$$

Let $\triangle_1$ denote the mean square deviation of log μ within one spectral subdivision; then the true dispersion of the log μ will be

$$\triangle_0 = \pm \sqrt{\triangle^2 + \triangle_1^2} \quad (14).$$

We may safely assume $\triangle_1 \leqslant \frac{1}{2}\triangle_0$; this would give

$$\triangle_0 \leqslant 0{,}31 \text{ and}$$
$$\triangle_1 \leqslant 0{,}16.$$

1) W. S. Adams, A. H. Joy, G. Strömberg and Cora G. Burwell. The Parallaxes of 1646 Stars Derived by the Spectroscopic Method. Mt. Wilson Contr. 199.

2) Groningen Publications 24.

We will further adopt the value $\triangle_0 = 0{,}31$; in any case this value will be rather over-estimated, since the effect of selection of luminous and, followingly, massive stars in table 6 has without doubt increased the dispersion; however, a sensible change in the value of $\triangle_0$ will probably not take place, if data entirely freed from selection will be used. The order of the uncertainty in $\triangle_0$ may be estimated in the following way; let us reduce the number of the massive Giants and A-type stars to one half of the one adopted in table 6: then the dispersion will come out as

$$\triangle_0 = \pm 0{,}26;$$

thus the change is not considerable.

To what extent selection has influenced the material used may be judged from the distribution of the absolute magnitudes given in the following table:

Table 7.

M_{vis} ($\pi = 1''$)	−4	−3	−2	−1	0	+1	+2	+3	+4	+5	+6	Total
n_1	3	2	5	10	8	19	11	7	8	14	3	90
n_2	3	2	5	12	7	15	7	8	4	12	3	78

For convenience's sake the absolute magnitudes are reduced to $\pi = 1''$; n_1 denotes the number, if the companions of double stars are counted separately; n_2 — if each double pair is counted as one star having the integral luminosity of the system. From the table the conclusion may be drawn that the proportion of stars fainter than $+1.0$ is probably smaller than the true one. It must be pointed out that the effect of selection will influence the distribution of the spectral types in a smaller degree than the distribution of the luminosities.

The dispersion of the masses has been discussed with the chief purpose to obtain a basis for an a priori deduction of the distribution of the Initial Magnitudes, $X(M_0)$, mentioned in the preceding section. According to A. S. Eddington[1]), the total luminosities of giant stars must vary approximately as the square of their masses — the constancy of a certain factor k

1) Monthly Notices 77 (1917), p. 604.

(the mass-coefficient of absorption) supposed. Since our conception of the Initial Luminosity must correspond to the giant stadium of a star's evolution, we infer that a Gaussian distribution of the logarithms of the masses with the dispersion $\triangle_0$ will correspond to a Gaussian distribution of the Initial Magnitudes with a dispersion

$$r = 5 \triangle_0 ;$$

we will asume therefore

$$X(M_0) = \frac{1}{r\sqrt{2\pi}} e^{-\frac{(M_0 - A)^2}{2r^2}} \quad \ldots . \quad (15).$$

If $\triangle_0 = \pm 0{,}31$, we have $r = \pm 1{,}55$ magnitudes. A denotes the most frequent Initial Magnitude, and must not be confounded with the most frequent absolute magnitude.

§ 4. According to the equation (11′), the distribution of luminosities within a stellar universe can be found if the functions $X(M_0)$, $V(M, M_0)$, $\Theta(M_0)$ and the frequency of the catastrophes q are given. As to V and Θ, they are not independent functions but form part of the general law of the variation of the luminosity with time.

We shall at first consider the case when the energy of the star is supplied by a process like radioactivity; it must be pointed out that in this case the logical process by which we connected the dispersion of the Initial Luminosities with the dispersion of the masses loses its foundation. For it would be difficult to understand why the radioactive energy, suddenly increasing during the „catastrophe“, would attain always the same intensity for stars of equal mass, and proportional to the square of the mass. The difficulties arising when the radioactive theory of stellar energy meets with the theory of the Radiative Equilibrium were already mentioned by Eddington [1]; our second note will deal with this question.

Notwithstanding these difficulties we will accept equation (15), representing the frequency-function of the Initial Magnitudes, as a pure formal law and the most natural one, with the value of the dispersion adopted in the preceding section.

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1) Loc. cit. p. 611.

Let E be the amount of radioactive matter within the star; then the production of energy during the unit of time or the intensity of radiation will be

$$I = cE \quad \ldots . \quad (16),$$

c being the energy-production of the radioactive matter per unit of mass and time; we shall assume c equal for all stars. The rate of diminution of E with time will be proportional to E, whence

$$\frac{dE}{d\tau} = -c_1 E \quad \ldots . \quad (17), \quad \text{and, according to (16),}$$

$$\frac{dI}{d\tau} = -c_1 I \quad \ldots . \quad (17');$$

introducing the bolometric magnitude $M = 2{,}5 \log I$, it is easy to obtain

$$V = \frac{\delta M}{\delta \tau} = 2{,}5 \; c_1 \log e = \text{constant for all stars} \quad \ldots . \quad (18).$$

Thus, in this case the magnitude will increase uniformly with the time, the rate of increase being independent of the magnitude — a known property of radioactivity.

Neglecting other sources of energy, like gravitational contraction, we must assume that the time during which the Initial Magnitude remains unaltered is equal to zero, $\Theta(M_0) = 0$; thus from (7′) we have $n_1 = 0$, and from (6′) with $n = 1$ we obtain the Elementary Luminosity-Curves of the form

$$\psi(M) = \frac{q}{v} \, e^{-\frac{q}{v}(M - M_0)} \quad \ldots . \quad (19),$$

limited by the condition

$$M \geqslant M_0.$$

The logarithms of $\psi(M)$, defined by (19), if plotted against M as abscissae turn out to be parallel straight lines differing by the parameter M_0.

Substituting (19) and (15) into (11) and taking into account that $n_1 = 0$, we obtain the following Luminosity-Curve for the case of the „Radioactive variation of luminosity“:

$$\left.\begin{aligned} &\varphi(M) = k\,e^{-k\left(M - A - \frac{kr^2}{2}\right)}\,\vartheta(t), \\ &\text{where } \vartheta(t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t} e^{-t^2}\,dt, \\ &t = \frac{(M - A - kr^2)}{r\sqrt{2}} \text{ and} \\ &k = \frac{q}{v}. \end{aligned}\right\} \quad (20).$$

The Luminosity-Curve found depends on the three following constants:

1) the dispersion of the Initial Magnitudes r; this quantity has been already assumed as equal to 1,55;

2) the ratio $k = \frac{q}{v}$ of the frequency of the catastrophes to the rate of cooling;

3) the most frequent Initial Magnitude A.

To obtain a plausible value of k it may be observed that for great values of M and t, when $\vartheta(t)$ approaches unity, the character of the curve depends only upon this constant; for $\vartheta(t) = 1$ we have

$$\log \varphi(M) = -k\,M \log e + \text{const.}$$

This property of the curve may be even regarded as a criterion whether the „radioactive law of cooling" holds for stars: for stars of a very low absolute luminosity the logarithm of the frequency-function must approach a straight line. The inclination of the straight line may furnish the value of k. The right-hand side half of the curve II, fig. 2, representing Kapteyn's Luminosity-Curve arranged according to the bolometric magnitude, may be fairly regarded as asymptotically approaching a straight line, whose inclination would correspond to $k = 0{,}6$; we will further adopt this value of k; it would mean that for an interval of time, during which the brightness decreases by $\frac{1}{60}$ of a stellar magnitude, the probability of a catastrophe will attain 0,01.

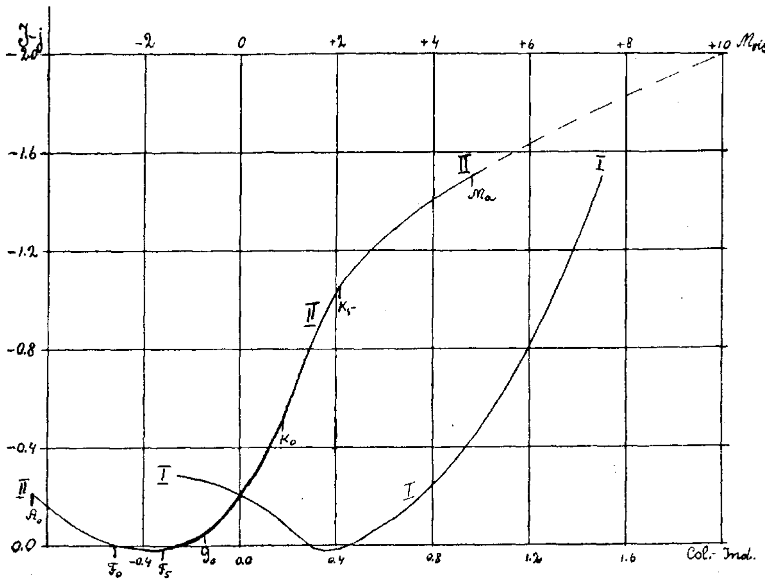
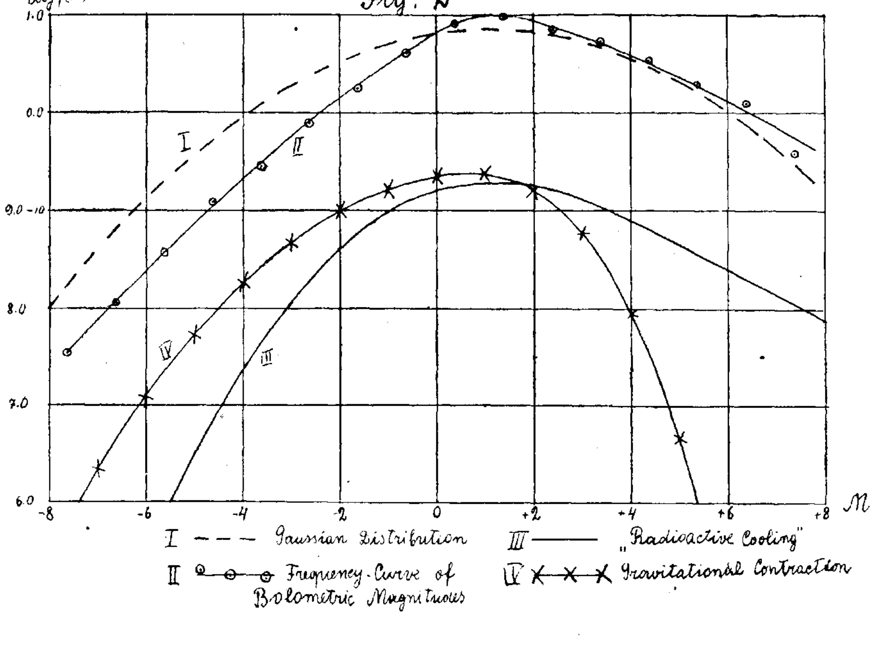

J-j
-2
0
+2
+4
+6
+8
+10
-1.6
-1.2
-0.8
-0.4
0.0
0.0
0.4
0.8
1.2
1.6
Col.-Ind.
I
II

Fig. 1

Fig. 2


1.0
0.0
8.0
7.0
6.0
-8
-6
-4
-2
0
+2
+4
+6
+8
I
II
III
IV
I - - - Gaussian Distribution
II Frequency-Curve of Bolometric Magnitudes
III —— „Radioactive Cooling"
IV Gravitational Contraction

To determine A, we shall make use of the condition that the maximum of the curve $y = \log \varphi(M)$ takes place, when

$$\frac{d\, lg\, \vartheta(t)}{dt} = kr\sqrt{2} \quad \ldots . (21).$$

With $k = 0.6$ and $r = 1.55$ it may be found (with the aid of the tables of the integral $\vartheta(t)$), that $t^{max} = -0{,}145$, which gives $M^{max} - A = +1{,}12$ st. mg.; assuming $M^{max} = +1{,}1$ (see section 3 of this paper), we have

$$A = 0{,}0 \; mg.$$

Thus the most frequent Initial Magnitude must be approximately equal to the absolute magnitude of our sun. The average absolute brightness of the observed Novae at maximum is without doubt considerably higher; e. g. C. Luplau-Janssen and G. Haarh[1]) estimate the absolute magnitude of the Novae as about $-6{,}6$; nevertheless, this may not be regarded as contradicting our result for A, since 1) among the New Stars observed there must act without doubt a strong selection of the most luminous objects and 2) the Initial Magnitude of our theory denotes the magnitude when a steady state is attained and the radioactive energy alone will furnish the energy available for radiation; the violent disturbances and the rapid change of magnitude during the catastrophe have been disregarded in our theory and schematically substituted by an instantaneous restoration of a steady state with a slow „radioactive“ diminution of the brightness; thus, say, the first thousand years of the existence of a Nova are omitted in our theory, and the Initial Magnitude will refer to the magnitude attained after this roughly estimated period of time, great in comparison with human measures, but fairly well represented as one moment when the whole life-time of a star is considered; and, owing to the rapid decrease of the brightness of the Novae after their maximum, the Initial Magnitude thus defined will be considerably fainter than at the maximum light of a Nova.

With the aid of the constants adopted table 8 has been computed; for this purpose formula (20) was remplaced by

$$\log \varphi(M) = -0{,}034 - 0{,}2606\, M + \log \vartheta\left(\frac{M - 1{,}44}{2{,}19}\right) \quad \ldots . (20^a).$$

1) Über die räumliche Verteilung neuer Sterne. Astronomische Nachrichten, 5045 (1920), pp. 89—94.

Table 8.
Theoretical Luminosity-Curve.
Radioactive Variation of Luminosity.

$M_{bolom.}$ ($\pi = 1''$)	Log $\varphi(M)$.	$\varphi(M)$
—6	[5.6] —10	[0.00004]
—5	6.445	0.00028
—4	7.370	0.00234
—3	8.066	0.01164
—2	8.606	0.04036
—1	8.987	0.09705
0(☉)	9.212	0.1629
+1	9.294	0.1968
+2	9.252	0.1786
+3	9.108	0.1282
+4	8.902	0.0798
+5	8.659	0.0456
+6	8.402	0.0252
+7	8.142	0.0139
+8	7.881	0.0076
+9	7.621	0.0042
+10	7.360	0.0023
+11	7.099	0.0013
+12	6.838	0.0007
+13	6.577	0.0004
+14	6.316	0.0002
+15	6.055	0.0001
	Sum . . .	0.9994

The log $\varphi(M)$ of table 8 are plotted on fig 2, curve III. It may be noted that the general features of this curve are very like curve II; a little greater value of the dispersion r would bring both curves to a very satisfactory agreement[1]).

§ 5. An other alternative of a star's evolution is represented by the theory of Gravitational Contraction; this theory presents the advantages of being easily reconcilable with the theory of the Radiative Equilibrium, and the disadvantage of

1) The difference of the ordinates is a matter of mere convenience.

too scant an energy-production, insufficient to cover the radiation towards space during the believed enormous intervals of time[1]). Besides, the Contraction-Theory is in good agreement with the method of deduction of the distribution of the Initial Magnitudes applied in § 3. But the exact solution of the problem which forms the subject of this paper is much more complicate in this case than in the preceding one; since high precision is not required, we shall introduce certain simplifying and schematizing assumptions as to the law of variation of the magnitude with the time, without affecting seriously the general features of the law.

Let I be the energy created by contraction during some interval of time $\triangle\tau$, i_1 and i_2 — the total radiation at the beginning and the end of the interval; if the latter is chosen short enough, we may write

$$\left.\begin{aligned} \triangle\tau &= c\frac{I}{i}, \\ \text{where } i &= \frac{i_1+i_2}{2} \end{aligned}\right\} \quad (22).$$

If ϱ_1 and ϱ_2 represent the density, r_1 and r_2 — the radius of the star of a mass μ at the beginning and the end of a certain interval of time, we may assume

$$I \sim \mu^2\left(\frac{1}{r_2}-\frac{1}{r_1}\right),$$

or, since

$$r \sim \left(\frac{\mu}{\varrho}\right)^{\frac{1}{3}},$$

$$I \sim \mu^{\frac{5}{3}}\left(\varrho_2^{\frac{1}{3}}-\varrho_1^{\frac{1}{3}}\right) \quad \ldots (23).$$

If T denotes the effective temperature of the star, we have

$$i \sim r^2\, T^4 \quad \ldots. (24);$$

assuming as unit of density the density of water, as unit of mass the mass of the sun, and taking for the sun the following data:

absolute magnitude	$M\odot = -0{,}2$;
Temperature	$T\odot = 6000^0$;
Density	$\varrho\odot = 1{,}4$,

1) See A. S. Eddington, loc. cit., p. 611.

we obtain for the absolute magnitude of a star from (24) the following expression:

$$M = 7{,}82 - \tfrac{5}{3}\log\mu + 10\left(\tfrac{1}{6}\log\varrho - \log\frac{T}{1000}\right) \ldots . \; (25).$$

If we take as unit of the radiation i the radiation of a star of the absolute magnitude zero, and if instead of (23) we write

$$I = \mu^{\frac{5}{3}}\left(\varrho_2^{\frac{1}{3}} - \varrho_1^{\frac{1}{3}}\right) \ldots . \; (23'),$$

the unit of time will be determined: this unit will be equal to the time during which a star of the mass $= 1$ and of the constant absolute magnitude $= 0$ will contract from $\varrho = 0$ to $\varrho = 1$; with the value of the solar constant available this time comes out to be about 13 million years.

Taking for simplicity's sake instead of the arithmetical mean of the luminosities, used in (22), their geometrical mean, defined by the mean absolute magnitude

$$\overline{M} = \tfrac{1}{2}(M_1 + M_2) \ldots . \; (26),$$

and combining the modified equation (22) with (23′), we obtain

$$\log \triangle\tau = \tfrac{5}{3}\log\mu + \log\left(\varrho_2^{\frac{1}{3}} - \varrho_1^{\frac{1}{3}}\right) + 0{,}4\,\overline{M} \ldots . \; (27),$$

where $\overline{M}$ can be found from (26) and (25).

The results of the computation for different values of μ and ϱ are contained in table 9; the cases considered are those treated by A. S. Eddington [1]) in the theory of Radiative Equilibrium for dwarf stars, whence the effective temperatures necessary to calculate the M were taken. The chief interest of the table lies in the $\log v$, the $v = v(M, M_0) = \dfrac{\triangle M}{\triangle\tau}$ denoting, according to § 2, the rate of increase of the absolute magnitude with the time. If the $\log v$ are plotted with the $\overline{M}$ as abscissae, one may observe that for greater values of M the curves obtained approach nearly parallel straight lines which may be fairly represented by

$$\log_{10} v = C - \tfrac{1}{3}(M - M_0) \ldots . \; (28),$$

C depending upon the mass and, followingly, upon the Initial Luminosity.

1) Monthly Notices 77, p. 606, table IV (Effective Temperatures of Stars).

Table 9.
Variation of Magnitude with Time. Gravitational Contraction. „Molecular Weight 2".

ϱ	$\mu = 0,2$ M	$\overline{M}$	$\triangle\tau$	$lg\ v$	$\mu = 0,5$ M	$\overline{M}$	$\triangle\tau$	$lg\ v$
0	+ 0.67				— 2.18			
		+ 0.67	0.016	— ∞		— 2.18	0.005	— ∞
0.002	+ 0.67				— 2.18			
		+ 0.87	0.030	1.12		— 2.00	0.010	1.54
0.035	+ 1.07				— 1.83			
		+ 1.45	0.046	1.21		— 1.48	0.014	1.69
0.13	+ 1.82				— 1.12			
		+ 2.27	0.103	0.94		— 0.68	0.031	1.44
0.33	+ 2.72				— 0.25			
		+ 3.21	0.226	0.64		+ 0.22	0.067	1.17
0.65	+ 3.70				+ 0.72			
		+ 4.16	0.402	0.35		+ 1.18	0.119	0.88
0.97	+ 4.62				+ 1.63			
						+ 2.36	0.429	0.53
1.53					+ 3.08			
						+ 3.90	1.496	0.04
2.11					+ 4.72			

ϱ	$\mu = 1.0$ M	$\overline{M}$	$\triangle\tau$	$lg\ v$	$\mu = 1.5$ M	$\overline{M}$	$\triangle\tau$	$lg\ v$
0	— 4.12				— 5.10			
		— 4.12	0.003	— ∞		— 5.10	0.002	— ∞
0.002	— 4.12				— 5.10			
		— 3.98	0.005	1.75		— 4.98	0.004	1.78
0.035	— 3.83				— 4.86			
		— 3.53	0.007	1.94		— 4.61	0.005	2.00
0.13	— 3.23				— 4.36			
		— 2.83	0.014	1.76		— 4.00	0.009	1.89
0·33	— 2.43				— 3.64			
		— 1.96	0.028	1.51		— 3.20	0.018	1.69
0.65	— 1.50				— 2.77			
		— 1.06	0.048	1.26		— 2.34	0.029	1.46
0.97	— 0.63				— 1.92			
		+ 0.10	0.170	0.93		— 1.20	0.101	1.15
1.53	+ 0.83				— 0.49			
		+ 1.64	0.593	0.44		+ 0.32	0.345	0.67
2.11	+ 2.46				+ 1.14			

ϱ	$\mu = 3.0$ M	$\overline{M}$	$\triangle\tau$	$lg\ v$	$\mu = 4.5$ M	$\overline{M}$	$\triangle\tau$	$lg\ v$
0	— 6.52				— 7.22			
		— 6.52	0.002	— ∞		— 7.22	0.002	— ∞
0.002	— 6.52				— 7.22			
		— 6.44	0.003	1.66		— 7.16	0.003	1.52
0.035	— 6.37				— 7.11			
		— 6.20	0.004	1.96		— 6.98	0.004	1.87
0.13	— 6.03				— 6.85			
		— 5.78	0.006	1.95		— 6.66	0.005	1.90
0.33	— 5.52				— 6.46			
		— 5.18	0.009	1.88		— 6.18	0.007	1.90
0.65	— 4.83				— 5.89			
		— 4.46	0.013	1.76		— 5.57	0.009	1.84
0.97	— 4.08				— 5.25			
		— 3.40	0.042	1.51		— 4.62	0.027	1.67
1.53	— 2.71				— 3.99			
		— 1.91	0.141	1.06		— 3.20	0.084	1.27
2.11	— 1.11				— 2.42			

For purposes of schematization we will adopt formula (28) instead of the actual log v, contained in table 9; the departures from this formula are considerable only for high luminosities, where the time $\triangle \tau$ is small and where these departures cannot, therefore, affect seriously the final result. The following scheme of the variation of M with the time has been adopted finally: for the time Θ, during which the star contracts from infinity to the density $\varrho = 0{,}035$, the absolute magnitude remains constant and equal to the Initial Magnitude M_0; Θ may be called the duration of the giant stage; from $\varrho = 0{,}035$ begins the decrease of brightness according to formula (28).

It may be noted that in the case of a body of uniform temperature, whose radiation is furnished by its capacity of heat only, we would obtain a similar law for the variation of the magnitude with the time, namely

$$\log_{10} v = -0{,}3\, M + \text{Const.}$$

Here the coefficient of M is very near the value found in formula (28), and thus the law expressed by this formula may be regarded as a typical representative of a whole category of different laws of „cooling".

The values of C, fitting best the log v of table 9, and the Θ for different values of μ and M_0 are given in table 10; these

Table 10.

μ	M_0	C	Θ
0.2	+0.67	1.48	0.0462
0.5	−2.18	1.98	0.0153
1.0	−4.12	2.26	0.00795
1.5	−5.10	2.42	0.00625
3.0	−6.52	2.58	0.00525
4.5	−7.22	2.58	0.00535

quantities were plotted against the M_0 as abscissae, and from the curves drawn the corresponding values were read for equidistant values of M_0; the result is contained in table 11; the data for $M_0 = -8$ and for $M_0 > +1$ are extrapolated values; they may be regarded, nevertheless, as fairly reliable, since the extrapolation for $M_0 > +1$ is linear and has theoretical foundation in the theory of Radiative Equilibrium (for $\mu < 0{,}5$ the luminosity during the giant stage varies as the third power of μ); instead of the extrapolation the data could be calculated directly, but the labour of the computation has been considered not worth while to undertake; the use of the extrapolated values having been very limited (the data for $M_0 = -8$ and $M_0 \leqslant +3$ not being used at all), we shall no longer linger upon this question.

Table 11.

M_0	+6	+5	+4	+3	+2	+1	0	−1
C	0.55	0.72	0.90	1.08	1.25	1.4	1.60	1.77
Θ	0.418	0.277	0.184	0.122	0.0811	0.0538	0.0357	0.0243

M_0	−2	−3	−4	−5	−6	−7	−8
C	1.95	2.10	2.25	2.40	2.53	2.59	2.54
Θ	0.0165	0.0115	0.00832	0.00640	0.00547	0.00513	0.00594

For a given M_0 formula (28) leads, according to (6′) and (7′), to the following Elementary Luminosity-Curve ($n = 1$):

$$\left.\begin{aligned} \log \psi(M) = \left(\log \frac{q}{1 + q\,\Theta} - C\right) + \tfrac{1}{3}(M - M_0) - \\ - 3\,q \log^2 e \,.\, 10^{\frac{1}{3}(M - M_0) - C} \\ n_1 = \frac{q\,\Theta}{1 + q\,\Theta} \end{aligned}\right\} \quad (29).$$

Equation (29), together with (15), gives the solution of (11) or (11′); the computation, however, must be executed numerically. As has been already noted, in the case of Gravitational Contraction the considerations by which we connected the dispersion of the masses with the dispersion of the Initial Luminosities obtain real foundation; thus we cannot assume an arbitrary value of the most frequent Initial Magnitude A; assuming the most frequent $\log \mu$ correspond to $\mu = 0{,}6$ ($\Theta = 1$), we obtain (interpolating table 10) $A = -2{,}5$. The constant q we shall define so that for stars with $M_0 = A$ the Elementary curve (29) will have the maximum at $M = +1{,}1$; the corresponding value of q results as $q = 6$ [1]).

The result of the numerical computation according to (29), (15) and (11) is given in table 12.

Table 12.

Theoretical Luminosity-Curve. Contraction-Theory.

M	$\varphi(M)$	Log $\varphi(M)$
— 7	0.000225	6.352 — 10
— 6	0.001188	7.075
— 5	0.00538	7.731
— 4	0.01847	8.267
— 3	0.04750	8.677
— 2	0.0970	8.987
— 1	0.1641	9.215
0	0.2269	9.356
+ 1	0.2371	9.375
+ 2	0.1609	9.207
+ 3	0.0580	8.763
+ 4	0.00880	7.944
+ 5	0.000444	6.647

Sum . . . 1.026

Here $\varphi(M)$ was calculated simply as the following sum:

$$\varphi(M) = \sum_{M_0=-\infty}^{M_0=M} \psi(M)\, P(M_0) + n_1\, P(M) \quad \ldots . \ (30),$$

$$\text{where } P(M_0) = \int_{M_0-\frac{1}{2}}^{M_0+\frac{1}{2}} X(M_0)\, d\, M_0$$

1) It may be noted that if M_{max} represents the position of the maximum of the Elementary curve, then $q = - \frac{d\,v}{d\,M}\Big|\, M = M_{max}$.

and the sum is taken over even values of M and M_0. The departure of the sum of the $\varphi(M)$ in table (12) from unity (see condition (13)) is due to the rough method of calculating the mechanical quadratures; this difference has, however, no serious significance, the relative values of $\varphi(M)$ presenting the chief interest. The log $\varphi(M)$, contained in table 12, are plotted on fig. 2, curve IV; this curve differs decidedly from curve III, exhibiting a very steep decrease of the relative number of stars for low luminosities and a greater number of highly luminous stars, the dispersion of the Initial Luminosities adopted being equal in both cases; thus if one of our hypotheses would hold for some stellar universe, where the distribution of the luminosities is known from observation, there would be no difficulty to perform the choice between the two hypotheses. As to the theory of Gravitational Contraction, the decrease of the number of faint stars required by this theory is so enormous, that beginning from a given limiting magnitude the fainter stars must practically be entirely wanting; since many stars of excessively low luminosity are known in the immediate neighbourhood of the sun, and since the observed curve deduced from the data of Kapteyn and Van Rhijn differs too widely from curve IV, the applicability of the theory of Gravitational Contraction, at least — to stars of low luminosity, becomes improbable from this standpoint too. We believe, that if the general theory of the reiteration of the consecutive stages of stellar evolution will prove to be fit to account for the frequency-function of stellar luminosities, a combination of both hypotheses will be necessary, the energy of contraction supplying a considerable part of the stellar radiation during the initial giant stage, and the radioactive energy prevailing during the dense dwarf stadium.

§ 6. In connection with the possibilities of stellar evolution discussed above a few words concerning the theories explaining the apparition of the New Stars will not be superfluous. All such theories may be subdivided into two classes: I) the theories of collision and II) the theories of an internal source of the catastrophe. The first class of hypotheses, proposed the most frequently, may be subdivided into two sub-classes: I^a) external collisions, as the collision of two stars (Vogel) or of star and nebulosity (Seeliger) and I^b) internal collisions, as planets [1]) or huge comets [2]) falling on the central sun. As to the external collisions, the hypothesis of Vogel and its modification given by Klinkerfues and Wilsing (tidal action) has been abandoned on account of the enormous number of stars and the impossible total mass of the stellar universe required by it; but it appears, that analogous difficulties arising in the case of the theory of Seeliger have been overlooked. Nölke [3]) observes that, owing to the sudden increase in brightness, the short duration of the maximum light and the moderate average velocity of the stellar motions, the dimensions of the hypothetical nebula must be very small, so that all advantages of Seeliger's theory (the great extent of a nebula increasing the probability of collision) must disappear. Nölke considered the case of a uniform velocity of the star moving through the nebula equal to 20 $\frac{\text{km}}{\text{sec.}}$. It will be shown that, though the actual case must be more favourable to Seeliger's hypothesis than has thought Nölke, this hypothesis must, nevertheless, be abandoned for similar reasons as the theory of Vogel. It may be noted here that several authors, treating the problem of collision of cosmical bodies moving through the interstellar space, take into account only the cosmical velocity of the star, and overlook the gravitational action, which for masses of the order of our sun and greater ones may give origin to velocities of the order of 1000 $\frac{\text{km}}{\text{sec}}$; so did Luplau-Janssen and Haarh [4]), attempting to prove that the greater space-velocity of the less massive stars would equalize the luminosities of Novae of different masses; and similarly did so Dr. Freundlich in the article upon New Stars on page 675 of Newcomb-Engelmann's „Populäre Astronomie", 6te Auflage, 1921. If the Novae are regarded as the result of the intrusion of a star into a cosmical cloud, the sphere of action and, thus, the maximum dimensions of the nebular (or meteoric) cloud which may participate in the catastrophe, may be assumed of the order of the distance which a particle falling from infinity would

1) Fr. Nölke. Über die Entstehung der neuen Sterne. Astronomische Nachrichten 5110 (1921), pp 345—352.

2) T. J. J. See. The Cause of Temporary Stars. Astronomische Nachrichten, Jubiläumsnummer 1921, pp 23—25.

3) Loc. cit. p. 90.

4) Loc. cit.

cover before reaching the surface of the star during an interval equal to the duration of the catastrophe; now, the observational data concerning the Novae indicate that if these phenomena are the result of some collision, the duration of the collision cannot be great — not more than, say, 2 days. For a star of the mass of the sun this interval gives a sphere of action equal to 25 million kilometers or about 10^{-6} parsecs; a star moving with a velocity of 30 km. per second or $0{,}3.10^{-4}$ parsec per year will thus cover a total sphere of action equal to $4\pi.(10^{-6})^2.0{,}3.10^{-4} = 4.10^{-16}$ cubic parsec per year; assuming the probability of a catastrophe equal to 10^{-9} for the interval ot 1 year, which corresponds to 1 yearly catastrophe for a universe of 1000 million stars, we find that one cubic parsec must contain $\frac{10^{-9}}{4.10^{-16}} = 2\,500\,000$ masses able to produce a catastrophe. The minimum mass of the hypothetical nebular matter which actually reaches the surface of the star may be estimated as equal to the earth's mass: such a mass, falling upon the central sun, would only account for the radiative energy of the New Star lost towards space. Adopting the minimum mass, we find that the nebular or meteoric mass contained in 1 cubic parsec must be equal to 8 times the sun's mass: that is about 50 times greater than the maximum possible mass deduced from the observed motions of the stars [1]). But even this must be regarded as a minimum, since in the case of a New Star the energy lost through radiation forms probably a small fraction of the whole energy involved in the catastrophe; as the result of the latter the New Star expands into an extended gaseous nebula (small nebula around Nova Persei 2, Nova Aquilae 3 and others) with an appreciable diameter of a few seconds of arc, whose actual dimensions must be many times greater than the orbit of Neptune; the energy necessary to produce such an effect must be of the order of the energy liberated from the collision

1) One cubic parsec contains, according to the data of Kapteyn and Van Rhijn (loc. cit.) 0,045 stars brighter than absolute magnitude 9.8; J. H. Jeans has found („The Motions of the Stars in a Kapteyn-Universe", Monthly Notices 82 (1922) pp 122—132) that the observed motions of the stars may be accounted for by an average mass attributed to each counted star equal to 2.4—3.2 times the sun's mass, or by a mass of about 0,1—0,15 per cubic parsec.

of two masses of stellar order. Further, in the case of the collision with a cosmical cloud only a small fraction of the mass contained within the sphere of action mentioned above will actually meet the star's surface, the major part being left behind; thus the hypothetical nebulous mass required to account for the observed frequency of Novae must be increased some million times. As the final result we may state that all theories of external collisions must be entirely abandoned in the explanation of New Stars. Internal collisions (the hypotheses of See, Nölke and others) meet with less serious objections; however, the mechanical energy necessary to produce the expansion of the star into a nebula seems to limit this case with the collision of two components of a binary star; if we would retain the conception of small planets or comets falling upon the star, we must regard the collision only as an impulse disturbing the state of equilibrium of the star, in consequence of which some unknown sources of energy — probably of intraatomical character — will be opened; but then the collision becomes a non-substantial feature, and may be substituted simply by some unknown disturbing factor acting upon the star.

The following scheme of the catastrophe may be imagined [1]. Let the energy of the star be supplied by certain atomic changes of a definite element, which may be called the „active matter", and let the conditions favouring these changes (e. g. a sufficiently high temperature) exist only in the interior parts of the star; these interior parts are thought to be in a perfect state of equilibrium (Radiative Equilibrium) without any convection currents for millions of years; as has been shown by S. Chapman [2], gaseous diffusion cannot produce sensible changes in the constitution of the inner parts for milliards of years. Thus a limited amount of the active matter being found within the central parts, the radiative energy of the star lost towards space will be supplied on the expense of only a fraction of the total amount of the active matter; as the quantity of the latter, found within

1) The principal features of the scheme are suggested by the ideas of H. N. Russel as expressed in a paper by S. Chapman: Diffusion and Viscosity in Giant Stars. Monthly Notices, 82 (1922) p. 295. Similar ideas are found in the paper of E. Wiechert, mentioned above (Vierteljahrschrift 56, 3 pp. 185—191).

2) Monthly Notices, 82, p. 297.

the central parts, will decrease with the time, the luminosity will decrease, too, according to a law similar to the one called above „the radioactive law of cooling". If then, under the action of some disturbing factor (say — a huge comet falling upon the star), violent convective currents appear, fresh amounts of the active matter from the outer parts will be transported towards the centre, the energy-production will suddenly increase and will give origin to the internal explosion which transforms the star into an extended nebula; a part of the star's mass will be probably lost, being thrown out and dispersed over the interstellar space. But the expansion of the remaining part will cease after the temperature in the centre will fall below the limit when the intraatomic changes occur; after a short period of pulsations the nebula will begin gradually to contract and will go again trough the consecutive stages of evolution.

Our hypothesis seems to be in no contradiction with the spaces of time required by Geology; the theory of a „radioactive cooling" (see § 4) leads to the conclusion that for an interval of time during which the luminosity diminishes by $\frac{1}{60}$ mg, the probability of a catastrophe attains 0,01; assuming this probability equal to the relative frequency of the Novae adopted above, or to 10^{-9} per year, we obtain that the decrease of brightness of an average star must be about 0,17 st. magn. per 100 million years; thus the relative constancy of the average climate on the earth's surface during the past time is in agreement with these numerical estimations. Let us assume that life began on earth when the mean temperature of its surface attained $+50^0$ *C*, (at the poles the temperature would be some $+20^0$ — $+30^0$ *C*, the difference of temperature between the poles and the equator being smaller than at present on account of the greater humidity of the air and the absence of the highly reflecting polar caps), and that the radiation of the sun was proportional to the fourth power of the effective temperature of the earth[1]); then we obtain, that the sun at the epoch considered should be by $10 \log \left(\frac{273+50}{273+15}\right) = 0{,}50$ stellar magnitudes brighter than now;

1) It is supposed here that the amount of energy radiated by the earth towards space is proportional to the energy received from the sun; in this case the effective temperatures of the earth and the sun will be proportional too.

if we assume for the sun the rate of „cooling“ given above for an average star, we obtain for the space of time elapsed since life appeared on earth's surface the round value of 300 million years[1]: a value in good agreement with the age of the radioactive rocks and with several estimations of the duration of Geological periods.

1) According to S. J. Bailey, „there appear between 9 and 25 Novae per year of the ninth magnitude and brighter“ (The Observatory, № 574 (1922, May), p. 93); this would give a considerably greater frequency of the Novae and proportionally a shorter duration of a star's life-time; but, probably, the faint Novae of the ninth magnitude must be regarded as members of a wider universe, including not 1 milliard of stars, as has been assumed above, but many milliards; probably, the frequency of the catastrophes estimated above as 10^{-9} per year represents the order of the quantity well enough.

II.

Note on some Consequences of the Theory of the Radiative Equilibrium of the Stars.

The theory of the Radiative Equilibrium of the Stars, first proposed by Schwarzschild[1]) for the outer layers of the sun and discussed by Emden[2]) as one of the possible forms of equilibrium for the interior of a star, has gained its actual importance for astronomical research through the investigations of A. S. Eddington; the chief results of these investigations are exposed in two consecutive papers[3]), on the basis of which the following considerations on stellar evolution will be developed.

Let L be the absolute bolometric luminosity, μ — the mass of the star; then

$$E = \frac{L}{\mu} \quad \dots . (1)$$

will represent the loss of energy towards space per unit of mass and time; the corresponding quantity expressed in absolute units has been denoted by Eddington in his first paper by $4\pi\varepsilon$; formula (26 a) of this paper gives

$$\frac{a k \varepsilon}{\pi G \mu} = (1 - \beta) \text{ or, with } \mu = \frac{a c}{4\pi}{}^{4)},$$

$$\frac{4 k \varepsilon}{G C} = (1 - \beta) \quad \dots . (2),$$

where C is the velocity of light, G — the constant of gravitation, k — the effective mass-coefficient of absorption in absolute units

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1) Über das Gleichgewicht der Sonnenatmosphäre. Nachrichten d. K. Gesellschaft der Wissenschaften zu Göttingen. Math. physik. Kl. H. 1. 1906.

2) Gaskugeln (Leipzig u. Berlin, 1907) pp. 320—332.

3) On the Radiative Equilibrium of Stars. Monthly Notices 77 (1916), pp. 16—35. Further Notes on the Radiative Equilibrium of the Stars. Ibidem, 77 (1917) pp. 596—612.

4) Loc. cit. p. 598; μ denotes here a universal constant, not the mass.

and $1 - \beta$ the ratio of the radiation-pressure to gravitation within the star. Instead of ε, we will take the value of E, defined by (1) with our sun as unit of luminosity and mass; then from (2) we have

$$kE = (1 - \beta) \times \text{const.};$$

the constant, which may be found with the aid of the universal constants G and c and the value of the solar constant, is more easily determined from the data for Eddington's typical Giant star[1]) („molecular weight 2"):

$$\text{Density} = 0{,}002;\ k = 5{,}4\ (\text{C. G. S.});$$

absolute bolometric magnitude $= -5{,}1$; and absolute magnitude of the sun $= -0{,}2$.

We find the constant $= 2070$; thus

$$kE = 2070 (1 - \beta) \ \ldots\ldots \ (3),$$

k being expressed in C. G. S. units and E — in terms of our sun.

The combination of (3) and (1) gives the following expression for the luminosity

$$L = \frac{2070 (1 - \beta)}{k} \mu \ \ldots\ldots \ (4),$$

which is equivalent to the corresponding formulae given by Eddington[2]).

Equations (3) and (4) are general and equally applicable to giant and dwarf stars: the difference between these two cases will be due to the different values of $(1 - \beta)$. Let $(1 - \beta_0)$ denote the radiation-pressure for a perfectly gaseous (giant) star; then Eddington's equation (44)[3]) gives

$$(1 - \beta_0) = \beta_0^4 \,.\, m^2 \mu^4 \times \text{const.} \ \ldots\ldots \ (5);$$

thus $(1 - \beta_0)$ depends upon the mass μ and the molecular weight m alone. For dwarf stars of the same mass (and molecular weight)

$$(1 - \beta) < (1 - \beta_0) \ \ldots\ldots \ (6),$$

the radiation-pressure diminishing with increasing density. Thus, whereas

$$(1 - \beta_0) = f(\mu, m) \ \ldots\ldots \ (7), \text{ we have}$$

$$(1 - \beta) = f_1 (\mu, m, \varrho) \ \ldots\ldots \ (8),$$

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1) Loc. cit. pp. 601—602.

2) Loc. cit. p. 601, second footnote.

3) Loc. cit. p. 601.

ϱ denoting the density. Since $(1-\beta)$ does not depend neither on k nor upon E, one of these two quantities, by virtue of (3), cannot be chosen arbitrarily and must be the function of the other and of $(1-\beta)$. Eddington finds k for his typical giant star, the luminosity and, consequently, the energy-loss E being given; the value of k found he applies then to all stars and calculates their hypothetical absolute luminosities; a similar computation of the hypothetical luminosities with a constant value of k we find by Fr. H. Seares[1]). But the data now available, especially those found by Seares, allow of a determination of the values of k for various typical stars separately. From the paper by Fr. H. Seares already cited were taken the absolute luminosities, masses[2]) and mean densities[3]) and with the mass and density as argument, the hypothetical temperature according to Eddington[4]) and the corresponding h y p o t h e t i c a l l u m i n o s i t y were computed; for this purpose the absolute magnitude[5]) was plotted as function of density for different masses, so that the values needed could be directly read from the curves.

If M is the observed, M_H — the hypothetical absolute magnitude, then, taking into account that $(1-\beta)$ for the given mass and density remains unaltered, it is easy to obtain from (4) the following equation:

$$\log k = \log 5{,}4 - 0{,}4\,(M_H - M) \quad \ldots . \ (9).$$

Table 1 contains the result. Since the actual values of the luminosity and mass, on which the computations were based, possess a considerable degree of accuracy, and since the constancy of the average molecular weight for different stars seems to be very probable[6]), the wide range of variation of the k for different stages of the dwarf series as revealed by table 1 may be regarded as an established fact.

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1) The Masses and Densities of Stars. Astrophysical Journal 55 (1922), pp. 233—237.

2) Ibidem, p. 179, table IV.

3) Ibidem, p. 202, table XIV.

4) Monthly Notices 77, p. 606.

5) A formula for the computation is given in the preceding paper (form. 25) in § 5.

6) See Fr. H. Seares, loc. cit., pp. 227—233.

Table 1.

The Mass-Coefficient of Absorption and Spectral Type.
„Molecular weight 2". Absolute Magnitudes reduced to $\pi = 1''$.

Spectrum	M_{vis}	M_{Bol}	μ ($\odot = 1$)	Density	M_H Bolom.	K (CGS)	E ($\odot = 1$)	$k:E$
A_5	-3.50	-3.60	4.0	0.40	-6.2	59	5.8	10
F_0	-2.60	-2.60	2.5	0.40	-5.0	49	3.6	14
F_5	-1.68	-1.66	1.5	0.39	-3.7	34	2.5	14
G_0	-0.65	-0.70	1.0	0.68	-1.6	12	1.6	8
G_5	$+0.20$	-0.08	0.76	1.2	$+0.4$	3.4	1.2	3
K_0	$+0.90$	$+0.39$	0.68	1.3	$+1.4$	2.1	0.85	3
K_5	$+2.10$	$+1.11$	0.62	1.4	$+2.0$	2.3	0.49	5

The variation is of such a character that for the hotter and more massive stars larger values of k occur; this is probably due to the different character of the radiation in the interior of stars of different temperatures, the high-temperature radiation being more strongly absorbed than that of low temperature. The 8^{th} column of the table gives the energy-loss per unit of mass; it must be remarked that the range of variation of this quantity as represented by the table is considerably smaller than the actual range, since E may vary from many thousand (Rigel) to less than $\frac{1}{10000}$ (companion to Procyon) times the corresponding value for the sun.

Taking into account this possible variety of the k and E, we may proceed to the discussion of the bearing of the theory of Radiative Equilibrium on stellar evolution. Let us consider a gaseous mass of a given density and a given k; $(1-\beta)$ being determined by (8), the energy-loss E will be a fixed quantity as long as the physical conditions remain unaltered; let us suppose that some source — say, radioactivity — supplies an amount of energy per unit of mass and time equal to E_0, and let the energy created by gravitational contraction be E_1; if $E_1 > 0$, the star will contract, if $E_1 < 0$ — expand. E_0 is thought to be either constant for the given star or vary with its density and temperature, whereas E_1 may assume arbitrary values and determines the direction of the star's evolution. We shall always have

$$E = E_0 + E_1 \ldots . (10) \text{ or}$$
$$E_1 = E - E_0 \ldots . (10').$$

Thus, if $E_0 < E$, the density will increase with the time and the direction of the evolution will be a normal one; but if the radioactive energy is greater than the amount which the star is able to emit towards space, an expansion will take place. Let us at first consider the case when k remains constant as well as the effective molecular weight. If $E_0 > E$, then the coefficient of the radiation-pressure $1 - \beta$ will increase with the decreasing density and simultaneously E must increase (according to (3)); if E becomes equal to E_0, a state of equilibrium will be established; but $(1 - \beta)$ cannot surpass the value $(1 - \beta_0)$, corresponding to the „perfect gaseous conditions", and thus for given k, μ and m there must exist an upper limit of the energy-loss,

$$E_{max} = \frac{2070\ (1 - \beta_0)}{k} \ldots . (11);$$

if $E_0 > E_{max}$, the matter of the star, producing too large an amount of energy, will expand infinitely: such stars cannot, therefore, exist for a long time. Since $(1 - \beta_0)$ increases with the mass, formula (11) furnishes us an inferior limit of the mass of a stable gaseous body with a given E_0. If a star, for which the condition of stability $E_0 < E_{max}$ is fulfilled, divides into smaller parts, it may happen that the E_{max} for each part separately will be smaller than the radioactive energy E_0 (supposed to remain unaltered) and then the division must go on until the whole star will convert into a meteoric cloud, the solid particles of the cloud being prevented from further division by the forces of cohesion. It may be possible that the extended cosmical clouds and dark nebulosities of the Milky Way are aggregations of matter which cannot unite into more compact bodies by virtue of its excessive energy-production.

On the contrary, in those cases when $E_0 < E_{max}$, a steady state will be soon attained as the result of contraction, and the greater the difference $E_{max} - E_0$, the denser will be the star.

However, the invariability of E_0 and k throughout the enormous range of temperatures, which various stages of the evolution must exhibit, seems to be highly improbable; as to k, we found above that this quantity undergoes substantial changes with the spectral type and suggested that these changes are the result of varying temperature; if so, during the expansion of the unstable star we must expect a decrease of k as the temperature diminishes, and this decrease will tend to restore the stability. But we cannot be satisfied with such a variation of k alone; the decrease of k must have definite limits, and even a periodical fall and increase of this value may be expected over the enormous range which is covered by the stellar and nebular temperatures and densities; e. g. for the products of vaporisation of shooting stars (Perseids) the writer estimates the value of k as about 1000, at a temperature of 6000—7000°; and such high values of k may therefore occur in masses which are small or rarified enough, so that their central temperature be of the order given above. Thus a dependence of E_0 upon the physical conditions, chiefly upon the temperature, is necessary too, if we should explain the existence of stellar masses at all. The values of E in table 1 may be fairly regarded as representative of the E_0, since it is generally believed that the energy of contraction forms a very small fraction of the total energy; and these relatively great values of the energy-production which we called for simplicity's sake the radioactive energy, make it difficult to understand why the hypothetical initial cosmical cloud has condensed into individual stars instead of tending to disperse over the infinite space: for the condensation will be stable only after masses of the stellar order are formed, whereas smaller intermediate condensations cannot exist. Unless the general opinion concerning the part played by nebular matter in cosmic evolution be altered, we must conclude that the nebular or meteoric matter which gave origin to the individual stars, must have possessed a value of the „radioactive“ energy-production considerably lower than the corresponding value actually observed in stars, or even a value equal to zero.

The dependence of E_0 upon the physical conditions of the interior of a star must therefore be assumed as the most plausible assumption to account for the observed luminosities and masses of the stars of different spectral types and the only one which seems to be in satisfactory agreement with the generally accepted direction of stellar evolution. The rough parallelity of the variation of the E and k, shown by the last column of table 1, suggests the action of the same physical factors upon both quantities.

III.

Summary and Conclusions.

The results of the preceding notes, dealing chiefly with the luminosity-distribution of the fixed stars in connection with the theories of stellar evolution, may be summarized as follows:

1) A theory is proposed which explains the frequency-function of stellar luminosities, called the Luminosity-Curve, by certain assumptions as to the laws of stellar evolution. These assumptions are: that there exists a reiteration of the stages of evolution of a single star, periods of „cooling" or of a gradual decrease of the luminosity with the time being interrupted by catastrophes, in consequence of which the luminosity is suddenly increased up to a limiting value defined by the Initial Magnitude (M_0); during the consecutive periods of cooling the law of decrease of the luminosity, or the „rate of cooling" is supposed to remain invariable;

2) the Luminosity-Curve within a stellar universe can be determined if the law of the variation of the luminosity with the time or the rate of cooling $V(M, M_0)$, the frequency of the catastrophes q and the dispersion of the Initial Luminosities are known;

3) theoretical Luminosity-Curves are constructed for the following two assumptions as to the function $V(M, M_0)$:

a) when the energy of the star is supplied by a process like radioactivity, so that the bolometric magnitude increases uniformly with the time; that is the so called law of „radioactive cooling";

b) when the energy of a star during the periods of undisturbed cooling is supplied by Gravitational contraction alone; the energy producing the catastrophe may be of radioactive as well as of gravitational character (collision);

4) with a certain choice of the dispersion of the Initial Luminosities the theoretical curve for the radioactive law of cooling may be made to agree closely enough with the Luminosity-Curve given by Kapteyn and Van Rhijn, if the observed

curve is arranged according to the bolometric magnitudes. The observed curve would lead to a rate of the radioactive cooling equal to 0,17 stellar magnitudes in 100 million years, if the frequency of the catastrophes is assumed equal to one Nova per annum for a universe of 1000 million stars;

5) the Gravitational Contraction as the only source of the radiative energy would lead to an improbably rapid decrease of the number of faint stars with decreasing luminosity, which contradicts observational evidences;

6) a hypothesis as to the cause of the New Stars is proposed, these phenomena being thought to represent explosions of intra-atomic character, occurring in consequence of convection currents through which large amounts of some active matter, supplying energy only at high temperatures, are transported towards the hot central parts of the star; the expansion of the New Star into a nebula is the consequence of the increased radiation-pressure;

7) assuming certain intraatomic changes or the activity of matter to be the main source of stellar energy, the theory of the Radiative Equilibrium leads to the conclusion that a plausible explanation of the mere existence of stellar masses can be given only if the assumption of a variation of the activity (or the energy-production) of the matter with the physical conditions, chiefly with the temperature is made, the activity being favoured by the higher temperatures;

8) observational data indicate a variation with the spectral type of the effective mass-coefficient of absorption k, playing an important part in the theory of Radiative Equilibrium, the values of k decreasing for the „later“ types with the decreasing mass and temperature;

9) the dispersion of the logarithms of the masses of all stars has been determined provisionally as $\pm 0{,}31$, this value being probably a little over-estimated.

May 1922.

Abbreviations used:

Mt Wilson Contr. = Contributions from the Mount Wilson Observatory.
Monthly Notices = Monthly Notices of the Royal Astronomical Society.
Groningen Publications = Publications of the Astronomical Laboratory at Groningen.